Universal dynamics for the logarithmic Schrödinger equation

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Based on a joint work with
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Logarithmic nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

- Physical motivation $\lambda < 0$: nonlinear wave mechanics, optics.
- $\lambda > 0$: interesting mathematical toy.

Formal conservations:

- Mass: $M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}$.
- Energy (Hamiltonian):

$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx.$$

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Weakly nonlinear? Just the opposite

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Lemma

Consider the ODE:

$$\ddot{\tau} = \frac{2\lambda}{\tau}\,, \quad \tau(0) = 1\,, \quad \dot{\tau}(0) = 0\,.$$

It has a unique solution $\tau \in C^2(0,\infty)$, *and, as* $t \to \infty$,

$$\tau(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda \ln t}, \quad \dot{\tau}(t) \underset{t\to\infty}{\sim} 2\sqrt{\lambda \ln t}.$$

Roughly speaking, every solution to logNLS disperses like $\tau^{-d/2}$.

$\rightsquigarrow$ Faster than usual, by a logarithmic factor.

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Define

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \exp\left(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\right)\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}},$$

where $\gamma(y) = e^{-|y|^2/2}$. Then:

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d).$$

Universal dynamics, more similar to the heat equation.

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Positive Sobolev norms

Corollary

Let $u_0 \in H^1 \cap \mathcal{F}(H^1)$, *and* $0 < s \leqslant 1$. *As* $t \to \infty$,

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot{H}^s(\mathbb{R}^d)$ *denotes the standard homogeneous Sobolev space.*

Stability of this regime

Same results with a defocusing, energy-subcritical, power-like perturbation,

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u + \mu |u|^{2\sigma} u\,, \quad u_{|t=0} = u_0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d,$$

with $\lambda > 0$, $\mu > 0$ and $0 < \sigma < \frac{2}{(d-2)_+}$.