Flash Presentation

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Research area: Control of PDE and related topics

Controllability, Stability, Inverse Problems.
Waves, heat and Navier-Stokes equations.

A typical question

Null-controllability problem for the heat equation

Let $\Omega \subset \mathbb{R}^d$, $\Gamma \subset \partial\Omega$.
The heat equation with control function $v \in L^2(0, T; L^2(\Gamma))$:

$$\begin{cases} \partial_t u - \Delta_x u = 0, & \text{in } (0, T) \times \Omega, \\ u(t, x) = v(t, x) \mathbf{1}_\Gamma(x), & \text{in } (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Given $u_0 \in L^2(\Omega)$, can we find a control function $v \in L^2(0, T; L^2(\Gamma))$ s.t. $u(T, \cdot) = 0$?

$\longrightarrow$ YES [Fursikov Imanuvilov '96, Lebeau Robbiano '95].

$\rightsquigarrow$ Based on Carleman estimates.

Despite important numbers of subsequent works based on deep Carleman estimates, there are still some open problems even for the 1-d heat equation with constant coefficient.

Coron Guerrero's problem (2005): Consider the null-controllability of the viscous transport equation:

$$\begin{cases} \partial_t u + \partial_x u - \varepsilon \partial_{xx} u = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = v(t), \quad u(t,L) = 0, & \text{in } (0,T), \\ u(0,\cdot) = u_0(\cdot), & \text{in } (0,L), \\ u(T,\cdot) = 0 & \text{in } (0,L), \end{cases}$$

The limit process is controllable with $v = 0$ iff $T > L$:

$$\begin{cases} \partial_t u + \partial_x u = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = v(t), & \text{in } (0,T), \\ u(0,\cdot) = u_0(\cdot), & \text{in } (0,L), \\ u(T,\cdot) = 0 & \text{in } (0,L), \end{cases}$$

Define $C(\varepsilon, L, T)$ as the cost of the null-control map for the viscous transport eq. :

$$\begin{cases} \partial_t u + \partial_x u - \varepsilon \partial_{xx} u = 0 & \text{in } (0,T) \times (0,L), \\ u(t,0) = v(t), \quad u(t,L) = 0, & \text{in } (0,T), \\ u(0,\cdot) = u_0(\cdot), & \text{in } (0,L), \\ u(T,\cdot) = 0 & \text{in } (0,L), \end{cases}$$

[Coron Guerrero '05]

- If $T < L$, then $\liminf_{\varepsilon \to 0} C(\varepsilon, L, T) = +\infty$.
- If $T > KL$, then $\limsup_{\varepsilon \to 0} C(\varepsilon, L, T) = 0$, for K large enough.

$K = 4.3$ [Coron Guerrero '05],
$K = 4.2$ [Glass '10],
$K = 2\sqrt{3}$ [Lissy '12]. $\longrightarrow$ Optimal K? Is $K = 1^+$?

A related issue

The problem

Let $A > 0$ and take a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that

- f is supported in $(-1, 1)$.
- $\hat{f}$ is known on $\mathbb{R} \setminus (-A, A)$.

- *Recovering f?* Possible as $\hat{f}$ is analytic.
- *Estimating f?*

$$\|f\|_{L^2(-1,1)} \leq C(A) \|\hat{f}\|_{L^2(\mathbb{R}\setminus(-A,A))}.$$

Claim: $C(A) \simeq \exp(A/2)$ as $A \to \infty$. [Landau, Slepian, Pollak, ... '60es]

$$K_A(\psi)(x) = \frac{1}{\pi} \int_{-1}^{1} \psi(y) \frac{\sin(A(x-y))}{x-y} \, dy.$$

$$L_A(\varphi)(x) = -(1-x^2)\frac{d^2\varphi}{dx^2} + 2x\frac{d\varphi}{dx} + A^2 x^2 \varphi.$$

Key ingredient: $K_A \circ L_A = L_A \circ K_A$ on $(-1, 1)$.

Thank you for your attention!