Asymptotics for solutions of the Cauchy problem with step-like initial data for integrable equations

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We study the long-time asymptotics of the Cauchy problem solutions.

modified Korteweg – de Vries equation

$$q_t(x,t)+6q^2(x,t)q_x(x,t)+q_{xxx}(x,t)=0,\ x\in\mathbb{R},\ t\geq 0, \tag{1}$$

$$q(x,0)=q_0(x)\sim\begin{cases}c_{\mathfrak{l}}, & x\to-\infty,\\ c_{\mathfrak{r}}, & x\to+\infty.\end{cases}$$

Camassa-Holm equation

$$u_t-u_{txx}+2\omega u_x+3uu_x=2u_xu_{xx}+uu_{xxx},\ x\in\mathbb{R},t\geq 0, \tag{2}$$

$$u(x,0)=u_0(x)\sim\begin{cases}c_{\mathfrak{l}}, & x\to-\infty,\\ c_{\mathfrak{r}}, & x\to+\infty,\end{cases}\qquad \frac{c_{\mathfrak{l}}+\omega}{c_{\mathfrak{r}}+\omega}>0.$$

Cauchy problem for the modified Korteweg – de Vries equation

The large time asymptotic behavior of the solution of the Cauchy problem for the modified Korteweg – de Vries equation (MKdV)

$$q_t(x,t) + 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0, \; x \in \mathbb{R}, \; t \geq 0, \tag{1}$$

$$q(x,0) = q_0(x), \tag{3}$$

with step-like initial data $q_0(x)$, which tends rapidly to constants as $x \to \pm\infty$:

$$q_0(x) \to c_{\mathfrak{l}}, \; x \to -\infty, \quad q_0(x) \to c_{\mathfrak{r}}, \; x \to +\infty, \qquad l > c_{\mathfrak{l}} > c_{\mathfrak{r}} \geq 0.$$

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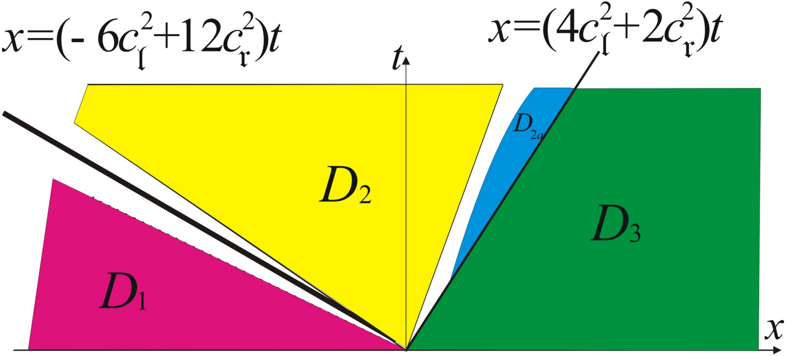
$$\int_{-\infty}^{0} |q_0(x) - c_{\mathfrak{l}}| \mathrm{e}^{2|x|l} \mathrm{d}x + \int_{0}^{+\infty} |q_0(x) - c_{\mathfrak{r}}| \mathrm{e}^{2xl} \mathrm{d}x < \infty.$$

$$\int_{-\infty}^{0} |x|\, |q(x,t) - c_{\mathfrak{l}}| \mathrm{d}x + \int_{0}^{+\infty} x |q(x,t) - c_{\mathfrak{r}}| \mathrm{d}x < \infty.$$



Regions in (x,t)–half-plane

$D_1 = \{(x,t): \quad x < (-6c_{\mathfrak l}^2 + 12c_{\mathfrak r}^2 - \varepsilon)t\}$ $q = c_{\mathfrak l} + o(1)$

$D_2 = \{(x,t): \quad (-6c_{\mathfrak l}^2 + 12c_{\mathfrak r}^2 + \varepsilon)t < x < (4c_{\mathfrak l}^2 + 2c_{\mathfrak r}^2 - \varepsilon)t\}$ modulated elliptic wave

$D_{2a} = \{(x,t): \quad (4c_{\mathfrak l}^2 + 2c_{\mathfrak r}^2 - \varepsilon)t - N\log t < x \le (4c_{\mathfrak l}^2 + 2c_{\mathfrak r}^2 -)t\}$ asymptotic solitons

$D_3 = \{(x,t): \quad x \ge (4c_{\mathfrak l}^2 + 2c_{\mathfrak r}^2 -)t\}$ solitons on background $c_{\mathfrak r}$

Theorem (Kotlyarov, M: 2010) Take $c_r = 0, c_l \equiv c > 0.$ Then
II. for $(x,t) \in D_2 = \{(x,t): -(6c^2-\varepsilon)t < x < (4c^2-\varepsilon)t\}$ and $t \to +\infty$

$$q(x,t) = q_{el}(\xi,t) + \mathrm{O}(t^{-2/3}) \ , \qquad \xi = \frac{x}{12t},$$

where

$$q_{el}(\xi,t) = \sqrt{c^2 - d^2(\xi)}\,\frac{\Theta(\pi i + itB(\xi) + i\Delta(\xi))}{\Theta(itB(t,\xi) + i\Delta(\xi))} \ . \qquad (4)$$

Here $\Theta(z)$ is the Jacobi theta-function, associated with the Riemann surface of function $\mathrm{w}(k,\xi) = \sqrt{(k^2+c^2)(k^2+d^2(\xi))}$, $d = d(\xi)$ is increasing function, defined for $\xi \in (\frac{-c^2}{2}, \frac{c^2}{3})$, determined by c,
$d\left(\frac{-c^2}{2}\right) = 0, d\left(\frac{c^2}{3}\right) = c.$
$B(\xi)$ is determined by c, $\Delta(\xi)$ is determined by $r(.)$ and c.

$c_{\mathfrak{l}} > c_{\mathfrak{r}} > 0, \quad t \to -\infty$

Theorem 4.2.1 rarefaction wave

II. For $(x,t) \in \{(x,t) : -(6c_{\mathfrak{l}}^2 - \varepsilon)t < x < (6c_{\mathfrak{r}}^2 + \varepsilon)t\}$ and $t \to -\infty$

$$q(x,t) = \sqrt{\frac{x}{6t}} + o(1).$$

Theorem 4.3.1 dispersive tail domain

I. For $(x,t) \in D_1 = \{(x,t) : x > -(6c_{\mathfrak{r}}^2 + \varepsilon)t\}$ and $t \to -\infty$

$$q(x,t) = c_{\mathfrak{r}} + \sqrt{\frac{\nu(\xi)\sqrt{\frac{c_{\mathfrak{r}}^2}{2} - \xi}}{3|t|\left(-\xi - \frac{c_{\mathfrak{r}}^2}{2}\right)}} \times$$

$$\times \cos\left(16|t|\left(\frac{c_{\mathfrak{r}}^2}{2} - \xi\right)^{3/2} + \nu(\xi)\log\left(\frac{192|t|\left(\xi + \frac{c_{\mathfrak{r}}^2}{2}\right)^2}{\sqrt{\frac{c_{\mathfrak{r}}^2}{2} - \xi}}\right) - \phi(\xi)\right) + \mathcal{O}\left(t^{-1}\right).$$

Here $\xi = \dfrac{x}{12t}, \nu(\xi) = \dfrac{1}{2\pi}\log\left(1 + \left|r\left(\sqrt{-\xi - \dfrac{c^2}{2}}\right)\right|^2\right)$,

$\phi(\xi)$ is a bounded function of ξ, determined by $r(.)$.

The Cauchy problem for the Camassa-Holm equation

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \; x \in \mathbb{R}, t \geq 0, \tag{2}$$

$$u(x,0) = u_0(x) \to \begin{cases} c_{\mathfrak{r}}, & x \to +\infty, \\ c_{\mathfrak{l}}, & x \to -\infty, \end{cases} \quad \frac{c_{\mathfrak{l}} + \omega}{c_{\mathfrak{r}} + \omega} > 0. \tag{2a}$$

Without loss of generality we can assume that $c_{\mathfrak{r}} = 0, \omega = 1, c_{\mathfrak{l}} = c > -1.$

$$c \in (-1,0) - \text{rarefaction}, \quad c \in (0,+\infty)\text{compressive}$$

$$m(x,t) = u(x,t) - u_{xx}(x,t). \qquad \text{(a "momentum" variable)}$$

We suppose that there exists a global classical solution that satisfies

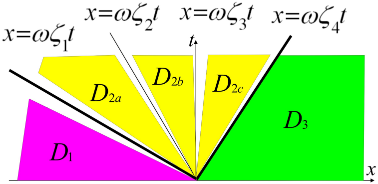
$$m(x,0) + \omega > 0 \quad (\Rightarrow \forall x,t: \quad m(x,t) + \omega > 0.) \; (T = \infty), \quad c > m(x,0) \tag{5}$$

$$\int_{\mathbb{R}} e^{C_0|x|}(|m(x,0) - cH(-x)| + |m_x(x,0)| + |m_{xx}(x,0)|)\, dx < \infty, \; C_0 > \hat{c}^2 := \frac{c}{4(c+\omega)},$$

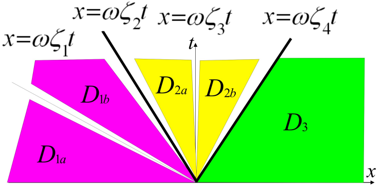
$$\max_{t\in[0,T]} \int_{\mathbb{R}} (1+|x|)^2 (|m(x,t) - cH(-x)| + |m_x(x,t)| + |m_{xx}(x,t)|)\, dx < \infty, \; H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

A.Fokas, B.Fuchssteiner, R.Camassa, D.Holm, R.Beals, D.H. Sattinger, J.Szmigielski,

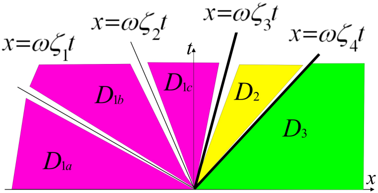
A.Constantin, D.Lannes, R.S.Johnson, A.Bressan, W.A.Strauss, H.Holden, X.Raynaud,



$\frac{c}{\omega} > 3$



$1 < \frac{c}{\omega} < 3$



$0 < \frac{c}{\omega} < 1$

$D_1 : q(x,t) = c_l + o(1)$

D_1 : modulated elliptic asymptotics

D_3 : possible solitons, otherwise $q(x,t) = c_r + o(1)$

Ellitpic sector

Theorem (oscillatory sector)
II. For $(x,t) \in D_2 = \{(x,t) : (\zeta_2+\varepsilon)t < x < 2(c+\omega-\varepsilon)t\}$ and $t \to +\infty$ asymptotics is given parametrically by formulas

$$\begin{cases} x = y - 2it(g - g_{\mathfrak{r}})(i/2) + c(\xi) + E(tB(\xi) + \Delta(\xi)) + o(1), \\ u(y,t) = \frac{c(1-d^2(\xi)\hat{c}^{-2})}{(1-4d^2(\xi))} + \Gamma(\xi)\left(E'\left(tB(\xi)+\Delta(\xi)\right) - E'(0)\right) + o(1), \quad \xi = \frac{y}{t}, \end{cases}$$

$$\Downarrow$$

$$u(x,t) = \hat{u}(y(x,t),t)$$

where

$$E(U) = \log \frac{\Theta(iU - A(i/2) + \pi i)\Theta(iU - A(i/2))}{\Theta(iU + A(i/2) + \pi i)\Theta(iU + A(i/2))},$$

$$\Gamma(\xi) = \frac{2\pi^2\omega}{\left(\int_0^d \frac{\mathrm{d}k}{\mathrm{w}(k)}\right)^2 \mathrm{w}(i/2)}, \hat{c} = \sqrt{\frac{c}{4(c+\omega)}},$$

Here $\Theta(z)$ is the Jacobi theta-function, associated with the Riemann surface of function $\mathrm{w}(k,\xi) = \sqrt{(k^2+\hat{c}^2)(k^2+d^2(\xi))}$, $d = d(\xi)$ is increasing function, defined for $\xi \in (\xi_2, \xi_1)$, determined by c,
$d(\xi_2) = 0, d(\xi_1) = \hat{c}$.

$B(\xi)$ is determined by c, $\Delta(\xi)$ is determined by $r(.)$ and c.