Ground states and dynamics for perturbed critical NLS

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Perturbed energy-critical NLS

$$(NLS_\varepsilon) \qquad \boxed{-iu_t = \Delta u + |u|^4 u + \varepsilon |u|^{p-1} u} \qquad x \in \mathbb{R}^3, \quad 0 < \varepsilon \ll 1$$

- conserved quantities:

$$M(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx, \quad E_\varepsilon(u) = \int_{\mathbb{R}^3} \left\{ \frac{1}{2}|\nabla u|^2 - \frac{1}{6}|u|^6 - \frac{\varepsilon}{p+1}|u|^{p+1} \right\} dx$$

- (NLS_0) **energy critical**: $u(x,t) \mapsto \lambda^{\frac{1}{2}} u(\lambda x, \lambda^2 t)$ preserve $(NLS_0), E_0$
- **solitary waves**: $u(x,t) = e^{i\omega t} v_\omega(x)$, v_ω a c.p. of $S_{\varepsilon,\omega} = E_\varepsilon + \omega M$:

$$\boxed{0 = -\Delta v_\omega - |v_\omega|^4 v_\omega - \varepsilon |v_\omega|^{p-1} v_\omega + \omega v_\omega = S'_{\varepsilon,\omega}(v_\omega)}$$

- **ground states**: (non-trivial) solitary waves of least "action" $S_{\varepsilon,\omega}$
- (NLS_0) admits no solitary wave, only static solutions $(\varepsilon = \omega = 0)$

$$\boxed{W(x) = \left(1 + \tfrac{|x|^2}{3}\right)^{-\frac{1}{2}} \notin L^2(\mathbb{R}^3), \quad \Delta W + W^5 = 0}$$

Question: does (NLS_ε) have solitary waves/ground states for $\varepsilon \neq 0$?

Outline

1. Perturbative construction of solitary waves (from $W(x)$)
2. "Ground state" variational characterization
3. Implications for $(NLS)_\varepsilon$ dynamics

1: Perturbative construction of solitary waves

Solitary waves as perturbations of $W(x)$

We seek real, radially-symmetric solutions $v = v(|x|) \in \mathbb{R}$ of

$$0 = -S'_{\varepsilon,\omega}(v) = \Delta v + v^5 + \varepsilon |v|^{p-1} v - \omega v \quad (*)$$

Since $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$ solves the unperturbed problem

$$0 = -S'_{0,0}(W) = \Delta W + W^5$$

it is natural to seek solutions v as small perturbations of W:

Thm[G-Coles]: for $0 < \varepsilon \ll 1, \quad 2 < p < 5$, there are smooth solutions $v_\varepsilon \in H^1(\mathbb{R}^3)$ of $(*)$ with

$$\|v_\varepsilon - W\|_{\dot{H}^1} \lesssim \varepsilon^{\frac{1}{2}}, \quad \|v_\varepsilon - W\|_r \lesssim \varepsilon^{1-\frac{3}{r}}, \quad 3 < r \leq \infty$$

$$\omega = \omega(\varepsilon) = \lambda_1^2 \varepsilon^2 + o(\varepsilon^2), \qquad \lambda_1 = \frac{5-p}{12\pi(p+1)} \int W^{p+1}$$

- also works for $p > 5$, $\varepsilon < 0$ (supercritical, defocusing perturbation)
- [Davila-delPino-Guerra 12]: similar perturbation result
- [Akahori-Ibrahim-Kikuchi-Nawa 12]: variational method in $\mathbb{R}^{\geq 4}$

Perturbation PDE, linear resonance, resolvent expansion

- plug $v_\varepsilon(x) = W(x) + \eta(x), \quad \omega = \lambda^2$ into (*) find equation for η:

$$\boxed{(H + \lambda^2)\eta = \varepsilon W^p - \lambda^2 W + N(\eta, \varepsilon) =: f(\varepsilon, \lambda, \eta)}$$

  where $H = -\Delta - 5W^4$ is the *linearized operator*.

- scale invariance $\implies$ H has a threshold **resonance**:

$$H\,\Lambda W = 0, \quad \Lambda W(x) := \frac{d}{d\lambda}\lambda^{1/2} W(\lambda x)\Big|_{\lambda=1} \notin L^2(\mathbb{R}^3)$$

- with *free resolvent* $R_0(\lambda) := (-\Delta + \lambda^2)^{-1}$, the resolvent expansion of [Jensen-Kato 79] for H: as $\lambda \to 0+$,

$$(H + \lambda^2)^{-1} = \left[\frac{1}{\lambda}\Big(\cdot\,,\ cW^4\Lambda W\Big)\Lambda W + O(1)\right] R_0(\lambda)$$

  (in weighted spaces – can convert to Lebesgue spaces).

- **orthogonality condition** $\boxed{f \perp R_0(\lambda) W^4 \Lambda W}$ removes singularity:

$$(H + \lambda^2)^{-1} f = \left[(1 - 5(-\Delta)^{-1} W^4)^{-1} + O_{L^r \to L^r}(\lambda^{1-\frac{3}{r}})\right] R_0(\lambda) f.$$

Solve the perturbation equation: Lyapunov-Schmidt

So we solve, by Lyapunov-Schmidt-type method, the system

$$\begin{cases} \eta = (H+\lambda^2)^{-1} f(\varepsilon, \lambda, \eta) \\ R_0(\lambda) W^4 \Lambda W \perp f(\varepsilon, \lambda\eta) = \varepsilon W^p - \lambda^2 W + N(\eta, \varepsilon) \end{cases}$$

Step 1: given $\|\eta\|_\infty \lesssim \varepsilon$, solve the orthogonality condition

$$0 = (R_0(\lambda) W^4 \Lambda W, f) \approx (R_0(\lambda) W^4 \Lambda W, \varepsilon W^p - \lambda^2 W)$$

$$\approx \varepsilon \cdot \frac{1}{5}\Big(\Lambda W, W^p\Big) - \lambda \cdot \lambda\Big(R_0(\lambda) W^4 \Lambda W, W\Big)$$

Lemma: if $\langle x \rangle^{1+} g \in L^1(\mathbb{R}^3)$, $\quad h - \frac{1}{|x|} \in L^{\frac{3}{2}}(\mathbb{R}^3)$, then

$$\lambda\Big(R_0(\lambda) g, h\Big) = \left(\int_{\mathbb{R}^3} g\right) + O(\lambda).$$

So: $\qquad 0 \approx \varepsilon \cdot \frac{1}{5}\Big(\Lambda W, W^p\Big) - \lambda \cdot \sqrt{3} \int W^4 \Lambda W$

$$\Longrightarrow \boxed{\lambda = \lambda(\varepsilon) \approx \lambda_1 \varepsilon} \qquad \lambda_1 = \frac{5-p}{12\pi(p+1)} \int W^{p+1} > 0$$

Solve the perturbation equation: Lyapunov-Schmidt

$$\begin{cases} \eta = (H+\lambda^2)^{-1} f(\varepsilon,\lambda,\eta) \\ R_0(\lambda) W^4 \Lambda W \perp f(\varepsilon,\lambda\eta) = \varepsilon W^p - \lambda^2 W + N(\eta,\varepsilon) \end{cases}$$

Step 2: with orthogonality satisfied, $\|(H+\lambda^2)^{-1} f\|_r \lesssim \|R_0(\lambda) f\|_r$, so

$$\|\eta\|_r \lesssim \varepsilon + \lambda^{1-3/r} + \|R_0(\lambda) N\|_r \lesssim \varepsilon^{1-3/r}$$

which can be used (together with difference estimates) in a fixed-point argument to give existence of a solution.

So we have solitary waves for $\varepsilon \ll 1$

$$\Delta v_\varepsilon + v_\varepsilon^5 + \varepsilon |v_\varepsilon|^{p-1} v_\varepsilon = \omega(\varepsilon) v_\varepsilon$$
$$\|v_\varepsilon - W\|_{L^r} \lesssim \varepsilon^{1-\frac{3}{r}}, \quad \omega(\varepsilon) \approx \lambda_1^2 \varepsilon^2.$$

Question: are (some of) these "ground states"? Is even $v_\varepsilon(x) > 0$?

2. “Ground state” variational characterization

Energy critical (NLS_0): ground-state static solutions

$$(NLS_0) \quad \boxed{-iu_t = \Delta u + |u|^4 u} \quad E_0(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{6}\int |u|^6$$

- *Pohozaev relations* $\implies$ solitary waves are **static** ($\omega = 0$):

$$\begin{cases} 0 = K_0^0(v) := \partial_\lambda S_{0,\omega}(\lambda v)|_{\lambda=1} = \int |\nabla v|^2 - \int |v|^6 + \omega \int |v|^2 \\ 0 = K_0(v) := \partial_\lambda S_{0,\omega}(\lambda^{\frac{3}{2}} v(\lambda \cdot))|_{\lambda=1} = \int |\nabla v|^2 - \int |v|^6 \end{cases}$$

- static solution $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$ (up to scaling, spatial translation, phase rotation) is the unique **ground state** in the sense

$$\boxed{E_0(W) = \min\{E_0(v) \mid 0 \neq v \in \dot{H}^1(\mathbb{R}^3),\ K_0(v) = 0\}} = E_0(\lambda^{\frac{1}{2}} W(\lambda \cdot))$$

- up to scaling, multiples and translates, W is the unique Sobolev maximizer [Aubin, Talenti 76]:

$$\max_{0 \neq v \in \dot{H}^1(\mathbb{R}^3)} \frac{\int |u|^6}{(\int |\nabla u|^2)^3} = \frac{\int |W|^6}{(\int |\nabla W|^2)^3} = \frac{1}{(\int W^6)^2}$$

Perturbed critical (NLS_ε): ground-state solitary waves

$$(NLS_\varepsilon) \quad \boxed{-iu_t = \Delta u + |u|^4 u + \varepsilon |u|^{p-1} u}$$

$$S_{\varepsilon,\omega}(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{6}\int |u|^6 - \varepsilon \frac{1}{p+1}\int |u|^{p+1} + \frac{\omega}{2}\int |u|^2$$

- Pohozaev relations $\implies$ solitary waves have $\omega = \frac{\varepsilon(5-p)}{2(p+1)} \frac{\int |v|^{p+1}}{\int |v|^2} > 0$:

$$0 = K^0_{\varepsilon,\omega}(v) := \partial_\lambda S_{\varepsilon,\omega}(\lambda v)|_{\lambda=1} = \int \{|\nabla v|^2 - |v|^6 - \varepsilon |v|^{p+1} + \omega |v|^2\}$$

$$0 = K_\varepsilon(v) := \partial_\lambda S_{\varepsilon,\omega}(\lambda^{\frac{3}{2}} v(\lambda\cdot))|_{\lambda=1} = \int \left\{|\nabla v|^2 - |v|^6 - \varepsilon \frac{3(p-1)}{2(p+1)} |v|^{p+1}\right\}$$

- variational problem for **ground states**:

$$\boxed{m_{\varepsilon,\omega} := \inf\{S_{\varepsilon,\omega}(v) \mid 0 \neq v \in H^1(\mathbb{R}^3),\ K_\varepsilon(v) = 0\}}$$

Perturbed critical (NLS_ε): ground-state variational problem

$$S_{\varepsilon,\omega}(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{6}\int |u|^6 - \varepsilon\frac{1}{p+1}\int |u|^{p+1} + \frac{\omega}{2}\int |u|^2$$

$$\boxed{m_{\varepsilon,\omega} := \inf\{S_{\varepsilon,\omega}(v) \mid 0 \neq v \in H^1(\mathbb{R}^3),\ K_\varepsilon(v) = 0\}}$$

'Classical' facts:

- $m_{\varepsilon,\omega} \leq m_{0,0} = E_0(W)$: use cut-off, rescaled $W(x)$ as test function
- if $\boxed{m_{\varepsilon,\omega} < E_0(W)}$ $m_{\varepsilon,\omega}$ is attained [Brezis-Nirenberg 83]

  — prevents "bubbling" $\lambda^{\frac{1}{2}} W(\lambda x)$, $\lambda \to \infty$
- $m_{\varepsilon,\omega} = \inf\{\frac{p-\frac{7}{3}}{2(p-1)}\|\nabla u\|_2^2 + \frac{5-p}{6(p-1)}\|u\|_6^6 + \frac{\omega}{2}\|u\|_2^2 \mid K_\varepsilon(v) \leq 0\}$ and minimizers agree [AIKN 12]
- yields positive, radial ground state $v = v(|x|) > 0$, $S'_{\varepsilon,\omega}(v) = 0$

Existence of ground states for $p > 3$

- [Akahori-Ibrahim-Kikuchi-Nawa 12]: in $\mathbb{R}^N$, $N \geq 4$, $m_{\varepsilon,\omega} < m_{0,0}$ for $1 + \frac{4}{N} < p < 1 + \frac{4}{N-2}$. Hence ground states exist.
- back to $\mathbb{R}^3$: compute (delicate) for constructed solitary waves v_ε:

$$S_{\varepsilon,\omega(\varepsilon)}(v_\varepsilon) - E_0(W) = \frac{(3-p)\varepsilon}{2(p+1)} \int W^{p+1} + o(\varepsilon) \begin{cases} < E_0(W) & p > 3 \\ > E_0(W) & p < 3 \\ ?? & p = 3 \end{cases}$$

- so for $S_{\varepsilon,\omega(\varepsilon)}$, $\begin{cases} 3 < p < 5 & \text{ground states **exist**} \\ 2 < p < 3 & v_\varepsilon \text{ are **not** ground states} \\ p = 3 & ?? \end{cases}$

Natural questions:

1. are v_ε indeed ground states for $3 < p < 5$?

   – yes!

2. do ground states exist for $p = 3$? $p < 3$?

   – ??

Constructed solitary waves are ground states for $p > 3$

Thm[G-Coles]: for $3 < p < 5$, $\varepsilon \ll 1$, v_ε is the unique ground state (up to phase and translation) for $S_{\varepsilon,\omega(\varepsilon)}$.

Strategy:

1. as $\varepsilon \to 0$, ground states w_ε converge, up to rescaling, to W
2. after further rescaling, the orthogonality condition for $w_\varepsilon - W$ holds
3. the difference estimates of the fixed-point argument show that (rescaled) w_ε and v_ε must agree.

Rescaled ground states converge to W

- $S_{\varepsilon,\omega(\varepsilon)}(w_\varepsilon) = m_{\varepsilon,\omega(\varepsilon)} < E_0(W) \implies$ for $w = w_\varepsilon$,

$$\int |\nabla w|^2, \int |w|^6, \varepsilon \int |w|^{p+1}, \omega(\varepsilon) \int |w|^2 \lesssim 1$$

- moreover interpolation gives

$$\varepsilon \int |w|^{p+1} \lesssim \varepsilon \omega^{-\frac{5-p}{4}} \left(\int |w|^6 \right)^{\frac{p-1}{4}} \left(\omega \int |w|^2 \right)^{\frac{5-p}{4}} \to 0$$

  if $\varepsilon^{\frac{4}{5-p}} \ll \omega(\varepsilon) \approx \lambda_1^2 \varepsilon^2$, i.e. $p > 3$.

- so w_ε is a maximizing sequence for the critcal (Sobolev) problem. Concentration-compactness (eg. [Gérard 98]) $\implies$ convergence mod scaling:

$$\mu_\varepsilon^{\frac{1}{2}} w_\varepsilon(\mu_\varepsilon \cdot) \to W \text{ in } \dot{H}^1$$

Rescaled ground states converge to W

- after further scaling $\tilde{\mu}_\varepsilon = \mu_\varepsilon(1 + o(1))$, orthogonality holds:

$$\tilde{\mu}_\varepsilon^{\frac{1}{2}} w_\varepsilon(\tilde{\mu}_\varepsilon \cdot) - W \perp R_0(\tilde{\lambda}_\varepsilon) W^4 \Lambda W, \qquad \tilde{\lambda}_\varepsilon = \tilde{\mu}_\varepsilon \omega(\varepsilon)$$

- then we can quantify the convergence:

$$\| \tilde{\mu}_\varepsilon^{\frac{1}{2}} w_\varepsilon(\tilde{\mu}_\varepsilon \cdot) - W \|_r \lesssim \tilde{\varepsilon}^{1-\frac{3}{r}}, \qquad \tilde{\varepsilon} = \tilde{\mu}_\varepsilon^{\frac{5-p}{2}} \varepsilon$$

- local uniqueness from the fixed-point construction $\implies$

$$\tilde{\mu}_\varepsilon^{\frac{1}{2}} w_\varepsilon(\tilde{\mu}_\varepsilon \cdot) = v_\varepsilon \quad \text{for } \varepsilon \text{ sufficiently small.}$$

3. Implications for $(NLS)_\varepsilon$ dynamics

Energy critical (NLS_0): dynamics below the ground state

$$\boxed{-iu_t = \Delta u + |u|^4 u, \quad u(x,0) = u_0(x) \in \dot{H}^1(\mathbb{R}^3)}$$

$$E_0(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{6}\int |u|^6, \quad K_0(u) = \int |\nabla u|^2 - \int |u|^6$$

If $E(u(\cdot,t)) = E(u_0) < E(W)$, the sets

$$\{\|\nabla u\|_2 \lessgtr \|\nabla W\|_2\} \;\leftrightarrow\; \{K_0(u) \lessgtr 0\}$$

are invariant (Sobolev inequality). Moreover, if $\boxed{E_0(u_0) < E_0(W)}$:

- [Kenig-Merle 06]: u_0 radial, $\|\nabla u_0\|_2 < \|\nabla W\|_2 \implies$ solution **scatters** ([Killip-Visan 10] $N \geq 5$; [Dodson 15] N=4)
- u_0 radial, $\|\nabla u_0\|_2 > \|\nabla W\|_2 \implies$ **blow-up** [Ogawa-Tsutsumi 91]

Among the tools: concentration-compactness, and **virial identity**:

$$\frac{d^2}{dt^2}\int |x|^2 |u(x,t)|^2 dx = 4K_0(u).$$

Perturbed critical (NLS_ε): dynamics below ground state

$$-iu_t = \Delta u + |u|^4 u + \varepsilon |u|^{p-1} u, \quad u_0 \in H^1(\mathbb{R}^3), \;\; 0 < \varepsilon \ll 1$$

- $S_{\varepsilon,\omega}(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{6}\int |u|^6 - \frac{\varepsilon}{p+1}\int |u|^{p+1} + \frac{\omega}{2}\int |u|^2$
- $K_\varepsilon(u) = \int |\nabla u|^2 - \int |u|^6 - \frac{3\varepsilon(p-1)}{2(p+1)}\int |u|^{p+1}$
- $m_{\varepsilon,\omega} := \inf\{S_{\varepsilon,\omega}(v) \mid 0 \neq v \in H^1(\mathbb{R}^3),\; K_\varepsilon(v) = 0\}$

If $S_{\varepsilon,\omega}(u(\cdot,t)) = S_{\varepsilon,\omega}(u_0) < m_{\varepsilon,\omega}$, the sets $\{K_\varepsilon(u_0) \lessgtr 0\}$ are invariant, &

Thm: for $u_0 = u_0(|x|) \in H^1(\mathbb{R}^3)$ radial,

$$S_{\varepsilon,\omega}(u_0) < m_{\varepsilon,\omega} \implies \begin{cases} K_\varepsilon(u_0) > 0 \implies & \text{scattering} \\ K_\varepsilon(u_0) < 0 \implies & \text{blow up} \end{cases}$$

- [Akahori-Ibrahim-Kikuchi-Nawa 12], [Killip-Oh-Pocovnicu-Visan 14]
- [Kenig-Merle 06]
- virial identity: $\frac{d^2}{dt^2}\int |x|^2 |u(x,t)|^2 dx = 4K_\varepsilon(u)$.

Conclusions

$$(NLS_\varepsilon) \qquad \boxed{-iu_t = \Delta u + |u|^4 u + \varepsilon |u|^{p-1} u} \qquad x \in \mathbb{R}^3, \;\; 0 < \varepsilon \ll 1$$

- solitary waves $e^{-i\omega t} v_\varepsilon(x)$ constructed perturbatively:
  $$v_\varepsilon(x) = W(x) + o(1), \quad \omega = \omega(\varepsilon) \sim \varepsilon^2$$
  for $2 < p < 5$
- for $3 < p < 5$, **ground states** exist for $\omega \leq \omega(\varepsilon) \sim \varepsilon^2$, and for $\omega = \omega(\varepsilon)$, v_ε is the **unique** ground state
- v_ε is **not** a ground state for $p < 3$
- dynamics of radial solutions of (NLS_ε) below these ground states is classified into scattering and blow-up sets
- the ground state situation for $p = 3$, $p < 3$ is unclear