Scattering for Nonlinear Klein-Gordon equations posed on product spaces.

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joint work with N. Visciglia (Unipi Pisa)

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laboratoiredemathématiquesdebesançon

Plan

1. Introduction
2. What happens for (NLS) posed on $\mathbf{R}^d$ and $\mathcal{M}^k$?
3. What happens in "mixed" settings ?
4. Same questions for the Klein-Gordon equation.

In this talk, total dimension $= 3$.

The equations

$$\text{(NLS):} \quad i\partial_t u + \Delta_X u = \pm|u|^\alpha u \quad ; \quad u(0,.) = u_0 \in H^1(X),$$

$$\text{(NLKG):} \quad \begin{cases} \partial_{tt} u - \Delta_X u + u = \pm|u|^\alpha u, \\ (u(0,.), \partial_t u(0,.)) = (u_0, u_1) \in H^1(X) \times L^2(X). \end{cases}$$

Question 1: According to the choices of X and α, do we have **global** solutions ?

Question 2: For the global solutions, what is the behaviour when $|t| \to +\infty$?

Aim: compare solutions to (NLS) or (NLKG) with "linear" solutions.

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The Schrödinger equation on $\mathbf{R}^3$

$\frac{4}{3} \leq \alpha \leq 4$

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Study of the equation thanks to **Strichartz estimates:** Consider *admissible* pairs: $0 \leq 2/q_j = 3/r_j - 3/2 < 1$. Then

1. $\|e^{it\Delta} f\|_{L_t^q L_x^r} \leq C(r)\|f\|_{L_x^2}$,
2. $\|e^{it\Delta} *_t f\|_{L_t^{q_1} L_x^{r_1}} \leq C(r_1, r_2)\|f\|_{L_t^{q'_2} L_x^{r'_2}}$.

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Used to prove local existence with **fixed point argument**.

Also used to prove "scattering".

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Every u_0 in H^1 gives a unique global solution u to (NLS), with

$$u, \nabla u \in C(\mathbf{R}, L^2) \cap L^q(\mathbf{R}, L^r), \quad \text{for some } (q, r).$$

Moreover

Asymptotic completeness: For all $u_0 \in H^1$, one can produce a $u_\pm \in H^1$ s.t. $(**)$ is satisfied.

Existence of the wave operator: For all $u_\pm \in H^1$, one can associate a solution $u(t)$ to (NLS), satisfying $(**)$.

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and $e^{-it\Delta}u(t)$ has to converge in H^1.

Duhamel $\to$

$$u(t) = e^{it\Delta}u_0 - i\kappa \int_0^t e^{i(t-s)\Delta}|u|^\alpha u(s)\, ds$$

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H^1–scattering if and only if $\kappa \int_0^\infty e^{-is\Delta}|u|^\alpha u(s)\, ds$ converges in H^1.

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On $(\mathcal{M}^k, g)$

See works done by J. Bourgain, N. Burq-P.Gérard-N.Tzvetkov...

Ex.: $\mathcal{M}^k$ is the flat torus, the sphere...

$$i\partial_t u + \Delta_{\mathcal{M}^k} u = \kappa |u|^\alpha u \quad ; \quad u(0,\cdot) = u_0 \in H^1(\mathcal{M}^k);$$

Basis of $L^2(\mathcal{M}^k)$ given by $(\Phi_j(y))_{j\in\mathbf{N}}$, $-\Delta_{\mathcal{M}^k}\Phi_j = \lambda_j \Phi_j$.

Existence of linear periodic solutions s.t.: forall K compact subset, $\|1_K u_{lin}(t)\|_{L^2} = C$, whereas ; $\lim_{|t|\to\infty} \|1_K u_{lin}(t)\|_{L^2} = 0$ on $\mathbf{R}^3$.

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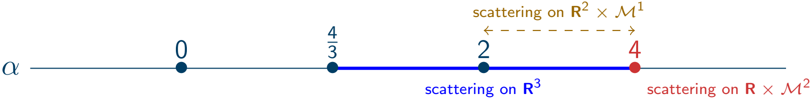
On a product space

What we expect for $d + k = 3$,

$$i\partial_t u + \Delta_{\mathbf{R}^d \times \mathcal{M}^k} u = \kappa |u|^\alpha u \quad ; \quad u(0,.) = u_0 \in H^1(\mathbf{R}^d \times \mathcal{M}^k);$$

Natural restrictions on α:



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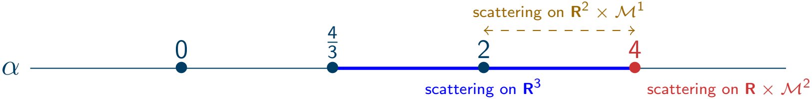

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Idea of proof

Key argument: Use of the $L^2(\mathcal{M}^k)$ basis, with $-\Delta_{\mathcal{M}^k}\Phi_k = \lambda_k \Phi_k$.

Then: $u(t,x,y) = \sum_k u_k(t,x)\Phi_k(y)$.

each u_k is solution to (NLS) posed on $\mathbf{R}^d$:

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Summing in k (ℓ_k^2−norm), one has:

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Theorem

Consider one of the following situations

(1) $\mathbf{R}^2 \times \mathcal{M}^1$ and $\alpha \in [2,4], X_{data} = H^1, X_{GWP} = L_t^q L_x^r H_y^{\frac{1}{2}+}$

(2) $\mathbf{R} \times \mathbb{T}^2$ and $\alpha = 4, X_{data} = H^1, X_{GWP} =$ "modified atomic space"

(3) $\mathbf{R} \times \mathcal{M}^2$ and $\alpha = 4, X_{data} = L_x^2 H_y^{1+}, X_{GWP} = L_t^q L_x^q H_y^{1+}$

Then, there exists $\delta > 0$ s.t. every data u_0 satisfying $\|u_0\|_{X_{data}} < \delta$ produces a unique global solution in $u \in C^0(\mathbf{R}, H^1) \cap X_{GWP}$ that scatters to a linear solution in H^1.

(Tzvetkov-Visciglia '11, Hani-Pausader '14, Tarulli '16).

Remarques:

- More general results : large data scattering available on $\mathbf{R}^d \times \mathcal{M}^1$ for $4/d \leq \alpha < 4/(d-1)$.

- Several works on product spaces that will not be described here (GWP, modified scattering...)

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Same role of parameter α.

- $X = \mathbf{R}^d \to$ P.Brenner, H.Pecher, C.Morawetz, C.Morawetz-W.Strauss, J.Ginibre-G.Velo, K.Nakanishi... global existence + scattering (use of smallness of a Strichartz norm)
- $X = \mathcal{M}^k \to$ global existence (J.-M. Delort, J.-M.Delort-J.Szeftel,D.Fang-Q.Zang...) but no scattering is proved.
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The difficulties

- Order 2 in time: one need to work with $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, in $H^1 \times L^2$.
- The propagator is unitary on $H^1 \times L^2$, but not scaling invariant

$$S(t) = \begin{pmatrix} \cos\left(t \cdot \sqrt{1-\Delta}\right) & \dfrac{\sin\left(t \cdot \sqrt{1-\Delta}\right)}{\sqrt{1-\Delta}} \\ -\sin\left(t \cdot \sqrt{1-\Delta}\right).\left(\sqrt{1-\Delta}\right) & \cos\left(t \cdot \sqrt{1-\Delta}\right) \end{pmatrix}$$

We want to prove

$$\lim_{|t| \to \pm\infty} \left\| U(t) - S(t) \begin{pmatrix} f_\pm \\ g_\pm \end{pmatrix} \right\|_{H^1 \times L^2} = 0.$$

- Strichartz estimates on $\mathbf{R}^3$ exist but are stated in Besov spaces: $0 \leq 2/q_j = 3/r_j - 3/2 < 1$, $s_j = s_j(r_j)$

$$\|u\|_{L^{q_1} B^s_{r_1,2}} \leq C(r_1, r_2) \left(\|u_0\|_{H^1} + \|u_1\|_{L^2} + \|F\|_{L^{p_1'} B^{1-s_j}_{r_2' 2}} \right).$$

Idea of proof

We still work on the basis of $L^2(\mathcal{M}^k)$ given by $-\Delta_{\mathcal{M}^k}\Phi_k = \lambda_k \Phi_k$:

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Each u_k is solution to

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Problems: estimates will depend on λ_k. *Scaling* type argument needed to quantify that dependence $\rightarrow$ homogeneous spaces are needed: embeddings from Besov to **Lebesgue**.

For each k

$$C_0(\lambda_k)\|u_k\|_{L_t^{q_1} L_x^{r_1}} \leq C\left[\sqrt{1+\lambda_k}\|u_{k,0}\|_{L^2} + \|u_{k,0}\|_{\dot{H}^1} + \|u_{k,1}\|_{L^2} + \|F_k\|_{L_t^1 L_x^2}\right].$$

Consequence: for some particular pairs, such that the embeddings are valid,

$$\|u\|_{L_t^{q_1} L_{x,y}^{r_1}} \leq \|u\|_{L_t^{q_1} L_x^{r_1} H_y^{\gamma}} \leq C\left[\|u_0\|_{H^1} + \|u_1\|_{L^2} + \|F\|_{L_t^1 L_{x,y}^2}\right].$$

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$$u(t,x,y) = \sum_k u_k(t,x)\Phi_k(y).$$

Each u_k is solution to

$$\partial_{tt} u_k - \Delta_{\mathbf{R}^d} u_k + u_k + \lambda_k u_k = F_k,\ u_k(0,\cdot) = u_{k,0}(\cdot),\ \partial_t u_k(0,\cdot) = u_{k,1}(\cdot)$$

Problems: estimates will depend on λ_k. *Scaling* type argument needed to quantify that dependence $\to$ homogeneous spaces are needed: embeddings from Besov to **Lebesgue**.

For each k

$$C_0(\lambda_k)\|u_k\|_{L_t^{q_1} L_x^{r_1}} \leq C \left[\sqrt{1+\lambda_k}\|u_{k,0}\|_{L^2} + \|u_{k,0}\|_{\dot{H}^1} + \|u_{k,1}\|_{L^2} + \|F_k\|_{L_t^1 L_x^2}\right].$$

Consequence: for some particular pairs, such that the embeddings are valid,

$$\|u\|_{L_t^{q_1} L_{x,y}^{r_1}} \leq \|u\|_{L_t^{q_1} L_x^{r_1} H_y^{\gamma}} \leq C \left[\|u_0\|_{H^1} + \|u_1\|_{L^2} + \|F\|_{L_t^1 L_{x,y}^2}\right].$$

Theorem (H'-Visciglia '17)

Consider one of the following situations

$$\mathbf{R} \times \mathcal{M}^2 \text{ and } \alpha = 4,$$
$$\mathbf{R}^2 \times \mathcal{M}^1 \text{ and } \alpha \in [2,4]$$

then there exists $\delta > 0$ s.t. any data (u_0, u_1) with $\|u_0\|_{H^1_{x,y}} + \|u_1\|_{L^2_{x,y}} < \delta$ produces a unique global solution

$$u \in C^0(\mathbf{R}, H^1) \cap C^1(\mathbf{R}, L^2) \cap L^{\alpha+1}(\mathbf{R}, L^{2\alpha+2}).$$

Moreover, those solutions scatter to a linear solution in H^1.

General statement $k = 1, 2$ and $d + k \in [3, 6]$, and $\frac{4}{d} \leq \alpha \leq \frac{4}{d+k-2}$.

Scattering follow from $\|u\|_{L_t^{\alpha+1}L_{x,y}^{2\alpha+2}} < \infty$:

$$U(t) = S(t)\begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} 0 \\ \pm|u|^\alpha u \end{pmatrix} ds$$

$$V(t) = S(-t)U(t) = \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t S(-s)\begin{pmatrix} 0 \\ \pm|u|^\alpha u \end{pmatrix} ds.$$

$V(t)$ exists/has some sense if it converges in $H^1 \times L^2$. We prove that $\lim_{t,\tau\to\infty} \|V(t) - V(\tau)\|_{H^1\times L^2} = 0$:

$$\begin{aligned}\|V(t) - V(\tau)\|_{H^1\times L^2} &\leq C \int_t^\tau \left\| \begin{pmatrix} 0 \\ \pm|u|^\alpha u \; ds \end{pmatrix} \right\|_{H^1\times L^2} ds \\ &\leq C \int_t^\tau \| \, |u|^\alpha u\|_{L^2} ds \\ &\leq C\|u\|^{\alpha+1}_{L^{\alpha+1}([t,\tau],L^{2\alpha+2})}\end{aligned}$$

which tends to zero as t, τ tend to infinity.

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Conclusion

Ongoing work (with L. Forcella - SNS, Pisa): what about large data ?
"simpler" case: defocusing, H^1-subcritical α.

Try to exploit the "flat" variables carrying the dispersive behaviour.
Use of concentration-compactness method ("à la Kenig-Merle").

- **Prove that for $\|u_0\|_{H^1} < E_0$ small enough, H^1- holds.**
- Assume there is no H^1-scattering for solutions above some critical energy $E_c \geq E_0$. For those solutions $\|u\|_{L_t^{\alpha+1} L_{x,y}^{2\alpha+2}} = +\infty$.
- Build such critical element with profile decomposition and try to understand its particular properties (compactness of trajectory). Bahouri-Gérard, Ibrahim-Masmoudi-Nakanishi, Nakanishi-Schlag, Banica-Visciglia.
- Exploit those properties, together with adapted "Morawetz estimates" instead of Virial estimates, to obtain a contradiction and deduce that $E_c = +\infty$.

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THANK YOU