Nonlinear effect in the Exciton-Polariton System

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[1] Joint work with Stephen Shipman(LSU), Joaquin Delgado(UAM-I México)

Exciton-Polariton

Semiconductor cavities : Half-light and Half-matter

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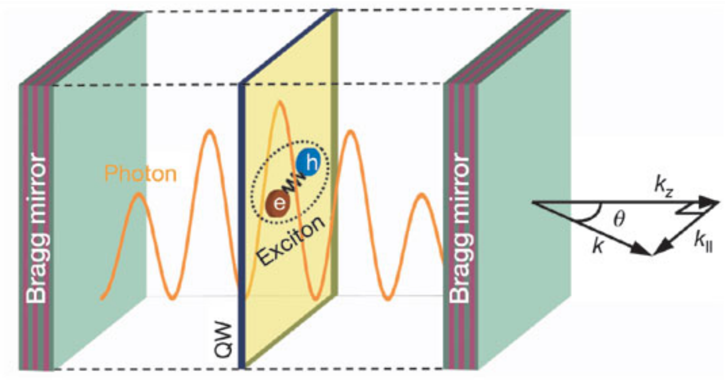

Figure: Kasprzak et al. Nature 443 (2006)

Exciton-Polariton

molybdenum diselenide (MoSe2)

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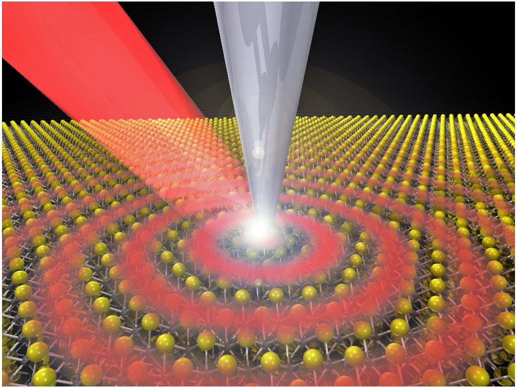

Figure: Zhe Fei - Molybdenum diselenide (MoSe2)-June,2017

$\phi(x,t)$: photons $\qquad$ $\psi(x,t)$: excitons

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$\phi(x,t)$: photons $\qquad \psi(x,t)$: excitons

$$\left\{ \begin{array}{rcl} i\partial_t \phi & = & (\omega_c - i\kappa_c - \frac{\hbar}{2m_C}\Delta)\phi + \gamma\psi \\ & & \end{array} \right.$$

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$$

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Nonlinear Schrödinger Equation

Nonlinearity : p

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Parameters :

Space : $x \in \mathbb{R}^n$

Time : $t \in \mathbb{R}$

$$g = \begin{cases} -1, & \text{focusing} \\ +1, & \text{defocusing} \end{cases}$$

Initial data : $u(x,0) = u_0(x) \in H^1(\mathbb{R}^n)$

- Number of particles (Mass)

$$M[u](t) = \int_{\mathbb{R}^n} |u(x,t)|^2 dx = \|u_0\|^2_{L^2(\mathbb{R}^n)}$$

- Hamiltonian (Energy)

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx - \frac{g}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx.$$

- Momentum

$$P[u](t) = \mathrm{Im}\int_{\mathbb{R}^n} \bar{u}\nabla u$$

Invariance/Symmetries

- Spatial translation

$$u(x,t) \iff u(x+x_0,t)$$

- Time translation

$$u(x,t) \iff u(x,t+t_0)$$

- Galilean transformation

$$u(x,t) \iff u(x-\xi t,t)e^{i(k\cdot\mathbf{x}-\omega t)}$$

- Scaling

$$u(x,t) \iff \lambda u(\lambda^2 x,\lambda t)$$

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Local theory

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- $T = +\infty$ global
- $T < +\infty$ finite blowup $\sim \; \|\nabla u\|_{L^2} \nearrow +\infty$

- Standard tools : Strichartz estimates

$$\left\| e^{it\triangle} u_0 \right\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2}$$

$$\left\| \int_{\mathbb{R}^n} e^{i(t-\tau)\triangle} F(\tau) d\tau \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s, \qquad \text{with} \quad 2 \leq q, r \leq \infty \quad \text{and} \quad (q, r, n) \neq (2, \infty, 2)$$

$\mathrm{NLS}_p(\mathbb{R}^n)$

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-\tau)\Delta}|u|^{p-1}u(\tau)d\tau \equiv \mathrm{NLS}(t)u_0$$

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When do solutions scatter?

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$$\|\nabla u(t)\|_{L^2} \to \infty \qquad \text{as} \qquad t \to T^*$$

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Conserved quantities

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Invariance

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- Time translation

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Break of Invariance

Lossless Polariton

Invariance

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Break of Invariance

- Scaling

$$\begin{pmatrix} \phi(\mathbf{x},t) \\ \psi(\mathbf{x},t) \end{pmatrix} \quad \mapsto \quad \lambda \begin{pmatrix} \phi(\lambda\mathbf{x},\lambda^2 t) \\ \psi(\lambda\mathbf{x},\lambda^2 t) \end{pmatrix},$$

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$$\begin{pmatrix} \omega_x - g|\psi|^2 & \gamma \\ \gamma & \omega_c - \Delta \end{pmatrix} \mapsto \begin{pmatrix} \lambda^2\omega_c - \Delta & \lambda^2\gamma \\ \lambda^2\gamma & \lambda^2\omega_x - g|\psi|^2 \end{pmatrix}$$

- Galilean transformation

$$\begin{pmatrix} \phi(\mathbf{x},t) \\ \psi(\mathbf{x},t) \end{pmatrix} \mapsto \begin{pmatrix} \phi(\mathbf{x}-\xi t,t) \\ \psi(\mathbf{x}-\xi t,t) \end{pmatrix} e^{i(\xi\cdot\mathbf{x}-|\xi|^2 t)},$$

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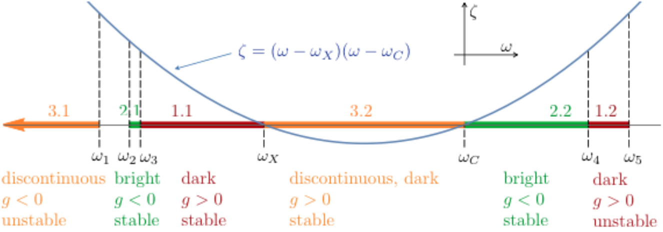
$$\downarrow$$

$$\begin{pmatrix} \omega_c - \Delta & \gamma \\ \gamma & \omega_x - |\xi|^2 - g|\psi|^2 - i\xi\cdot\nabla\psi \end{pmatrix}$$

Komineas, Shipman, Venakides' 14 : $x \in \mathbb{R}$

$$\phi(x,t) = \phi_c(x)e^{i(-\omega t)}$$
$$\psi(x,t) = \psi_x(x)e^{i(-\omega t)}$$



$$\left\{ \begin{array}{lcl} \phi(x,t) & = & \phi_c(x-ct)e^{i(kx-\omega t)} \\ \psi(x,t) & = & \psi_x(x-ct)e^{i(kx-\omega t)} \end{array} \right.$$

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- $\omega,\ k \neq 0$, and $c = 0$

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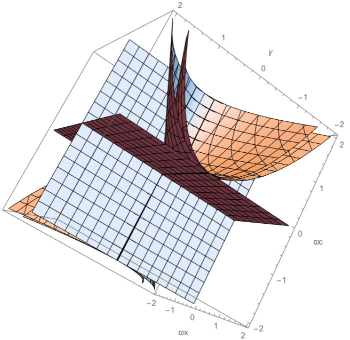
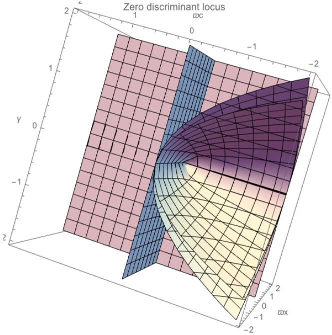
- $\omega,\ k \neq 0$, and $c = 0$

$$\omega-\omega_x, \quad \omega-\omega_c, \quad \gamma^2-2(\omega-\omega_x)(\omega-\omega_c), \quad 9\gamma^2-8(\omega-\omega_x)(\omega-\omega_c)$$





$$g > 0, \qquad \omega - \omega_c > 0 \qquad \text{and} \qquad \omega - \omega_x > 0$$

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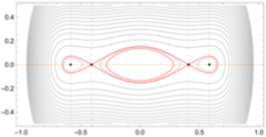
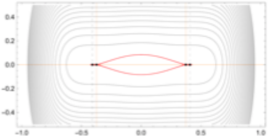
DoGraph[1, 1, .5, .4]

{0, 0.009248, 0.0118519}



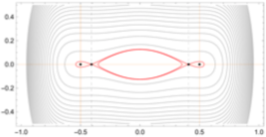
DoGraph[1, 1, .5, .6]

{0, 0.003528, 0.00351852}



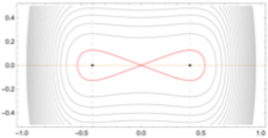
DoGraph[1, 1, .5, .5]

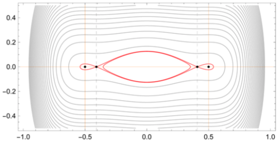
{0, 0.0078125, 0.00810185}

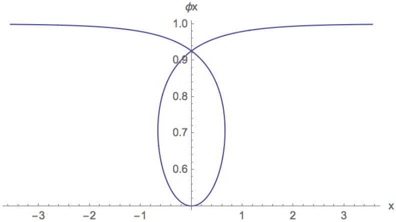


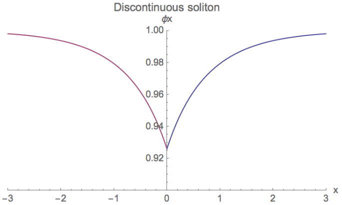
DoGraph[1, 1, .5, .8]

{0, -0.00814815}




0.4
0.2
0.0
−0.2
−0.4
−1.0
−0.5
0.0
0.5
1.0


ϕx
1.0
0.9
0.8
0.7
0.6
x
−3
−2
−1
1
2
3


Discontinuous soliton
ϕx
1.00
0.98
0.96
0.94
0.92
x
−3
−2
−1
0
1
2
3

Lossless Polariton with $\omega_c = 0$

Consider

$$\begin{aligned} i\phi_t &= -\Delta\phi + \gamma\psi \\ i\psi_t &= (\omega_x + g|\psi|^2)\psi + \gamma\phi \end{aligned}$$

and

$$\begin{pmatrix} \phi(x,0) \\ \psi(x,0) \end{pmatrix} = \begin{pmatrix} \phi_0(x) \\ 0 \end{pmatrix} \in H^s(\mathbb{R}^n) \qquad \text{with } s > \frac{n}{2}.$$

Existence

Given :

$$\|\phi_0\|_{H^s} \leq \alpha N \qquad \text{for} \qquad N > 0, \quad \alpha \in (0,1)$$

There exists a unique solution $\begin{pmatrix} \phi(x,t) \\ \psi(x,t) \end{pmatrix} \in C(I, H^s(\mathbb{R}^n))$ to the polariton system such that

$$\|\phi(t)\|_{H^s} < N \quad \text{and} \quad \|\psi(t)\|_{H^s} < N$$

for

$$0 \leq t \leq \frac{1-\alpha}{2\gamma + |g|N^2}.$$

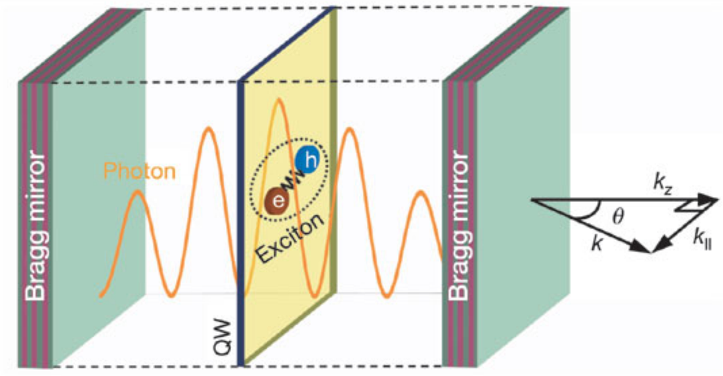


Figure: Kasprzak et al. Nature 443 (2006)

Short-time behavior

- Up to what time is the effect of the exciton on the photon field negligible?

$$\begin{aligned} i\phi_t &= -\Delta\phi \\ i\psi_t &= \omega_X\psi + \gamma\phi\,. \end{aligned} \qquad \text{(Approximation A)}$$

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$$\begin{aligned} i\phi_t &= -\Delta\phi + \gamma\psi \\ i\psi_t &= \omega_X\psi + \gamma\phi\,. \end{aligned} \qquad \text{(Approximation B)}$$

$0 < \epsilon \ll 1, \quad c_1, c_2 \in \mathbb{R}$ s.t. $c_2\epsilon^\beta < T$.

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} \leftrightarrow \text{ polariton,} \qquad \begin{pmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{pmatrix} \leftrightarrow \begin{cases} \text{approx. A} & [0, c_1\epsilon^{1/2}] \\ \text{approx. B} & [c_1\epsilon^{1/2}, c_2\epsilon^\beta] \end{cases}$$

$$\text{IC} \begin{pmatrix} \phi(0) \\ \psi(0) \end{pmatrix} = \begin{pmatrix} \tilde{\phi}(0) \\ \tilde{\psi}(0) \end{pmatrix} = \begin{pmatrix} \epsilon^\alpha \phi_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \|\phi_0\|_s = \epsilon^\alpha M \neq 0$$

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$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_s}{\|\phi(t)\|_s} \leq K_1\epsilon + O(\epsilon^q) \qquad 0 \leq t \leq c_1\epsilon^{1/2} \qquad (\epsilon \to 0),$$

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$3/2 < q = \min\left\{2,\ 1 + \frac{p}{2} + \alpha(p-1)\right\}$

$$\text{if} \quad 0 \leq \alpha < \frac{1}{p-1} \quad \text{then} \quad \beta = \frac{1}{p+2} - \frac{p-1}{p+2}\alpha$$

Compare the systems

$$\begin{aligned}\hat{\phi} &:= \tilde{\phi} - \phi\,,\\ \hat{\psi} &:= \tilde{\psi} - \psi\,.\end{aligned}$$

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if $t \in [0, c_1 \epsilon^{1/2}]$

$$\begin{aligned} i\hat{\phi}_t &= -\triangle\hat{\phi} + \gamma\psi(t) \\ i\hat{\psi}_t &= \omega_X\hat{\psi} + \gamma\hat{\phi} + g|\psi(t)|^2\psi(t) \end{aligned} \qquad \begin{cases} \hat{\phi}(0) = 0 \\ \hat{\psi}(0) = 0 \end{cases}$$

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$$\begin{aligned}\|\hat{\phi}(t)\| &\le \gamma\int_0^t \|\psi(\tau)\|d\tau \le \gamma\int_0^t y_1(\tau)\,d\tau \\ &= \frac{1}{2}\gamma^2 Mt^2 + O(t^4) \le \frac{1}{2}\gamma^2 MC_1^2\epsilon + O(\epsilon^2)\end{aligned}$$

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$$\frac{\|\hat{\phi}(t)\|}{\|\phi(t)\|} \leq \frac{1}{2}\gamma^2 C_1^2\epsilon + O(\epsilon^2).$$

THEOREM: $p = 3,\ \alpha = 0$

$$\begin{cases} \phi_t & = -\Delta\phi + \gamma\psi \\ \psi_t & = (\omega_o + g|\psi|^2)\psi + \gamma\phi \end{cases} \qquad \begin{cases} \phi(x,0) & = \phi_0 \\ \psi(x,0) & = 0 \end{cases} \qquad \|\phi_0\|_s = M$$

Then

$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_s}{\|\phi(t)\|_s} \leq K_1\epsilon + O(\epsilon^2) \qquad 0 \leq t \leq c_1\epsilon^{1/2}$$

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$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_s}{\|\phi(t)\|_s} \leq K_1\epsilon + O(\epsilon^2) \qquad 0 \leq t \leq c_1\epsilon^{1/2}$$

$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_s}{\|\phi(t)\|_s} \leq K_2\epsilon + O(\epsilon^{7/5}) \qquad c_1\epsilon^{1/2} \leq t \leq c_2\epsilon^{1/5}.$$

Numerics

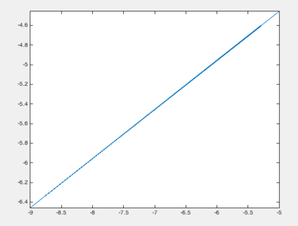
$\phi_0(r) = e^{-\frac{1}{2}\mu\, r^2}$ with $g = -1,$ $p = 3$ in $\mathbb{R}^2$

NUMERICS

$$\color{red}{\phi_0(r) = e^{-\frac{1}{2}\mu r^2}} \quad \text{with} \quad g = -1, \quad p = 3 \quad \text{in} \quad \mathbb{R}^2$$

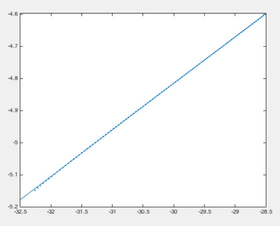
$$\log t \quad \text{vs.} \quad \log \frac{\|\tilde{\phi}(t) - \phi(t)\|_{H^2}}{\|\phi(t)\|_{H^2}}$$

approx. A



slope $= 0.5000 \approx 1/2$

approx. B



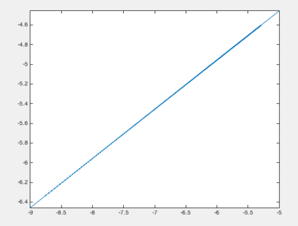
slope $= 0.2003 \approx 1/5$

Numerics

$\phi_0(r) = e^{-\frac{1}{2}\mu\, r^2}$ with $g = -1,$ $p = 3$ in $\mathbb{R}^2$

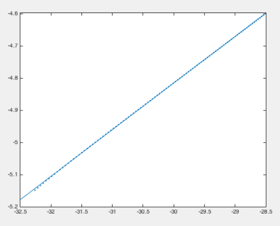
$$\log t \quad \text{vs.} \quad \log \frac{\|\tilde{\phi}(t) - \phi(t)\|_{H^2}}{\|\phi(t)\|_{H^2}}$$

approx. A



slope $= 0.5000 \approx 1/2$

approx. B



slope $= 0.2003 \approx 1/5$

Thanks Svetlana-Kai

NUMERICS: Cubic $\mathbb{R}^2$

$\phi_0(r) = e^{-\frac{1}{2}\mu r^2}, \quad g = 1$

–approx. A

μ	0.5	1.0	2.0	
slope	0.5000	0.5000	0.5000	$\approx 1/2$

–approx. B

μ	0.5	1.0	2.0	
slope	0.2002	0.2003	0.2008	$\approx 1/5$

NUMERICS: Cubic $\mathbb{R}^2$

$\phi_0(r) = e^{-\frac{1}{2}\mu\, r^2}, \quad g = 1$

–approx. A

μ	0.5	1.0	2.0	
slope	0.5000	0.5000	0.5000	$\approx 1/2$

–approx. B

μ	0.5	1.0	2.0	
slope	0.2002	0.2003	0.2008	$\approx 1/5$

$\phi_0(r) = e^{-\frac{1}{2}\mu\, r^2}\, e^{-i\frac{1}{2}\, r^2}, \quad g = -1$

– approx. A

μ	0.5	1.0	2.0	
slope	0.5000	0.5000	0.5000	$\approx 1/2$

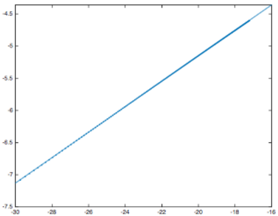
–approx. B

μ	0.5	1.0	2.0	
slope	0.1992	0.2036	0.2101	$\approx 1/5$

$$\begin{cases} i\phi_t &= -\Delta\phi + \gamma\psi \\ i\psi_t &= (\omega_0 + g|\psi|^{p-1})\psi + \gamma\phi \end{cases} \qquad \begin{cases} \phi(\mathbf{x},0) = \phi_0(\mathbf{x}) \\ \psi(\mathbf{x},0) = 0 \end{cases} \qquad (\alpha = 0)$$

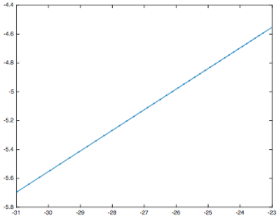
$$\frac{\|\tilde{\phi}(t) - \phi(t)\|_{L^2}}{\|\phi(t)\|_{L^2}} \leq \epsilon + O(\epsilon^2) \quad \text{for} \quad 0 \leq t \leq C\epsilon^{\beta}; \quad \beta = \max\left\{0, \frac{1}{p+2}\right\}$$

$$\log t \quad \text{vs.} \quad \log \frac{\|\tilde{\phi}(t) - \phi(t)\|_{L^2}}{\|\phi(t)\|_{L^2}}$$



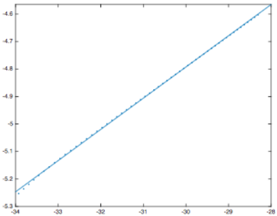
$p = 3,\ u_0 = e^{(-1-5i)r^2}$ in $\mathbb{R}^2$

slope $= 0.19821 \approx 1/5$



$p = 5,\ u_0 = e^{-0.25\,r^2}$ in $\mathbb{R}^3$

slope $= 0.14323 \approx 1/7$



$p = 7,\ u_0 = e^{(-0.5+0.5i)r^2}$ in $\mathbb{R}^2$

slope $= 0.11360 \approx 1/9$

General case-Numerics

$$\begin{cases} \phi_t & = -\Delta\phi + \gamma\psi \\ \psi_t & = (\omega_o + g|\psi|^2)\psi + \gamma\phi \end{cases} \qquad \|\phi_0\|_s = \epsilon^\alpha M$$

$$\text{if} \quad 0 < \alpha < \frac{1}{p-1} \quad \text{then} \quad \beta = \frac{1}{p+2} - \frac{p-1}{p+2}\alpha$$

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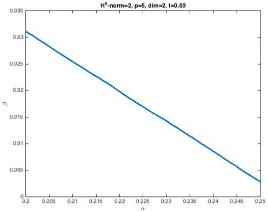
$$\text{if} \quad 0 < \alpha < \frac{1}{p-1} \quad \text{then} \quad \beta = \frac{1}{p+2} - \frac{p-1}{p+2}\alpha$$

$\phi_0(r) = e^{-\frac{1}{2}\mu\, r^2}$ $\qquad \alpha$ vs. β

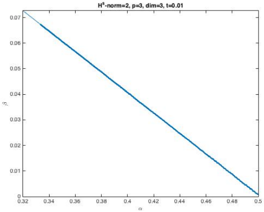
$PE_5(\mathbb{R}^2)$ in H^2



int. $= 0.1431 \approx 1/7$

slope $= -0.5734 \approx -4/7$

$PE_3(\mathbb{R}^3)$ in H^2



int $= 0.2006 \approx 1/5$

slope $= -0.3997 \approx -2/5$

- Exciton bounds

Comments

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Comments

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  - Approximation A : $0 < t \leq c_1 \epsilon^{1/2}$

$$\frac{\|\hat{\psi}\|_{L_t^\infty H_x^s}}{\|\psi\|_{L_t^\infty H_x^s}} \approx O(\epsilon)$$

Comments

- Exciton bounds
  - Approximation A: $0 < t \leq c_1\epsilon^{1/2}$

$$\frac{\|\hat{\psi}\|_{L_t^\infty H_x^s}}{\|\psi\|_{L_t^\infty H_x^s}} \approx O(\epsilon)$$

  - Approximation B: $c_1\epsilon^{1/2} \leq t \leq c_2\epsilon^{\beta}$

$$\frac{\|\hat{\psi}\|_{L_t^\infty H_x^s}}{\|\psi\|_{L_t^\infty H_x^s}} \approx O(\epsilon^{\beta})$$

Comments

- Exciton bounds
  - Approximation A: $0 < t \leq c_1 \epsilon^{1/2}$

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$$\frac{\|\hat{\psi}\|_{L_t^\infty H_x^s}}{\|\psi\|_{L_t^\infty H_x^s}} \approx O(\epsilon^{\beta})$$

- Are these optimal bounds?

Thank you !!