Bound from below of the exterior energy for the wave equation and applications

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CIRM. June 12th, 2017

Outline

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1. Focusing critical wave equation
2. Energy equirepartition

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2. Energy equirepartition
3. Improved energy equirepartition
4. Well-prepared initial data

Equation

(NLW) $$\begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u \\ \vec{u}_{\restriction t=0} = (u_0, u_1) \in \mathcal{H} = \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \end{cases}$$

where $u : [0, T[\times \mathbb{R}^N \to \mathbb{R}, \quad N \geq 3.$

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Conserved energy

$$E(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_x u(t)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\partial_t u(t)|^2 - \frac{N-2}{2N} \int_{\mathbb{R}^N} |u(t)|^{\frac{2N}{N-2}}.$$

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Scaling: $\quad u_\lambda(t,x) = \frac{1}{\lambda^{N/2-1}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right).$

The energy and the $\mathcal{H}$-norm are scale invariant.

Ground state

Stationary solutions of (NLW):

(E) $$-\Delta Q = |Q|^{\frac{4}{N-2}} Q, \quad Q : \mathbb{R}^N \to \mathbb{R}, \quad Q \in \dot{H}^1(\mathbb{R}^N).$$

"Unique" Minimal energy solution of (E) (ground state):

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{1-\frac{N}{2}}$$

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Threshold for the dynamics: [Kenig-Merle 2008].

$$T_+(u) < \infty \Longrightarrow \limsup_{t\to T_+} \|\nabla u\|_{L^2}^2 + \frac{N-2}{2}\|\partial_t u\|_{L^2}^2 \geq \|\nabla W\|_{L^2}^2$$

Existence of solutions of (E) with arbitrary large energy: [W.Y. Ding 1986], [Del Pino, Musso, Pacard, Pistoia 2013].

Other examples of solutions

Solitary waves (solitons): if $\mathbf{p} \in \mathbb{R}^3$ et $p = |\mathbf{p}| < 1$ and Q is a solution of (E):

$$Q_{\mathbf{p}}(t,x) = Q\left(\left(-\frac{t}{\sqrt{1-p^2}} + \frac{1}{p^2}\left(\frac{1}{\sqrt{1-p^2}} - 1\right)\mathbf{p}\cdot x\right)\mathbf{p} + x\right)$$

$$Q_{\mathbf{p}}(t,x) = Q_{\mathbf{p}}(0, x - t\mathbf{p}).$$

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Type II Blow-up solutions:

$$\vec{u}(t) = \left(\frac{1}{\lambda(t)^{\frac{N-2}{2}}} W\left(\frac{\cdot}{\lambda(t)}\right), 0\right) + (v_0, v_1) + o(1), \quad t \to T_+,$$

where $(v_0, v_1) \in \mathcal{H}$ and $\lambda(t) \ll T_+ - t$, see [Krieger Schlag & Tataru 09]. See also [Hillairet & Raphaël 2012, Krieger & Schlag 2014, Jendrej 2015].

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Open questions: solutions with more than one bubbles, or other bubbles than the ground state? See [Jendrej], [Martel & Merle] and also [Côte & Zaag 2012], [Côte & Martel].

Generalities on type II blow-up

Let u be a solution of (NLW) such that

$$T_+ = T_+(u) < \infty \text{ and } \limsup_{t\to T_+} \|\nabla u\|^2_{L^2} + \|\partial_t u\|^2_{L^2} < \infty.$$

Then there exist $k \geq 1$, k blow-up points $(x_1, \ldots, x_k) \in \left(\mathbb{R}^N\right)^k$ and $(v_0, v_1) \in \mathcal{H}$ such that

$$\vec{u}(t) \xrightharpoonup[t\to T_+]{} (v_0, v_1)$$

and, letting $\mathcal{R}_t = \left\{x \in \mathbb{R}^N \;:\; \forall j \in \{1, \ldots, k\},\ |x - x_j| > T_+ - t\right\}$,

$$\lim_{t\to T_+} \int_{\mathcal{R}_t} |\nabla u(t,x) - \nabla v_0(x)|^2\, dx + \int_{\mathcal{R}_t} |\partial_t u(t,x) - v_1(x)|^2\, dx = 0.$$

Outline

Linear wave equation

The asymptotic behavior for solutions of the linear wave equation:

$$\text{(LW)} \qquad \begin{cases} \partial_t^2 u_L - \Delta u_L = 0, \quad x \in \mathbb{R}^N \\ \vec{u}_{L\restriction t=0} = (u_0, u_1) \in \mathcal{H} = \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N). \end{cases}$$

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is well-known. Let $\not\partial$ be the tangential derivative. Then:

$$\lim_{t\to+\infty} \int \frac{1}{|x|^2}|u_L(t,x)|^2 + |\not\partial u_L(t,x)|^2 + |u_L(t,x)|^{\frac{2N}{N-2}}\, dx = 0$$

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and (see [Friedlander 70s]) there exists $G_\pm \in L^2(\mathbb{R} \times S^{N-1})$ such that

$$\lim_{t\to+\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_r u_L(t, r\omega) \mp G_\pm(r-t,\omega)\right|^2$$

$$+ \left| r^{\frac{N-1}{2}} \partial_t u_L(t, r\omega) + G_\pm(r-t,\omega)\right|^2 dr d\omega = 0.$$

Equirepartition for the linear equation

Theorem [TD, Kenig, Merle 2012]. *Assume that N is odd. Let u_L be a solution of the linear wave equation. Then the following holds for all $t \geq 0$ or for all $t \leq 0$:*

$$\int_{|x|\geq|t|} |\nabla_{t,x} u_L(t,x)|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_{t,x} u_L(0,x)|^2 \, dx.$$

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Question: for which solutions of (NLW) does there exists $\eta > 0$ such that

$$\forall t \geq 0 \text{ ou } \forall t \leq 0, \quad \int_{|x|\geq|t|} |\nabla_{t,x} u(t,x)|^2\,dx \geq \eta?$$

Equirepartition for critical focusing wave

Théorème [TD, Kenig, Merle 2012]. *Assume N is odd. There exists* $\varepsilon_0 > 0$ *such that if u is a solution of* (NLW) *with:*

$$\|(u_0, u_1)\|_{\mathcal{H}} < \varepsilon_0,$$

then the following holds for all $t \geq 0$ *or for all* $t \leq 0$*:*

$$\int_{|x|\geq|t|} |\nabla_{t,x} u(t,x)|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla_{t,x} u(0,x)|^2 \, dx.$$

Application

Theorem. There exists $\varepsilon_0 > 0$ such that, for any solution u of (NLW) such that $T_+(u) < \infty$ and

$$\limsup_{t\to T_+} \|\nabla u\|_{L^2}^2 + \frac{N-2}{2}\|\partial_t u\|_{L^2}^2 \leq \|\nabla W\|^2 + \varepsilon_0,$$

there exists $x_0 \in \mathbb{R}^N$, $(v_0, v_1) \in \mathcal{H}$, $\mathbf{p} \in \mathbb{R}^N$, $\lambda(t)$ and $x(t)$ such that

$$(u(t), \partial_t u(t)) - (v_0, v_1)$$
$$-\left(\frac{\iota_0}{\lambda(t)^{\frac{N}{2}-1}} W_{\mathbf{p}}\left(0, \frac{\cdot - x(t)}{\lambda(t)}\right), \frac{\iota_0}{\lambda(t)^{\frac{N}{2}}}(\partial_t W_{\mathbf{p}})\left(0, \frac{\cdot - x(t)}{\lambda(t)}\right)\right) \xrightarrow[t\to T_+]{} 0$$

in $\mathcal{H}$ and

$$\lim_{t\to T_+} \frac{\lambda(t)}{T_+ - t} = 0, \quad \lim_{t\to T_+} \frac{x(t) - x_0}{T_+ - t} = \mathbf{p}, \quad |\mathbf{p}| \leq C\varepsilon_0^{1/4}.$$

Outline

Exterior energy for radial data in 3D

Proposition. *Let u_L be a radial solution of the linear wave equation in space dimension 3. Let $A > 0$. Assume $u_0 \perp \frac{1}{r}$ in $\dot{H}^1(\{r > A\})$, i.e $u_0(A) = 0$. Then:*

$$\forall t \geq 0 \text{ or } \forall t \leq 0, \quad \int_{A+|t|}^{+\infty} (\partial_{t,r} u_L(t,r))^2 r^2 \, dr \geq \frac{1}{2} \int_A^{+\infty} (\partial_{t,r} u_L(0,r))^2 r^2 \, dr.$$

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Generalization to other odd dimensions: [Kenig, Lawrie, Baoping Liu, Schlag 2015]

Rigidity theorem

Theorem *Assume* $N = 3$*. Let* u *be a global and radial solution of (NLW). Assume*

$$\forall A > 0, \quad \liminf_{t \to \pm\infty} \int_{|x| \geq A + |t|} |\nabla_{t,x} u|^2 \, dx = 0.$$

Then $u = 0$ *or there exist* $\lambda > 0$, $\iota \in \{\pm 1\}$ *such that* $u(t,x) = \frac{\iota}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right)$.

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Recall that $W(x) = \frac{1}{\left(1+\frac{|x|^2}{3}\right)^{\frac{1}{2}}}$, so that

$$\frac{\iota}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right) \approx \frac{\sqrt{3}\lambda^{1/2}}{|x|}, \quad |x| \to \infty.$$

First step of the proof: $\exists \ell \in \mathbb{R}$ such that

$$\lim_{r\to\infty} r u_0(r) = \ell.$$

Classification of type II blow-up solution

Theorem. *Assume* $N = 3$. *Let* u *be a radial solution of* (NLW) *such that* $T_+(u) < +\infty$. *Then*

$$\lim_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{H}} = +\infty$$

or there exist $J \geq 1$,

- $(v_0, v_1) \in \mathcal{H}$,
- *signs* $\iota_j \in \{\pm 1\}$, $j = 1 \ldots J$,
- *parameters* $\lambda_j(t)$, $0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll T_+ - t$,

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such that

$$\vec{u}(t) = (v_0, v_1) + \sum_{j=1}^{J} \left(\frac{\iota_j}{\lambda_j^{\frac{1}{2}}(t)} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + \vec{\varepsilon}(t),$$

where $\lim_{t \to T_+} \|\vec{\varepsilon}(t)\|_{\mathcal{H}} = 0$.

Outline

Lower bound for the exterior energy

Lemma. *Let* $\gamma \in (0,1)$*. There exists* $\varepsilon = \varepsilon(\gamma) > 0$ *with the following property. Let* u_L *be a solution of* (LW) *with initial data* (u_0, u_1) *such that*

$$\begin{cases} (u_0,u_1) \in \dot{H}^1 \times L^2 & \text{if } N \geq 3 \\ |\nabla u_0| \in L^2, u_1 \in L^2 \text{ and } u_0 \equiv u_\infty & \text{si } N = 2 \end{cases}$$

(where $u_\infty \in \mathbb{R}$*) and*

$$\|(\nabla u_0, u_1)\|_{L^2(B^c_{1+\varepsilon} \cup B_{1-\varepsilon})} + \|\not{\partial} u_0\|_{L^2} + \|\partial_r u_0 + u_1\|_{L^2} \leq \varepsilon \|(\nabla u_0, u_1)\|_{L^2}.$$

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Then, for all $t \geq 0$,

$$\int_{|x| \geq \gamma + t} |\nabla_{x,t} u_L|^2(x,t)\, dx \geq \gamma \|(\nabla u_0,\, u_1)\|^2_{L^2}.$$

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Application for critical semilinear wave equation: lower bound of the exterior energy for well-prepared initial data, and soliton resolution along a sequence of times.

Wave maps

$$\text{(WM)} \qquad \begin{cases} \partial_t^2 u - \Delta u = \left(|\nabla u|^2 - |\partial_t u|^2\right) u, & x \in \mathbb{R}^2 \\ \vec{u}_{\restriction t=0} = (u_0, u_1), \quad u_0 \cdot u_1 = 0. \end{cases}$$

$$u : [0, T[\times \mathbb{R}^2 \to \mathbb{S}^2.$$

"Well-posedness" in $\mathcal{H} = \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ [Tao, Tataru].
To fix ideas, consider classical solutions: (u_0, u_1) C^∞, u_0 constant at infinity, $u_1 \equiv 0$ at infinity.

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$$E_M(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_x u(t)|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |\partial_t u(t)|^2$$

is conserved.

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Scaling: $u_\lambda(t, x) = u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$.
The energy is invariant by the scaling.

References for Wave Maps.

Global and local Cauchy theory in the critical space: [Tao 2001], [Tataru 2001 & 2005], [Sterbenz & Tataru 2010].

Global existence below the energy of the ground state: [Christodoulou, Tahvildar-Zadeh 1993], [Struwe 2003] (equivariant case), [Sterbenz & Tataru 2010]. See also [Tao], [Krieger & Schlag].

Explicit blow-up solutions: [Krieger, Schlag & Tataru 2008], [Raphaël & Rodnianski 2012], [Jendrej 2016].

Soliton resolution for equivariant solutions below a natural threshold: [Côte, Kenig, Lawrie & Schlag 2015].

Soliton resolution along a sequence of times for equivariant solutions: [Côte 2015].

Solition resolution strictly inside the wave cone for equivariant solutions: [Grinis 2016].

Well-prepared initial data for wave maps

Theorem. *Let* $\gamma \in (0,1)$*. There exists* $\varepsilon = \varepsilon(\gamma) > 0$ *with the following property. Let u be a classical solution of* (WM) *with initial data* (u_0, u_1) *such that*

$$E_M(u_0, u_1) \leq \varepsilon$$

and

$$\|(\nabla u_0, u_1)\|_{L^2(B^c_{1+\varepsilon} \cup B_{1-\varepsilon})} + \|\not\partial u_0\|_{L^2} + \|\partial_r u_0 + u_1\|_{L^2} \leq \varepsilon \|(\nabla u_0, u_1)\|_{L^2}.$$

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and

$$\|(\nabla u_0, u_1)\|_{L^2(B^c_{1+\varepsilon} \cup B_{1-\varepsilon})} + \|\not\partial u_0\|_{L^2} + \|\partial_r u_0 + u_1\|_{L^2} \leq \varepsilon \|(\nabla u_0, u_1)\|_{L^2}.$$

Then, for all $t \geq 0$,

$$\int_{|x| \geq \gamma + t} |\nabla_{x,t} u|^2(t,x)\, dx \geq \gamma \|(\nabla u_0,\, u_1)\|^2_{L^2}.$$

Small blow-up solutions for wave maps

Theorem *Let u be a classical of* (WM) *such that* $E_M(\vec{u}(0)) < E_M(W,0) + \epsilon_0^2$*, blowing-up in finite time* T_+ *at* $x = 0$*. Then* $\exists \mathbf{p} \in \mathbb{R}^2$ *such that* $|\mathbf{p}| \ll 1$*,* $x(t) \in \mathbb{R}^2$*,* $\lambda(t) > 0$ *with*

$$\lim_{t\to T_+} \frac{x(t)}{T_+ - t} = \mathbf{p}, \quad \lim_{t\to T_+} \frac{\lambda(t)}{T_+ - t} = 0,$$

and $(v_0,\, v_1) \in \mathcal{H} \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$ *with* $(v_0 - u_\infty,\, v_1)$ *compactly supported, such that*

$$\text{(i)} \qquad \inf\left\{ \left\| \vec{u}(t) - (v_0,\, v_1) - (Q_{\mathbf{p}},\, \partial_t Q_{\mathbf{p}}) \right\|_{\mathcal{H}} \, : \, Q_{\mathbf{p}} \in \mathcal{M}_{\mathbf{p}} \right\} \underset{t\to T_+}{\longrightarrow} 0,$$

$$\text{(ii)} \qquad \left\| (\nabla u(t), \partial_t u(t)) - (\nabla v_0,\, v_1) \right\|_{L^2\left(\mathbb{R}^2 \setminus B_{\lambda(t)}(x(t))\right)} \underset{t\to T_+}{\longrightarrow} 0,$$

where $B_{\lambda(t)}(x(t)) = \left\{ x \in \mathbb{R}^2 \, : \, |x - x(t)| < \lambda(t) \right\}$*,* $\mathcal{M}_{\mathbf{p}}$ *is the set of all geometrical transforms of* $W_{\mathbf{p}}$ *(space translation, scaling, and* $\mathbb{S}^2$ *isometries), and* W *is the ground state.*