Minimal mass blow up solutions for L^2 critical nonlinear dispersive equations

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CIRM flash presentation

For the L^2 critical nonlinear Schrödinger equation

$$\text{(NLS)} \quad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{N}} u = 0, \\ u(t_0) = u_0 \in H^1(\mathbb{R}^N). \end{cases}$$

Let Q_{NLS} be the unique radial ground state of (NLS) solution to

$$\Delta Q - Q + |Q|^{\frac{4}{N}} Q = 0, \quad Q > 0, \quad Q \in H^1(\mathbb{R}^N).$$

Any solution u of (NLS) which satisfies $\|u(t)\|_{L^2} < \|Q_{\mathrm{NLS}}\|_{L^2}$ is global.

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Let S_{NLS} be the solution of (NLS) defined for all $t > 0$ by

$$S_{\mathrm{NLS}}(t,x) = \frac{1}{t^{\frac{N}{2}}} e^{-i\frac{|x|^2}{4t} - \frac{i}{t}} Q_{\mathrm{NLS}}\left(\frac{x}{t}\right).$$

Theorem (Merle '93)

Up to the symmetries of the equation, S_{NLS} is the unique minimal mass blow up solution of (NLS) in $H^1(\mathbb{R}^N)$.

For an L^2 critical inhomogeneous NLS equation

$$\text{(INLS)} \quad \begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^{\frac{4-2b}{N}} u = 0, \\ u(t_0) = u_0 \in H^1(\mathbb{R}^N). \end{cases}$$

Let ψ be the unique radial ground state of (NLS) solution to

$$\Delta\varphi - \varphi + |x|^{-b}|\varphi|^{\frac{4-2b}{N}}\varphi = 0, \quad \varphi > 0, \quad \varphi \in H^1(\mathbb{R}^N).$$

Any solution u of (INLS) which satisfies $\|u(t)\|_{L^2} < \|\psi\|_{L^2}$ is global in time [Genoud '12].

For an L^2 critical inhomogeneous NLS equation

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Any solution u of (INLS) which satisfies $\|u(t)\|_{L^2} < \|\psi\|_{L^2}$ is global in time [Genoud '12].

Let S_{INLS} be the solution of (INLS) defined for all $t > 0$ by

$$S_{\text{INLS}}(t,x) = \frac{1}{t^{\frac{N}{2}}} e^{-i\frac{|x|^2}{4t} - \frac{i}{t}} \psi\left(\frac{x}{t}\right).$$

Theorem (C.–Genoud '16)

Let $N \geqslant 1$ and $0 < b < \min\{2, N\}$. Up to the symmetries of the equation, S_{INLS} is the unique minimal mass blow up solution of (INLS) in $H^1(\mathbb{R}^N)$.

For the L^2 critical generalized KdV equation

$$\text{(gKdV)} \begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^5) = 0, \\ u(t_0) = u_0 \in H^1(\mathbb{R}). \end{cases}$$

Let $Q \in H^1(\mathbb{R})$ be the unique positive solution to $Q'' - Q + Q^5 = 0$.
Any solution u of (gKdV) which satisfies $\|u(t)\|_{L^2} < \|Q\|_{L^2}$ is global.

For the L^2 critical generalized KdV equation

$$\text{(gKdV)} \quad \begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^5) = 0, \\ u(t_0) = u_0 \in H^1(\mathbb{R}). \end{cases}$$

Let $Q \in H^1(\mathbb{R})$ be the unique positive solution to $Q'' - Q + Q^5 = 0$. Any solution u of (gKdV) which satisfies $\|u(t)\|_{L^2} < \|Q\|_{L^2}$ is global.

Theorem (Martel–Merle–Raphaël '15)

There exist a solution $S \in \mathcal{C}((0,+\infty), H^1)$ to (gKdV) and a universal constant $c_0 \in \mathbb{R}$ such that $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ for all $t > 0$ and

$$S(t) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{\cdot + \frac{1}{t}}{t} + c_0\right) \to 0 \quad \text{in } L^2 \text{ as } t \downarrow 0.$$

Moreover, up to the symmetries of the equation, S is the unique minimal mass blow up solution of (gKdV) in $H^1(\mathbb{R})$.

For the L^2 critical generalized KdV equation (continued)

Theorem (C.–Martel '17)

There exist Schwartz functions $\{Q_k\}_{k\geqslant 0}$ such that, for all $m \geqslant 0$,

$$\partial_x^m S(t) - \sum_{k=0}^{[m/2]} \frac{1}{t^{\frac{1}{2}+m-2k}} Q_k^{(m-k)}\left(\frac{\cdot + \frac{1}{t}}{t} + c_0\right) \to 0 \quad \text{in} \quad L^2 \text{ as } \quad t \downarrow 0.$$

For the L^2 critical generalized KdV equation (continued)

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Remark

For (NLS), from the explicit formula satisfied by S_{NLS}, there exist Schwartz functions $\{\tilde{Q}_k\}_{k\geqslant 0}$ such that, for all $m \geqslant 0$,

$$\partial_x^m S_{\mathrm{NLS}}(t) - e^{-\frac{i}{t}} \sum_{k=0}^{m} \frac{1}{t^{\frac{1}{2}+m-k}} \tilde{Q}_k^{(m)}\left(\frac{\cdot}{t}\right) \to 0 \quad \textit{in} \quad L^2 \textit{ as } \quad t \downarrow 0.$$

For the L^2 critical generalized KdV equation (continued)

Theorem (C.–Martel '17)

For any $m \geqslant 0$, the following hold for all $0 < t \ll 1$.

1. For all $x \leqslant -\frac{1}{t} - 1$,

$$S(t,x) \sim -\frac{1}{2}\|Q\|_{L^1}|x|^{-\frac{3}{2}} \quad \text{and} \quad |\partial_x^m S(t,x)| \lesssim |x|^{-\frac{3}{2}-m}.$$

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2. There exists $\gamma_m > 0$ such that, for all $x \in \mathbb{R}$,

$$|\partial_x^m S(t,x)| \lesssim \frac{1}{t^{\frac{1}{2}+m}} \exp\left(-\gamma_m \frac{x+\frac{1}{t}}{t}\right).$$

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Theorem (C.–Martel '17)

For any $m \geqslant 0$, the following hold for all $0 < t \ll 1$.

1. For all $x \leqslant -\frac{1}{t} - 1$,
$$S(t,x) \sim -\frac{1}{2}\|Q\|_{L^1}|x|^{-\frac{3}{2}} \quad \text{and} \quad |\partial_x^m S(t,x)| \lesssim |x|^{-\frac{3}{2}-m}.$$
2. There exists $\gamma_m > 0$ such that, for all $x \in \mathbb{R}$,
$$|\partial_x^m S(t,x)| \lesssim \frac{1}{t^{\frac{1}{2}+m}} \exp\left(-\gamma_m \frac{x+\frac{1}{t}}{t}\right).$$
3. $S(t) \in L^1(\mathbb{R})$ and
$$\int_{\mathbb{R}} S(t,x)\,dx = 0.$$

Research interests

- L^2 critical and supercritical equations: gKdV, NLS, ...
- Solitons dynamics: stability, instability, blow up, ...
- Multi-solitons dynamics: existence and uniqueness, interactions, ...