Instability of solitary waves in Zakharov-Kuznetsov equation

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June 15, 2017 - CIRM French-American Conference on PDE

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  - $p < 5$ – L^2-subcritical case

    $p > 5$ – L^2-supercritical case

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  $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}\,x)$,
  and
  $Q(x) = Q_1(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right)$

Stability of solitons

Let $\alpha > 0$, $s \geq 0$ and define a "tube"

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- Asymptotic stability (Pego-Weinstein '94, Martel-Merle '01, Mizumachi '01, ...)

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  - Characterization of dispersion
    Obstacle: no virial

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- this yields a contradiction at a finite time $t(u_0)$.

Multidimensional model: ZK

- Generalization of gKdV to two dimensions

$$u_t + \partial_{x_1}(\Delta_{(x_1,x_2)} u + u^p) = 0$$

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- Traveling waves in x_1-direction:
  $u(t, x_1, x_2) = Q_c(x_1 - ct, x_2)$, $Q_c(x) \to 0$ as $|x| \to \infty$
  Q solves $\Delta Q - Q + Q^p = 0$ (take radial vanishing solution)

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- ▸ we address $p = 3$ case (L^2-critical case in 2d).

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if $u_0 = Q + \epsilon_0$, $\epsilon_0 \in H^1(\mathbb{R}^2)$, with

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Remark: Example $\epsilon_0^n = \frac{1}{n}(Q + a\chi_0)$, where $a = -\frac{\int \chi_0 Q}{\|\chi_0\|_2^2}$.

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$\Rightarrow$ $\exists\ \lambda(t), \vec{x}(t) = (x(t), 0)$ such that

$$\epsilon(t) = \lambda(t)\, u\,(t, \lambda(t)\vec{y} + \vec{x}(t)) - Q(\vec{y})$$

$$(\text{with } \vec{x}(0) = 0, \lambda(0) = 1)$$

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(with $\vec{x}(0) = 0, \lambda(0) = 1$)
and also satisfy $\epsilon(t) \perp \{Q_{x_1}, Q_{x_2}, \chi_0\}$

  - rescale time $t \mapsto s$: $\frac{ds}{dt} = \frac{1}{\lambda^3}$
    $+$ modulation of parameters (and $u \in U_{\alpha_0}$):
    $\|\epsilon(s)\|_{H^1} \lesssim \alpha_0$ and $|\lambda(s) - 1| \lesssim \alpha_0$

A la virial

$$\text{Set up } J(s) = \int_{\mathbb{R}^2} \epsilon(s, y_1, y_2) \left(\int_{-\infty}^{y_1} \Lambda Q(z, y_2)\, dz \right) dy_1 dy_2$$

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- furthermore, for more precise time control,
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- Issues : - how to get (2)?
  - control/independence of parameters α_0, b_0, A ?

Virial estimates

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- Problem to control the last two terms

3rd term: monotonicity

- Define (as in Martel-Merle)
  $I_{x_0,t_0}(t) = \int u^2(t,x,y)\psi(x - x(t_0) + \frac{1}{2}(t_0 - t) - x_0)dxdy$
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Prop: Suppose $|u_0(x,y)| \lesssim e^{-\delta|(x,y)|}$
Then for all $t \geq 0$ and $x_0 > 0$

$$\int_{\mathbb{R}} \int_{x>x_0} u^2(t, x + x(t), y)dxdy \leq C\, e^{-\frac{x_0}{M}}.$$

In particular, $\int_{\mathbb{R}} \int_{x>x_0} \epsilon^2(t,x,y)dxdy \leq C\, e^{-\frac{x_0}{2M}}$,
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- The 3rd term: $A^{1/2}\|\epsilon(s)\|_{L^2(x \geq A)} < \frac{1}{8}\int \epsilon_0 Q$.

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  Goal: pointwise decay on $\epsilon(s)$
- Allows to show smallness of the last term

Fundamental solution estimate

Prop: Let $A(x, y, t) = \iint_{\mathbb{R}^2} e^{i(t\xi^3 + t\xi\eta^2 + x\xi + y\eta)}\, d\xi d\eta$

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$$|A(x,y,t)| \lesssim t^{-2/3} \begin{cases} \langle\lambda\rangle^{-\alpha} & \text{if } |z| \leq 4, \forall\alpha \geq 0 \\ \langle\lambda\rangle^{-\alpha}\langle\lambda|z|^{3/2}\rangle^{-\beta} & \text{if } |z| \geq 4\,, \forall\alpha,\beta \geq 0 \end{cases}$$

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- Next, estimate on the linear solution

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- Propagate this to the nonlinear equation.
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for all $s \geq 0$, $y \in \mathbb{R}$, $x > 0$.

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- Thus, the 4th term:
$$\left| \int_{\mathbb{R}^2} y \left(\int_{-\infty}^{x} \Lambda Q \right)_y \epsilon \, \varphi_A \right| \leq \frac{1}{8} \int \epsilon_0 Q.$$

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- Contradiction with boundedness $|K_A(s)|$, finishes the proof!

THANK YOU for your attention!