Flash Presentation - Romain Duboscq

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Computing stationary states for GPE

Gross-Pitaevskii equation

$$\begin{cases} i\partial_t \psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x) + \beta|\psi|^2\psi(t,x), \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \psi(0,x) = \psi_0(x). \end{cases}$$

Models the evolution of a system of identical bosons using a Hartree-Fock approximation.

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Stationary state GPE

Find an eigenvalue $\mu \in \mathbb{R}$ and eigenfunction $\phi \in \mathbb{S}_{L^2} = \{\varphi \in L^2(\mathbb{R}^d); \|\varphi\|_{L^2} = 1\}$ such that

$$\mu\phi(x) = -\frac{1}{2}\Delta\phi(x) + V(x)\phi(x) + \beta|\phi|^2\phi(x).$$

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- Find efficient and fast numerical methods to compute (μ, ϕ),
- Compute the ground-state (global minimizer) *and* excited-states (other stationary states),
- Develop toolbox for physicists.

Works in collaboration with: X. Antoine, C. Besse, S. le Coz...

Stochastic regularization effects for transport equations

Stochastic Differential equation

$$\begin{cases} dX_t = b(t, X_t) + \sigma dB_t, \quad t \in \mathbb{R}^+, \\ X_0 = x. \end{cases}$$

If $\sigma = 0$, well-posedness holds under Lipschitz conditions on b. If $\sigma \neq 0$, well-posedness holds:

- [Veretennikov, 81'] if b bounded measurable,
- [Krylov & Röckner, 05'] if b in $L^q(\mathbb{R}^+, L^p(\mathbb{R}^d))$.

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Stochastic transport equation

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = \sigma \nabla u(t,x) \dot{B}_t, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 \in L^\infty(\mathbb{R}^d). \end{cases}$$

If $\sigma = 0$, well-posedness holds if $b \in L^1(\mathbb{R}^+, W^{1,1}_{\text{loc}}(\mathbb{R}^d)$, with linear growth, and $\operatorname{div} b \in L^1(\mathbb{R}^+, L^\infty(\mathbb{R}^d))$. If $\sigma \neq 0$, well-posedness holds if:

- [Flandoli, Gubinelli & Priola, '10] if $b \in L^\infty(\mathbb{R}^+, \mathcal{C}^\alpha(\mathbb{R}^d))$, $\alpha > 0$.

White-noise dispersion NLS equation

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$$\begin{cases} i\partial_t \psi(t,x) = -\Delta\psi(t,x)\dot{B}_t + \beta|\psi|^{2\sigma}\psi(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \psi(0,\cdot) = \psi_0 \in L^2(\mathbb{R}^d). \end{cases}$$

In the classical NLS equation, global well-posedness if $\sigma < \frac{2}{d}$.

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In the classical NLS equation, global well-posedness if $\sigma < \frac{2}{d}$. In the white-noise dispersion case, global well-posdness:

- [de Bouard & Debussche, 10'] if $\sigma < \frac{2}{d}$,
- [Debussche & Tsutsumi, 11'] if $d = 1$ and $\sigma = 2$.

Numerical simulations by [Belaouar, de Bouard & Debussche, 15'] indicates that global well-posedness should hold for $\sigma < \frac{4}{d}$ but there's no theoretical proof...