On the periodic Zakharov-Kuznetsov equation

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In this talk we will consider the Cauchy problem associated to the Zakharov-Kuznetsov equation in $\mathbb{T}^2$, that is,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u + u\partial_x u = 0, & (x,y) \in \mathbb{T}^2,\ t > 0, \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{1}$$

The main goal is to establish the local theory for data in $H^s(\mathbb{T}^2)$, $s > 3/2$.

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Outline

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- Main Result
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Motivation

We are concerned with solutions of the problem

$$\begin{cases} \partial_t u - iP(D)u = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{2}$$

where $x \in \mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and $t > 0$.
To explain the difference with the $\mathbb{R}^n$ setting we consider the operator $P(D) = \Delta$ and $n = 1$.

The solution $u(x,t)$ of the problem (2) can be obtained by Fourier series and written as

$$u(x,t) = \sum_k a_k e^{i(tk^2 + kx)}$$

with initial data $u_0(x) = \sum_k a_k e^{ikx}$ with $\sum_k |a_k|^2 < \infty$.

Observe that in this case the fundamental solution $k(x,t) \sim \sum_k e^{i(tk^2+kx)}$ is not in $L^\infty(\mathbb{S})$.

Moreover, the estimate

$$\|k(\cdot,t) * f\|_{L^\infty(\mathbb{S})} \leq c_t \|f\|_{L^1(\mathbb{S})} \tag{3}$$

for any fixed t is false. Indeed, take $f_{N,t}(x) = \sum_1^N e^{i(tk^2+kx)}$ and observe that if (3) holds

$$N = \|k(\cdot,t) * f_{N,t}\|_{L^\infty(\mathbb{S})} \leq c_t \|f_{N,t}\|_{L^1(\mathbb{S})} \leq c_t \|f_{N,t}\|_{L^2(\mathbb{S})} \leq c_t N^{1/2},$$

which is a contradiction.

Also notice that the local smoothness obtained for $\{e^{it\Delta}\}$ in $\mathbb{R}^n$ cannot occur in the periodic case because it would be a global smoothing in $L^2(\mathbb{T}^n)$ and $\{e^{it\Delta}\}$ is unitary in $L^2(\mathbb{T}^n)$.

A different kind of smoothing effect can be expected as the following result due to Zygmund.

Theorem 1.

$$\|\sum_k a_k e^{i(tk^2+kx)}\|_{L^4(\mathbb{T}^2)} \le c\Big(\sum |a_k|^2\Big)^{1/2}, \tag{4}$$

where $(x,t) \in \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$

Theorem 2 (Kenig-Ponce-Vega). *Given* $\alpha \in \mathbb{Z}^+$, $\alpha > 1$ *and* $\gamma \in \mathbb{R}^+$, *define*

$$|D|^\gamma S_N(t)u_0(x) = \sum_{|k|\leq N} a_k |k|^\gamma e^{i(t|k|^\alpha + kx)} \tag{5}$$

for $(x,t) \in \mathbb{T}^2$, *where* $u_0(x) = \sum_k a_k e^{ikx}$. *Then*

$$\||D|^{\gamma(\nu)} S_N(\cdot)u_0(\cdot)\|_{L_t^q(I_N;L^p(\mathbb{S}^1))} \leq c\Big(\sum |a_k|^2\Big)^{1/2}, \tag{6}$$

$$\|\int_{-\infty}^{\infty} |D|^{\gamma(\nu)} S_N(t-\tau)g(x,\tau)\,d\tau\|_{L_t^q(I_N;L^p(\mathbb{S}^1))} \leq c\|g\|_{L_t^{q'}(I_N;L^{p'}(\mathbb{S}^1))}, \tag{7}$$

$$\|\int_0^t |D|^{\gamma(\nu)} S_N(t-\tau)g(x,\tau)\,d\tau\|_{L_t^q(I_N;L^p(\mathbb{S}^1))} \leq c\|g\|_{L_t^{q'}(I_N;L^{p'}(\mathbb{S}^1))}, \tag{8}$$

where $(q,p) = (\frac{4}{\nu}, \frac{2}{1-\nu})$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, $\gamma(\nu) = \left(\frac{\alpha-2}{2}\right)\nu$ *and* $I_N = (0, N^{-(\alpha-1)})$ *for any* $\nu \in [0,1]$. *The constant* c *depends only on* α *and* γ.

Main Result

We consider the Cauchy problem for the Zakharov-Kuznetsov equation,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u + u\partial_x u = 0, \quad \mathbf{x} = (x,y) \in \mathbb{T}^2, \ \ t \in \mathbb{R}, \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{9}$$

The model presented in (9) is a bi-dimensional generalization of the Korteweg-de Vries (KdV) equation, and was introduced by Zakharov and Kuznetsov to describe the propagation of nonlinear ion-acoustic waves in magnetized plasma. Its rigurous derivation was shown by Lannes-L-Saut. This model is widely known as the Zakharov-Kuznetsov (ZK) equation and has been extensively studied in the literature. In the most of these works well-posedness issues are considered for the IVPs posed either on $\mathbb{R}^2$ or on $\mathbb{R}^3$.

Summary of well-posedness results

- Local well-posedness for data in $H^s(\mathbb{R}^2)$, $s > 1/2$, Molinet and Pilod, Grunrock and Herr independently.
- Global well-posedness for data in $H^l(\mathbb{R}^2)$, $l \in \mathbb{Z}^+$, Faminskii, for data in $H^s(\mathbb{R}^2)$, $s \geq 1$, L-Pastor.
- Local well-posedness for data in $H^s(\mathbb{R} \times \mathbb{T})$, $s > 1/2$, Molinet and Pilod.
- Local well-posedness for data in $H^s(\mathbb{R}^3)$, $s > 1$, Ribaud and Vento.
- Global well-posedness for data in $H^s(\mathbb{R}^3)$, $s > 1$, Molinet and Pilod.

Conjecture: A scaling argument suggest that in $\mathbb{R}^2$, local well-posedness shall be achieved in $H^s(\mathbb{R}^2)$, $s > -1$.

Theorem 3 (L-Panthee-Tzvetkov). *Let* $u_0 \in H^s(\mathbb{T}^2)$*,* $s > \frac{3}{2}$*, then there exist a time* $T = T(\|u_0\|_{H^s})$ *and a unique solution*

$$u \in C([0,T] : H^s(\mathbb{T}^2))$$

to the IVP (9) *such that*

$$u, \partial_x u, \partial_y u \in L^1_T L^\infty_{xy}.$$

Moreover, the application that takes the initial data to the solution

$$u_0 \mapsto u \in C([0,T] : H^s(\mathbb{T}^2))$$

is continuous.

Symmetrization

Following Grunröck and Herr we symmetrize the equation in (9) using the change of variables

$$x' = \frac{x}{2} + \frac{y}{2\lambda}, \qquad y' = \frac{x}{2} - \frac{y}{2\lambda}, \tag{10}$$

and define $v(x', y', t) = u(x, y, t)$. Choosing $\lambda^2 = \frac{1}{3}$, it can be seen that, if v is a solution of the IVP

$$\begin{cases} \partial_t v + \frac{1}{2}\Big(\partial_{x'}^3 v + \partial_{y'}^3 v + v(\partial_{x'} + \partial_{y'})v\Big) = 0 \\ v(x', y', 0) = v_0(x', y'), \end{cases} \tag{11}$$

then u given by

$$u(x, y, t) = v\big(\frac{x}{2} + \frac{y}{2\lambda}, \frac{x}{2} - \frac{y}{2\lambda}, t\big) \tag{12}$$

is a solution of the original IVP (9).

Now the periodicity of u, i.e., $u(x,y,t) = u(x+2\pi, y+2\pi, t)$, will require that

$$v\Big(\frac{x}{2}+\frac{y}{2\lambda}, \frac{x}{2}-\frac{y}{2\lambda}, t\Big) = v\Big(\frac{x}{2}+\frac{y}{2\lambda}+(1+\frac{1}{\lambda})\pi,\ \frac{x}{2}-\frac{y}{2\lambda}+(1-\frac{1}{\lambda})\pi, t\Big). \quad (13)$$

For the choice of $\lambda^2 = \frac{1}{3}$, (13) turns out to be

$$v(x', y') = v(x' + a2\pi, y' + b2\pi),$$

where $a := \frac{1}{2}(1+\sqrt{3})$ and $b := \frac{1}{2}(1-\sqrt{3})$.

This discussion shows that, in order to get the LWP of the original problem, it is enough to prove the LWP to the IVP (11) for solution satisfying $v(x', y', t) = v(x'+T_1, y'+T_2, t)$, for some T_1 and T_2. For this, we define $x = \frac{x'}{a}$, $y = \frac{y'}{b}$ and $w(x,y,t) = v(x',y',t)$, so that the function $w(x,y,t)$ is periodic on $[0,2\pi]\times[0,2\pi]$ and satisfies

$$\begin{cases} \partial_t w + \frac{1}{2a^3}\partial_x^3 w + \frac{1}{2b^3}\partial_y^3 w + \frac{1}{2a} w\partial_x w + \frac{1}{2b} w\partial_y w = 0 \\ w(x,y,0) = w_0(x,y). \end{cases} \quad (14)$$

To simplify the exposition, we define $\alpha := \frac{1}{2a^3}$, $\beta := \frac{1}{2b^3}$, $\gamma_1 := \frac{1}{2a}$ and $\gamma_2 := \frac{1}{2b}$. With these notations, (14) becomes,

$$\begin{cases} \partial_t w + \alpha\partial_x^3 w + \beta\partial_y^3 w + \gamma_1 w\partial_x w + \gamma_2 w\partial_y w = 0, & (x,y) \in \mathbb{T}^2, t \in \mathbb{R}, \\ w(x,y,0) = w_0(x,y). \end{cases} \tag{15}$$

In what follows we will study the local well-posedness to the IVP (15), which in turn implies the local well-posedness result stated in Theorem 3.

Theorem 4. *Let $w_0 \in H^s(\mathbb{T}^2)$, $s > \frac{3}{2}$, then there exist a time $T = T(\|w_0\|_{H^s})$ and a unique solution $w \in C([0,T] : H^s(\mathbb{T}^2))$ to the IVP (15) such that $w, \partial_x w, \partial_y w \in L_T^1 L_{xy}^\infty$. Moreover, the application that takes the initial data to the solution $w_0 \mapsto w \in C([0,T] : H^s(\mathbb{T}^2))$ is continuous.*

Localized Strichartz' estimate

Let $\phi \in C_0^\infty(-2,2)$ be such that $\phi(r) = 1$ for $|r| \leq 1$. For $k \in \mathbb{Z}$ consider a dyadic number $N := 2^k$ and define a partition of unity as follows

$$\phi_N(m) = \begin{cases} \phi(\frac{|m|}{N}) - \phi(\frac{2|m|}{N}), & N \geq 2, \\ \phi(|m|), & N = 1. \end{cases} \tag{16}$$

Now, we define a projection operator P_N, that localizes frequency in the level $|m| \sim N = 2^k$, as a Fourier multiplier operator by

$$\widehat{P_N f}(m) := \phi_N(m)\widehat{f}(m). \tag{17}$$

In terms of the projection operator we have

$$\|f\|^2_{H^s(\mathbb{T}^2)} := \sum_{m\in\mathbb{Z}^2} (1+|m|^2)^s |\widehat{f}(m)|^2 \sim \sum_{N\geq 1} N^{2s} \|P_N f\|^2_{L^2(\mathbb{T}^2)}. \tag{18}$$

Consider the linear problem

$$\begin{cases} \partial_t w + \alpha\partial_x^3 w + \beta\partial_y^3 w = 0, & (x,y) \in \mathbb{T}^2,\ t \in \mathbb{R}, \\ w(x,y,0) = w_0(x,y). \end{cases} \tag{19}$$

whose solution is given by the unitary group

$$W(t)w_0(x) = \sum_{m\in\mathbb{Z}^2} e^{i(m.\mathbf{x}+(\alpha m_1^3+\beta m_2^3)t)}\widehat{w}_0(m). \tag{20}$$

Lemma 1. *Let* $\delta \in [0,1/2)$ *and* P_N *be the operator defined in* (17)*. Then for any* $w_0 \in L^2_{xy}$*,*

$$\|W(t)P_N w_0\|_{L^{q_\delta}_{|t|\in[0,N^{-2}]}L^\infty_{xy}} \lesssim N^{-\delta}\|w_0\|_{L^2_{xy}} \tag{21}$$

where $\frac{1}{q_\delta} = \frac{1+\delta}{3}$*, and*

$$\|W(t)w_0\|_{L^{q_\delta}_{[0,1]}L^\infty_{xy}} \lesssim \|w_0\|_{H^{\frac{2}{q_\delta}-\delta}} = \|w_0\|_{H^{\frac{2-\delta}{3}}}. \tag{22}$$

Sketch of the Proof. To obtain the estimate (21), it is enough to show that

$$\|\chi_{[0,2^{-2k}]}(|t|) \sum_{m\in\mathbb{Z}^2} \widehat{w}_0(m) e^{i(m\cdot\mathbf{x}(t)+(\alpha m_1^3+\beta m_2^3)t)} \phi_N(m) \|_{L_t^{q_\delta}} \leq C N^{-\delta} \|w_0\|_{L^2}, \quad (23)$$

for any measurable function $\mathbf{x} : [-2^{-k}, 2^{-k}]^2 \to \mathbb{T}^2$.

After some some analysis the problem reduces to showing that

$$\Big| \sum_{m\in\mathbb{Z}^2} \phi_N^2(m) e^{i(m\cdot\mathbf{x}+(\alpha m_1^3+\beta m_2^3)t)} \Big| \lesssim 2^{\frac{4j}{q_\delta}} 2^{-2\delta k-\epsilon j}, \quad (24)$$

for any $\mathbf{x} \in \mathbb{T}^2$, $|t| \in [2^{-2j}, 2^{-2j+1}]$, $j \geq 2k$ and some small $0 < \epsilon \leq \frac{\delta}{3}$.

Poisson Summation Formula

$$\sum_{m\in\mathbb{Z}^2} F(m) = \sum_{n\in\mathbb{Z}^2} \widehat{F}(2\pi n), \ \forall F \in \mathcal{S}(\mathbb{R}^2). \tag{25}$$

Using (25) the left hand side in (24) becomes

$$\sum_{n\in\mathbb{Z}^2} \int_{\mathbb{R}^2} e^{-in\cdot\xi}\, \phi_N^2(\xi)\, e^{i\left(\mathbf{x}\cdot\xi+(\alpha\xi_1^3+\beta\xi_2^3)t\right)} d\xi$$

the whole matter reduces to estimating the term

$$\begin{aligned} \text{Term} &:= \sum_{|n|<M} \int_{\mathbb{R}^2} \phi_N^2(\xi) e^{i\left(\mathbf{x}\cdot\xi + (\alpha\xi_1^3 + \beta\xi_2^3)t\right)} d\xi \\ &\sim M \int_{\mathbb{R}^2} \phi_N^2(\xi) e^{i\left(\mathbf{x}\cdot\xi + (\alpha\xi_1^3 + \beta\xi_2^3)t\right)} d\xi, \end{aligned} \tag{26}$$

for some large constant $M > 0$, where $\xi = (\xi_1, \xi_2)$. Taking into account that the function $\phi_N^2(\xi)$ is bounded and writing the oscillatory integral in terms of the Airy function

$$\int_{\mathbb{R}^2} e^{i\left(\mathbf{x}\cdot\xi + (\alpha\xi_1^3 + \beta\xi_2^3)t\right)} d\xi = \frac{2\pi}{\sqrt[3]{3\alpha t}} Ai\Big(\frac{x}{\sqrt[3]{3\alpha t}}\Big) \frac{1}{\sqrt[3]{3\beta t}} Ai\Big(\frac{y}{\sqrt[3]{3\beta t}}\Big), \tag{27}$$

we obtain that

$$|\text{Term}| \lesssim |t|^{-\frac{2}{3}}. \tag{28}$$

Kato-Ponce Commutator estimate in $\mathbb{T}^2$

This estimate is obtained employing the idea of Ionescu and Kenig in $\mathbb{T}$-case extension of the Kato-Ponce commutator.

Lemma 2. *Let* $s \geq 1$, $\widehat{J^s f}(m) := \widehat{J^s_{\mathbb{T}^2} f}(m) = (1+|m|^2)^{\frac{s}{2}} \widehat{f}(m)$ *and* $f, g \in H^\infty(\mathbb{T}^2)$. *Then*

$$\begin{aligned} \|J^s(fg) - f J^s g\|_{L^2(\mathbb{T}^2)} \leq c \Big\{ & \|J^s f\|_{L^2(\mathbb{T}^2)} \|g\|_{L^\infty(\mathbb{T}^2)} \\ & + \left(\|f\|_{L^\infty(\mathbb{T}^2)} + \|\nabla f\|_{L^\infty(\mathbb{T}^2)}\right) \|J^{s-1} g\|_{L^2(\mathbb{T}^2)} \Big\}. \end{aligned} \tag{29}$$

A priori estimates

Lemma 3. *Let* $w_0 \in H^\infty(\mathbb{T}^2)$ *and* w *be the corresponding smooth solution to the IVP* (15)*. Then for any* $T \in [0,1]$ *and* $s \geq 1$*, we have*

$$\|w\|_{L^\infty_T H^s(\mathbb{T}^2)} \leq c_s \exp\left(c_s(\|w\|_{L^1_T L^\infty_{xy}} + \|\nabla w\|_{L^1_T L^\infty_{xy}})\right)\|w_0\|_{H^s(\mathbb{T}^2)}. \tag{30}$$

Proof.
We apply the operator J^s to the equation in (15), multiply the resulting equation by $J^s w$ and then integrate by parts, to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(J^s w)^2\,dxdy+\gamma_1\int_{\mathbb{T}^2} J^s(u\partial_x w)J^s w\,dxdy+\gamma_2\int_{\mathbb{T}^2} J^s(w\partial_y w)J^s w\,dxdy = 0.$$

Using the commutator notation $[A,B]f = A(Bf) - B(Af)$, integration by parts and Cauchy-Schwartz inequality, we obtain from () that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(J^s w)^2\,dxdy \le \frac{1}{2}(\|\partial_x w\|_{L^\infty(\mathbb{T}^2)} + \|\partial_y w\|_{L^\infty(\mathbb{T}^2)})\|J^s w\|^2_{L^2(\mathbb{T}^2)}$$
$$+ (\|[J^s, w]\partial_x w\|_{L^2(\mathbb{T}^2)} + \|[J^s, w]\partial_y w\|_{L^2(\mathbb{T}^2)})\|J^s w\|_{L^2(\mathbb{T}^2)}. \tag{31}$$

Applying Lemma 2 (Kato-Ponce commutator), we have

$$\|[J^s, w]\partial_x w\|_{L^2(\mathbb{T}^2)} \le c\{\|J^s w\|_{L^2(\mathbb{T}^2)}\|\partial_x w\|_{L^\infty(\mathbb{T}^2)}$$
$$+ (\|w\|_{L^\infty(\mathbb{T}^2)} + \|\nabla w\|_{L^\infty(\mathbb{T}^2)})\|J^{s-1}\partial_x w\|_{L^2(\mathbb{T}^2)}\}. \tag{32}$$

Similarly,

$$\|[J^s, w]\partial_y w\|_{L^2(\mathbb{T}^2)} \leq c\{\|J^s w\|_{L^2(\mathbb{T}^2)}\|\partial_y w\|_{L^\infty(\mathbb{T}^2)} + (\|w\|_{L^\infty(\mathbb{T}^2)} + \|\nabla w\|_{L^\infty(\mathbb{T}^2)})\|J^{s-1}\partial_y w\|_{L^2(\mathbb{T}^2)}\}. \tag{33}$$

Inserting (32) and (33) in (31), we get after simplification

$$\frac{d}{dt}\|w(t)\|^2_{H^s} \leq c\big(\|w\|_{L^\infty(\mathbb{T}^2)} + \|\nabla w\|_{L^\infty(\mathbb{T}^2)}\big)\|w(t)\|^2_{H^s}. \tag{34}$$

Using Gronwall's inequality, (34) yields

$$\|w\|^2_{L^\infty_T H^s} \leq \|w_0\|^2_{H^s} \exp\Big(c\int_0^T \big(\|w\|_{L^\infty(\mathbb{T}^2)} + \|\nabla w\|_{L^\infty(\mathbb{T}^2)}\big)dt\Big), \tag{35}$$

which gives the required estimate (30).

Refined Strichartz estimate

One of the key estimates in our analysis is the following estimate in the one dimensional continuos case it was first established by Koch and Tzvetkov for the Benjamin-Ono equation. It was extended by Kenig and its collaborators.

Lemma 4. *Let* $\delta \in [0, 1/2)$*. Then the solution of the equation*

$$i\partial_t w + \alpha\partial_x^3 w + \beta\partial_y^3 w + \gamma_1\partial_x F(w) + \gamma_2\partial_y F(w) = 0, \quad (x,y) \in \mathbb{T}^2, \quad t \in \mathbb{R}, \tag{36}$$

satisfies

$$\|w\|_{L_T^1 L_{xy}^\infty} \lesssim T^{\theta_\delta} \left(\|w\|_{L_T^\infty H^{\frac{2-\delta}{3}+}} + \|F\|_{L_T^1 L_{xy}^2} \right), \tag{37}$$

where $\theta_\delta = \frac{2-\delta}{3}$.

Proof. We divide the interval $[0,T]$ in subintervals $[a_k, a_{k+1})$ of size TN^{-2} for $k = 1, \dots, N^2$. Let P_N be the projector operator on frequencies of size $N \in \{1, 2, 4, 8, \dots\}$. Then, similarly to (3), one gets

$$\|P_N w\|_{L^1_T L^\infty_{xy}} \lesssim (TN^{-2})^{1/q'_\delta} \sum_{k=1}^{N^2} \|\chi_{[a_k, a_{k+1})}(t) P_N w\|_{L^{q_\delta}_T L^\infty_{xy}}. \tag{38}$$

where $1/q'_\delta = 1 - 1/q_\delta = \frac{2-\delta}{3}$.
On other hand, using Duhamel's formula, for $t \in [a_k, a_{k+1})$, we obtain

$$w(t) = W(t - a_k)\, u(a_k) - i \int_{a_k}^{t} W(t-s)[\partial_x F(w) + \partial_y F(w)]\, ds. \tag{39}$$

Using Lemma 1, one obtains

$$\begin{aligned} \|\chi_{[a_k, a_{k+1})}(t) P_N w\|_{L^{q_\delta}_T L^\infty_{xy}} &\lesssim N^{-\delta} \|P_N w(a_k)\|_{L^2_{xy}} \\ &\quad + N^{-\delta} N \|\chi_{[a_k, a_{k+1})}(t) P_N F\|_{L^1_T L^2_{xy}}. \end{aligned} \tag{40}$$

The following estimates is key to prove the local well-posedness result to the IVP (15).

Lemma 5. *Let* u *be a solution of* (15) *with* $u_0 \in H^\infty(\mathbb{T}^2)$ *defined in* $[0,T]$. *Then for any* $s > \frac{2-\delta}{3} + 1$, *there exist* $T = T(\|u_0\|_{H^s})$ *and a constant* $c_T(\|u_0\|_{H^s}, s)$ *such that*

$$f(T) := \int_0^T \left(\|u(t)\|_{L^\infty_{xy}} + \|\partial_x u(t)\|_{L^\infty_{xy}} + \|\partial_y u(t)\|_{L^\infty_{xy}}\right) dt \le c_T. \tag{41}$$

Proof.

Observe that u, $\partial_x u$ and $\partial_y u$ satisfy Lemma 4 with F given, respectively, by $u^2/2$, $\partial_x(u^2)/2$ and $\partial_y(u^2)/2$. Hence for $s' > s_\delta = \frac{2-\delta}{3}$ we obtain

$$\begin{aligned} f(T) \le c\,T^{\theta_\delta}\Big(&\|J^{s'}u\|_{L^2} + \|J^{s'}\partial_x u\|_{L^2} + \|J^{s'}\partial_y u\|_{L^2} \\ &+ \int_0^T \|u^2\|_{L^2}\,dt + \int_0^T \|\partial_x(u^2)\|_{L^2}\,dt + \int_0^T \|\partial_y(u^2)\|_{L^2}\,dt\Big). \end{aligned} \tag{42}$$

It follows that

$$\|J^{s'}u\|_{L^2} + \|J^{s'}\partial_x u\|_{L^2} + \|J^{s'}\partial_y u\|_{L^2} \leq c\,\|J^s u\|_{L^2_{xy}}, \tag{43}$$

for $s > s' + 1$.

On the other hand,

$$\begin{aligned} &\int_0^T (\|u^2\|_{L^2} + \|\partial_x(u^2)\|_{L^2} + \|\partial_y(u^2)\|_{L^2})\,dt \\ &\quad \leq c\ \|u\|_{L^\infty_T H^s} \int_0^T (\|u(t)\|_{L^\infty_{xy}} + \|\partial_x u(t)\|_{L^\infty_{xy}} + \|\partial_y u(t)\|_{L^\infty_{xy}})\,dt. \end{aligned} \tag{44}$$

Combining (42), (43), (44) with Lemma 3 we obtain the inequality

$$f(T) \leq cT^{\theta_\delta}\,\|u_0\|_{H^s}\ \exp\left(c\,f(T)\right)(1 + f(T)). \tag{45}$$

Sketch of the Proof of Theorem 4

The main tools in the proof are the results from Lemmas 3 and 5.
Let $\delta \in [0, \frac{1}{2})$ and consider an initial datum $w_0 \in H^s(\mathbb{T}^2)$ with $s > \frac{2-\delta}{3} + 1$. Note that, for sufficiently regular data (for example if $s > 2$) one can use a classical method to show that the IVP (15) is locally well-posed.

Consider $w_0 \in H^s(\mathbb{T}^2)$ with $\frac{2-\delta}{3} + 1 < s \leq 2$. Density of H^∞ in H^s allows one to find $w_0^\epsilon \in H^\infty$ such that $\|w_0^\epsilon - w_0\|_{H^s} \to 0$. Moreover, one can have $\|w_0^\epsilon\|_{H^s} \leq c\|w_0\|_{H^s}$.

For $0 < \epsilon < 1$, let w^ϵ be the solution to the regularized IVP (15) corresponding to the initial data $w_0^\epsilon \in H^\infty$ on $[0, T]$. Therefore, one can use Lemma 5 to conclude that there exist a time $T = T(\|w_0\|_{H^s}) > 0$ and a constant c_T such that

$$\int_0^T \left(\|u^\epsilon(t)\|_{L^\infty_{xy}} + \|\partial_x w^\epsilon(t)\|_{L^\infty_{xy}} + \|\partial_y w^\epsilon(t)\|_{L^\infty_{xy}}\right) dt \leq c_T. \tag{46}$$

Also, from Lemma 3, one has

$$\sup_{0\leq t\leq T} \|w^\epsilon(t)\|_{H^s(\mathbb{T}^2)} \leq c_T. \tag{47}$$

Now, using Gronwall's inequality and the estimate (46), one can show that

$$\sup_{0\leq t\leq T} \|w^\epsilon - w^{\epsilon'}\|_{L^2(\mathbb{T}^2)} \to 0 \qquad \text{as} \quad \epsilon,\ \epsilon' \to 0, \tag{48}$$

where $w^{\epsilon'}$ is the solution of the corresponding regularized problem for $0 < \epsilon' < \epsilon < 1$.

In view of (48) and (47), one can get, for $s' < s$,

$$w \in C([0,T]; H^{s'}) \cap L^\infty([0,T]; H^s)$$

such that

$$w^\epsilon \to w \text{ in } C([0,T]; H^{s'}).$$

Indeed, (48) implies that as $\epsilon \to 0$, $w^\epsilon \to w$ in $C([0,T], L^2(\mathbb{T}^2))$. In the light of estimate (47) we note that, $w^\epsilon \in L^\infty([0,T], H^s)$. Hence, by weak* compactness, $w \in L^\infty([0,T], H^s)$.

Once again, one can use Gronwall's inequality, to prove that w is the unique solution to the IVP (15).

Bona–Smith argument can be used to prove the continuity of the solution $w(t)$ and the continuity of the flow-map in H^s.

Note that, we considered $s > \frac{2-\delta}{3} + 1$. Since $\delta \in [0, \frac{1}{2})$, the maximum possible value of δ corresponds to the lowest Sobolev regularity ($s > \frac{3}{2}$) of the initial data to obtain local well-posedness result in H^s, and this completes the proof.

Final Remarks

- What can we say about the character of the equation in this setting?

- The method explained above can be used in different situations, for instance,

$$\begin{cases} i\partial_t u + |D_{x,y}|^3 u = iu\partial_x u, \quad \mathbf{x} = (x,y) \in \mathbb{T}^2, \ \ t \in \mathbb{R}, \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{49}$$

where $u : [0,2\pi] \times [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ is a real valued periodic function defined on the torus $\mathbb{T}^2$ and the operator $|D_{x,y}|^3$ is defined via the Fourier transform $\widehat{|D_{x,y}|^3 f}(m_1, m_2) = (m_1^2 + m_2^2)^{\frac{3}{2}} \widehat{f}(m_1, m_2)$.

Linearized versions of the model (49) already appear in its generalized form

$$\begin{cases} i\partial_t u + |D|^\alpha u = 0 \\ u(0) = u_0, \end{cases} \tag{50}$$

while studying water wave equation considering several values of $\alpha > 0$.

The local theory for the Cauchy problem is as follows.

Theorem 5. *Let* $u_0 \in H^s(\mathbb{T}^2)$, $s > \frac{3}{2}$, *then there exist a time* $T = T(\|u_0\|_{H^s})$ *and a unique solution* $u \in C([0,T] : H^s(\mathbb{T}^2))$ *to the IVP* (49) *such that* $u, \partial_x u, \partial_y u \in L^1_T L^\infty_{xy}$. *Moreover, the application that takes the initial data to the solution* $u_0 \mapsto u \in C([0,T] : H^s(\mathbb{T}^2))$ *is continuous.*

The previous analysis can be applied almost line by line, what matters is how to estimate the oscillatory integral

$$\sum_{n\in\mathbb{Z}^2}\int_{\mathbb{R}^2} e^{-in\cdot\xi}\phi_N^2(\xi)e^{i(\mathbf{x}\cdot\xi+|\xi|^3 t)}d\xi.$$

Thank You For Your Attention