Dispersion phenomena on $\mathbb{H}^d$ when the vertical frequency λ tends to 0

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Approach

• Frequency space : $(\widehat{\mathbb{H}}^d, \widehat{d})$

• Classical definition : the Fourier transform of an integrable function f on $\mathbb{H}^d$ is a family $(\mathcal{F}^{\mathbb{H}}(f)(\lambda))_{\lambda \in \mathbb{R} \setminus \{0\}}$ of bounded operators on $L^2(\mathbb{R}^d)$.

• New point of view : amounts to considering the 'infinite matrix' of $\mathcal{F}^{\mathbb{H}} f(\lambda)$ in the orthonormal basis of $L^2(\mathbb{R}^d)$ given by $(H_{n,\lambda})_{n \in \mathbb{N}}$ the rescaled Hermite functions.

• Function point of view : $\widehat{f}_{\mathbb{H}}$ is a map on the set $\widetilde{\mathbb{H}}^d = \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{R} \setminus \{0\}$

$$\widehat{f}_{\mathbb{H}}(n, m, \lambda) = \Big(\mathcal{F}^{\mathbb{H}}(f)(\lambda) H_{m,\lambda} | H_{n,\lambda}\Big)_{L^2}.$$

• Structure of $\widetilde{\mathbb{H}}^d$:

- suitable distance $\widehat{d}$

- $(\widetilde{\mathbb{H}}^d, \widehat{d})$ fails to be complete

• The completion of $\widetilde{\mathbb{H}}^d$: $\widehat{\mathbb{H}}^d = \widetilde{\mathbb{H}}^d \cup \widehat{\mathbb{H}}^d_0$ with $\widehat{\mathbb{H}}^d_0 = ((\mathbb{R}_-)^d \cup (\mathbb{R}_+)^d) \times \mathbb{Z}^d$.

• Key property : $\widehat{f}_{\mathbb{H}}$ is uniformly continuous on the set $(\widetilde{\mathbb{H}}^d, \widehat{d})$.

• Consequence : the above property prompts us to extend $\mathcal{F}_{\mathbb{H}} f$ to $\widehat{\mathbb{H}}^d$ which captures the limit behavior as λ tends to 0.

• Miracle : we get an explicit asymptotic description of the Fourier transform when the vertical frequency tends to 0.

• Applications :

- a handy characterization of $\mathcal{F}_{\mathbb{H}}(\mathcal{S}(\mathbb{H}^d))$ and extension to $\mathcal{S}'(\mathbb{H}^d)$ and in particular to functions independent of the vertical variable

- recover several fundamental results : Gaveau, Hulanicki,...

- dispersion phenomenon for evolution equations on $\mathbb{H}^d$

• Many fields ranging from biology, control theory and physics to complex analysis and PDEs

Basic facts about the Heisenberg group

$$\mathbb{H}^d = T^\star \mathbb{R}^d \times \mathbb{R}\,, w = (Y,s) = (y,\eta,s) \in \mathbb{H}^d$$

The non commutative law of product is

$$w\cdot w' = \big(Y+Y', s+s'+2\sigma(Y,Y')\big) \text{ with } \sigma(Y,Y') = \langle \eta, y'\rangle - \langle \eta', y\rangle$$

The center of $\mathbb{H}^d$:

$$\mathcal{C}(\mathbb{H}^d) = \{0_{T^\star\mathbb{R}^d}\}\times\mathbb{R}\,.$$

The Lie algebra of left invariant vector fields on $\mathbb{H}^d$ (commuting with any left translation : $(\mathcal{X}\cdot f)\circ\tau_h = \mathcal{X}\cdot(f\circ\tau_h)$) is generated by

$$X_j = \partial_{y_j} + 2\eta_j\partial_s\,,\ \Xi_j = \partial_{\eta_j} - 2y_j\partial_s,\ \ j=1,\cdots,d \text{ and } \partial_s = \frac{1}{4}[\Xi_j, X_j].$$

To see X_j and Ξ_j as constant coefficients vector fields.

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{d}\big(X_j^2+\Xi_j^2\big).$$

Several objects are linked to this group : Haar measure, non commutative convolution product with Young's inequalities, dilations, homogeneous distance,...

Fourier transform on the Heisenberg group

The set of characters of $\mathbb{H}^d \sim T^\star \mathbb{R}^d$, so just resorting to characters is not enough since the information pertaining to the vertical variable s is lost.

For $f \in L^1(\mathbb{H}^d)$ and $\lambda \in \mathbb{R}^*$, we define

$$\mathcal{F}^{\mathbb{H}}(f)(\lambda) = \int_{\mathbb{H}^d} f(w) U_w^\lambda dw,$$

where U^λ is the Schrödinger representation defined by

$$U^\lambda \begin{cases} \mathbb{H}^d & \longrightarrow & \mathcal{U}(L^2(\mathbb{R}^d)) \\ w & \longmapsto & U_w^\lambda \end{cases} \quad \text{with} \quad U_w^\lambda(\phi)(x) = e^{-is\lambda - 2i\lambda\langle \eta, x-y\rangle} \phi(x-2y).$$

• Family of bounded operators on $L^2(\mathbb{R}^d)$: $\|\mathcal{F}^{\mathbb{H}}(f)(\lambda)\|_{\mathcal{L}(L^2)} \leq \|f\|_{L^1(\mathbb{H}^d)}$.

• Inversion and Fourier-Plancherel formulas involving the trace and the Hilbert-Schmidt norm of $(\mathcal{F}^{\mathbb{H}}(f)(\lambda))_{\lambda \in \mathbb{R}^*}$

• $\mathcal{F}^{\mathbb{H}}$ exchanges convolution and composition : U^λ is a group homomorphism $U_w^\lambda \circ U_v^\lambda = U_{w \cdot v}^\lambda$.

Fourier transform and Laplacian

In $\mathbb{R}^n$, we have $\mathcal{F}_{\mathbb{R}^d}(-\Delta u)(\xi) = |\xi|^2 \widehat{u}(\xi)$.

In $\mathbb{H}^d$, we have the following result which gives the spectral representation of $\Delta_{\mathbb{H}}$: if f belongs to $\mathcal{S}(\mathbb{H}^d)$, then for any $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned} \mathcal{F}^{\mathbb{H}}(\Delta_{\mathbb{H}} f)(\lambda)(\phi) &= 4\,\mathcal{F}^{\mathbb{H}}(f)(\lambda)(\Delta^{\lambda}_{\mathrm{osc}}\phi) \quad \text{with} \\ \Delta^{\lambda}_{\mathrm{osc}}\phi(x) &= \Delta\phi(x) - \lambda^2|x|^2\phi(x). \end{aligned}$$

This relation follows from the straightforward formulae ($M_j\phi(x) = x_j\phi(x)$)

$$\mathcal{F}^{\mathbb{H}}(X_j f)(\lambda) = 2\mathcal{F}^{\mathbb{H}}(f)(\lambda) \circ \partial_j \quad \text{and} \quad \mathcal{F}^{\mathbb{H}}(\Xi_j f)(\lambda) = 2i\lambda\mathcal{F}^{\mathbb{H}}(f)(\lambda) \circ M_j.$$

The spectral theory of the harmonic oscillator is well known : If $(H_n)_{n\in\mathbb{N}^d}$ denotes the Hermite functions on $\mathbb{R}^d$, we define for $\lambda \in \mathbb{R}\setminus\{0\}$

$$H_{n,\lambda}(x) = |\lambda|^{\frac{d}{4}} H_n(|\lambda|^{\frac{1}{2}}x) \quad \text{which satisfies} \quad -\Delta^{\lambda}_{\mathrm{osc}} H_{n,\lambda} = |\lambda|(2|n|+d)H_{n,\lambda}.$$

This implies that

$$\mathcal{F}^{\mathbb{H}}(-\Delta_{\mathbb{H}} f)(\lambda)(H_{m,\lambda}) = 4\,|\lambda|(2|m|+d)\mathcal{F}^{\mathbb{H}}(f)(\lambda)(H_{m,\lambda}).$$

The Fourier transform as a function

For f in $L^1(\mathbb{H}^d)$, we define the map $\widehat{f}_{\mathbb{H}}(\mathcal{F}_{\mathbb{H}}(f))$:

$$\widehat{f}_{\mathbb{H}} : \begin{cases} \widetilde{\mathbb{H}}^d & \longrightarrow \mathbb{C} \\ \widehat{w} = (n, m, \lambda) & \longmapsto \displaystyle\int_{\mathbb{H}^d} \big(U_w^\lambda H_{m,\lambda} | H_{n,\lambda}\big)_{L^2} f(w)\, dw. \end{cases}$$

But

$$\big(U_w^\lambda H_{m,\lambda} | H_{n,\lambda}\big)_{L^2} = \overline{e^{is\lambda}\mathcal{W}(\widehat{w}, Y)} \quad \text{with}$$

$$\mathcal{W}(\widehat{w}, Y) = \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z\rangle} H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z)\, dz$$

the Wigner transform of the rescaled Hermite functions. Thus

$$\widehat{f}_{\mathbb{H}}(\widehat{w}) = \int_{\mathbb{H}^d} \overline{e^{is\lambda}\mathcal{W}(\widehat{w}, Y)}\, f(Y, s)\, dY\, ds$$

and obviously

$$|\widehat{f}_{\mathbb{H}}(n, m, \lambda)| \leq \|f\|_{L^1(\mathbb{H}^d)}.$$

One of the basic principles of the Fourier transform on $\mathbb{R}^n$ is that regularity implies decay. This remains true in the Heisenberg framework.

Let f in $\mathcal{S}(\mathbb{H}^d)$. For any integer N, there exists $C_N > 0$ such that

$$\Big(1 + |\lambda|(|n| + |m| + d) + |n - m|\Big)^N |\widehat{f}_{\mathbb{H}}(n, m, \lambda)| \leq C_N.$$

- One part of this decay inequality is given by the action of $\Delta_{\mathbb{H}}^N$.

- A second part is obtained by the action of the right-invariant vector fields

$$\widetilde{X}_j = \partial_{y_j} - 2\eta_j \partial_s \quad \text{and} \quad \widetilde{\Xi}_j = \partial_{\eta_j} + 2y_j \partial_s \quad \text{with} \quad j \in \{1, \cdots, d\},$$

which satisfy

$$4|\lambda|(2n_j + 1)\widehat{f}_{\mathbb{H}}(n, m, \lambda) = \mathcal{F}_{\mathbb{H}}\Big(-(\widetilde{X}_j^2 + \widetilde{\Xi}_j^2)f\Big)(n, m, \lambda).$$

- Finally the decay property with respect to $m - n$ stems from the following easy computations

$$-\widetilde{X}_j^2 + X_j^2 - \widetilde{\Xi}_j^2 + \Xi_j^2 = 8\partial_s \mathcal{T}_j \quad \text{with} \quad \mathcal{T}_j = \eta_j \partial_{y_j} - y_j \partial_{\eta_j},$$

which implies that

$$|\lambda|(n_j - m_j)\widehat{f}_{\mathbb{H}}(n, m, \lambda) = i\lambda \mathcal{F}_{\mathbb{H}}\Big(\mathcal{T}_j f\Big)(n, m, \lambda).$$

The metric space $(\widetilde{\mathbb{H}}^d, \widehat{d})$

It is natural to endow $\widetilde{\mathbb{H}}^d$ with the following distance $\widehat{d}$:

$$\widehat{d}(\widehat{w}, \widehat{w}') = |\lambda(n+m) - \lambda'(n'+m')|_1 + |(m-n) - (m'-n')|_1 + |\lambda - \lambda'|,$$

where $|\cdot|_1$ denotes the ℓ^1 norm.

• $(\widetilde{\mathbb{H}}^d, \widehat{d})$ seems to be the natural frequency space within our approach. However, it fails to be complete.

• $\widehat{\mathbb{H}}^d$ is the completion of $\widetilde{\mathbb{H}}^d$:

$$\widehat{\mathbb{H}}^d = \widetilde{\mathbb{H}}^d \cup \widehat{\mathbb{H}}^d_0 \quad \text{with} \quad \widehat{\mathbb{H}}^d_0 = ((\mathbb{R}_-)^d \cup (\mathbb{R}_+)^d) \times \mathbb{Z}^d.$$

• On $\widehat{\mathbb{H}}^d$, the extended distance $\widehat{d}$ is given by

$$\widehat{d}(\widehat{w}, \widehat{w}') = |\lambda(n+m) - \lambda'(n'+m')|_1 + |(m-n) - (m'-n')|_1 + |\lambda - \lambda'| \quad \text{if} \quad \lambda \neq 0 \quad \text{and} \quad \lambda' \neq 0,$$

$$\widehat{d}(\widehat{w}, (\dot{x}, k)) = |\lambda(n+m) - \dot{x}|_1 + |m-n-k|_1 + |\lambda| \quad \text{if} \quad \lambda \neq 0,$$

$$\widehat{d}((\dot{x}, k), (\dot{x}', k')) = |\dot{x} - \dot{x}'|_1 + |k-k'|_1.$$

Fundamental formulae

- The action on convolution rewrites as follows :

$$\mathcal{F}_{\mathbb{H}}(f\star g)(n,m,\lambda) = (\widehat{f}_{\mathbb{H}}\cdot\widehat{g}_{\mathbb{H}})(n,m,\lambda) \quad \text{with}$$

$$(\widehat{f}_{\mathbb{H}}\cdot\widehat{g}_{\mathbb{H}})(n,m,\lambda) = \sum_{\ell\in\mathbb{N}^d} \widehat{f}_{\mathbb{H}}(n,\ell,\lambda)\widehat{g}_{\mathbb{H}}(\ell,m,\lambda).$$

- If the set $\widetilde{\mathbb{H}}^d$ is endowed with the measure $d\widehat{w}$ defined by :

$$\int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{w})\,d\widehat{w} = \sum_{(n,m)\in\mathbb{N}^{2d}} \int_{\mathbb{R}} \theta(n,m,\lambda)|\lambda|^d d\lambda,$$

then the Fourier-Plancherel and inversion formulae recast as follows :

$$\|f\|^2_{L^2(\mathbb{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}}\|\widehat{f}_{\mathbb{H}}\|^2_{L^2(\widetilde{\mathbb{H}}^d)}$$

and

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}}\int_{\widetilde{\mathbb{H}}^d} e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\,\widehat{f}_{\mathbb{H}}(\widehat{w})\,d\widehat{w}\,.$$

Asymptotic behavior of $\widehat{f}_{\mathbb{H}}$ when the vertical frequency λ tends to 0

• The Fourier transform $\widehat{f}_{\mathbb{H}}$ of any integrable function on $\mathbb{H}^d$ is uniformly continuous on $\widetilde{\mathbb{H}}^d$, and thus may be extended continuously to $\widehat{\mathbb{H}}^d$.

• For all $(\dot{x}, k)$ in $\widehat{\mathbb{H}}^d_0$,

$$\mathcal{F}_{\mathbb{H}} f(\dot{x}, k) = \int_{T^\star \mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y) f(Y, s)\, dY\, ds \quad \text{with}$$

$$\mathcal{K}_d(\dot{x}, k, Y) = \bigotimes_{j=1}^{d} \mathcal{K}(\dot{x}_j, k_j, Y_j) \quad \text{and}$$

$$\mathcal{K}(\dot{x}, k, y, \eta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(2|\dot{x}|^{\frac{1}{2}}(y \sin z + \eta\, \mathrm{sgn}(\dot{x}) \cos z) + kz)}\, dz.$$

• $\mathcal{F}_{\mathbb{H}} : L^1(\mathbb{H}^d) \longrightarrow \mathcal{C}_0(\widehat{\mathbb{H}}^d)$.

$\mathcal{F}_{\mathbb{H}}$ of functions independent of the vertical variable

- $\forall g \in L^1(T^\star\mathbb{R}^d)$: $\mathcal{F}_{\mathbb{H}}(g \otimes \mathbf{1}) = 2\pi(\mathcal{G}_{\mathbb{H}} g)\mu_{\widehat{\mathbb{H}}^d_0}$, where (in the sense of measure)

$$(\mathcal{G}_{\mathbb{H}} g)(\dot{x}, k) = \int_{T^\star\mathbb{R}^d} \overline{\mathcal{K}}_d(\dot{x}, k, Y) g(Y) dY \quad \text{and}$$

$$\int_{\widehat{\mathbb{H}}^d_0} \theta(\dot{x}, k)\, d\mu_{\widehat{\mathbb{H}}^d_0} = 2^{-d} \sum_{k \in \mathbb{Z}^d} \Big(\int_{(\mathbb{R}_-)^d} \theta(\dot{x}, k)\, d\dot{x} + \int_{(\mathbb{R}_+)^d} \theta(\dot{x}, k)\, d\dot{x} \Big) \cdot$$

- Fourier-Plancherel and inversion formulae

$$\|g\|^2_{L^2(T^\star\mathbb{R}^d)} = \left(\frac{2}{\pi}\right)^d \|\mathcal{G}_{\mathbb{H}} g\|^2_{L^2(\widehat{\mathbb{H}}^d_0)},$$

$$g(Y) = \left(\frac{2}{\pi}\right)^d \int_{\widehat{\mathbb{H}}^d_0} \mathcal{K}_d(\dot{x}, k, Y) \mathcal{G}_{\mathbb{H}} g(\dot{x}, k)\, d\mu_{\widehat{\mathbb{H}}^d_0}.$$

- Commutative convolution identity

$$\mathcal{G}_{\mathbb{H}}(f \star g)(\dot{x}, k) = \sum_{k' \in \mathbb{Z}^d} (\mathcal{G}_{\mathbb{H}} f)(\dot{x}, k - k')\, (\mathcal{G}_{\mathbb{H}} g)(\dot{x}, k').$$

$\forall f \in L^1(\mathbb{H}^d)$, $\mathcal{F}_{\mathbb{H}} f(\dot{x}, k) = (\mathcal{G}_{\mathbb{H}} g)(\dot{x}, k)$, with g the vertical average of f.

Sketch of proof of the explicit formula

First step : study of the kernel $\mathcal{K}_d$.

- The symmetry identities :

$$\mathcal{K}(\dot{x}, -k, -Y) = \overline{\mathcal{K}(\dot{x}, k, Y)},\ \mathcal{K}(-\dot{x}, -k, Y) = (-1)^k \mathcal{K}(\dot{x}, k, Y) \quad \text{and}$$

$$\mathcal{K}(-\dot{x}, k, Y) = \overline{\mathcal{K}(\dot{x}, k, Y)},\ \mathcal{K}(\dot{x}, k, 0) = \delta_{0,k};$$

- The identities

$$\Delta_Y \mathcal{K}(\dot{x}, k, Y) = -4|\dot{x}|\mathcal{K}(\dot{x}, k, Y);$$

$$ik\mathcal{K}(\dot{x}, k, Y) = (\eta \partial_y \mathcal{K}(\dot{x}, k, Y) - y\partial_\eta \mathcal{K}(\dot{x}, k, Y)) \operatorname{sgn}(\dot{x});$$

- The convolution property

$$\mathcal{K}(\dot{x}, k, Y_1 + Y_2) = \sum_{k' \in \mathbb{Z}} \mathcal{K}(\dot{x}, k - k', Y_1)\mathcal{K}(\dot{x}, k', Y_2);$$

- and finally, the following relation for $\dot{x} > 0$:

$$|Y|^2 \mathcal{K} + \dot{x}\partial_{\dot{x}}^2 \mathcal{K} + \partial_{\dot{x}} \mathcal{K} - \frac{k^2}{4\dot{x}} \mathcal{K} = 0.$$

The last item is given by the study of $\mathcal{F}_{\mathbb{H}}(|Y|^2 f)$:

1) Integrating by parts, we get (where $\widehat{w}_j^{\pm} = (n \pm \delta_j, m \pm \delta_j, \lambda))$

$$|Y|^2 \mathcal{W}(\widehat{w}, Y) = -\widehat{\Delta} \mathcal{W}(\cdot, Y)(\widehat{w}) \quad \text{with}$$

$$\widehat{\Delta}\theta(\widehat{w}) = -\frac{1}{2|\lambda|}(|n+m|+d)\theta(\widehat{w}) + \frac{1}{2|\lambda|}\sum_{j=1}^{d}\Big\{\sqrt{(n_j+1)(m_j+1)}\,\theta(\widehat{w}_j^{+}) + \sqrt{n_j m_j}\,\theta(\widehat{w}_j^{-})\Big\}.$$

2) We have $\lim_{\varepsilon\to 0} \mathcal{B}_\varepsilon(g,\psi) = \mathcal{B}(g,\psi)$, (where $\Theta_\psi(\widehat{w}) = \psi(|\lambda|(n+m+1), m-n))$

$$\mathcal{B}_\varepsilon(g,\psi) = \int_{T^\star\mathbb{R}\times\widehat{\mathbb{H}}^1} |Y|^2 \mathcal{W}(\widehat{w}, Y) g(Y) \Theta_\psi(\widehat{w}) \frac{1}{\varepsilon}\widehat{\chi}\Big(\frac{\lambda}{\varepsilon}\Big)\, dY\, d\widehat{w},$$

$$\mathcal{B}(g,\psi) = \int_{T^\star\mathbb{R}\times\widehat{\mathbb{H}}^1_0} |Y|^2 \mathcal{K}(\dot{x}, k, Y) g(Y) \psi(\dot{x}, k)\, dY\, d\mu_{\widehat{\mathbb{H}}^1_0}.$$

3) Applying Taylor formula, we infer that for $\dot{x} > 0$

$$\widehat{\Delta}\Theta_\psi(\widehat{w}) \sim (L\psi)(\dot{x}, k) \ \text{ with } (L\psi)(\dot{x},k) = \dot{x}\psi''(\dot{x},k) + \psi'(\dot{x},k) - \frac{k^2}{4\dot{x}}\psi(\dot{x},k)\cdot$$

Second step : computation of the kernel.

To compute $\mathcal{K}$, it is wise to introduce the following function $\widetilde{\mathcal{K}}$ on $\mathbb{R}\times\mathbb{T}\times T^\star\mathbb{R}$, where $\mathbb{T}$ denotes the one-dimensional torus :

$$\widetilde{\mathcal{K}}(\dot{x},z,Y)=\sum_{k\in\mathbb{Z}}\mathcal{K}(\dot{x},k,Y)e^{ikz}.$$

According to properties of $\mathcal{K}$, we observe that

- $\widetilde{\mathcal{K}}(\dot{x},z,Y_1+Y_2)=\widetilde{\mathcal{K}}(\dot{x},z,Y_1)\,\widetilde{\mathcal{K}}(\dot{x},z,Y_2)$;
- $\widetilde{\mathcal{K}}(\dot{x},z,-Y)=\overline{\widetilde{\mathcal{K}}(\dot{x},z,Y)}$;
- $\widetilde{\mathcal{K}}(\dot{x},z,0)=1$.

This implies that the function $Y\mapsto\widetilde{\mathcal{K}}(\dot{x},z,Y)$ is a character of $\mathbb{R}^2$. Thus there exists a function $\Phi=(\Phi_y,\Phi_\eta)$ from $\mathbb{R}\times\mathbb{T}$ to $\mathbb{R}^2$ such that

$$\widetilde{\mathcal{K}}(\dot{x},z,Y)=e^{iY\cdot\Phi(\dot{x},z)}=e^{i(y\Phi_y(\dot{x},z)+\eta\Phi_\eta(\dot{x},z))}.$$

• In view of the third property of $\mathcal{K}$, we have for $\dot{x} > 0$

$$\partial_z \widetilde{\mathcal{K}}(\dot{x}, z, Y) = \eta \partial_y \widetilde{\mathcal{K}}(\dot{x}, z, Y) - y \partial_\eta \widetilde{\mathcal{K}}(\dot{x}, z, Y).$$

We deduce that (where $R(z)$ denotes the rotation of angle z)

$$\Phi(\dot{x}, z) = R(z) \widetilde{\Phi}(\dot{x})$$

• The second property of $\mathcal{K}$ ensures that $|\widetilde{\Phi}(\dot{x})| = 2|\dot{x}|^{\frac{1}{2}}$. This implies that

$$\widetilde{\mathcal{K}}(\dot{x}, z, Y) = e^{2i|\dot{x}|^{\frac{1}{2}} Y \cdot (R(z)\phi(\dot{x}))},$$

where ϕ is a function from $\mathbb{R}$ to the unit circle of $\mathbb{R}^2$.

• Translating the fourth property in terms of $\widetilde{\mathcal{K}}$, we discover that ($\dot{x} > 0$)

$$\frac{1}{4\dot{x}} \partial_z^2 \widetilde{\mathcal{K}} + \partial_{\dot{x}}(\dot{x} \partial_{\dot{x}} \widetilde{\mathcal{K}}) + |Y|^2 \widetilde{\mathcal{K}} = 0.$$

This implies that ϕ satisfies : $\dot{x}\phi''(\dot{x}) + 2\phi'(\dot{x}) = 0 \quad$ for $\dot{x} > 0$. As ϕ is valued in the unit circle, we infer that ϕ is a constant.

- Thus

$$\widetilde{\mathcal{K}}(\dot{x},z,Y) = e^{2i|\dot{x}|^{\frac{1}{2}}(y\cos(z+z_0)+\eta\sin(z+z_0))}.$$

- Taking advantage of the symmetry relation

$$\mathcal{K}(\dot{x},-k,y,-\eta) = (-1)^k\mathcal{K}(\dot{x},k,y,\eta),$$

we find that $z_0 \equiv \dfrac{\pi}{2}[\pi]$ and hence there exists $\varepsilon \in \{-1,1\}$ so that $(\dot{x}>0)$

$$\widetilde{\mathcal{K}}(\dot{x},z,Y) = e^{2i\varepsilon\sqrt{\dot{x}}(y\sin z-\eta\cos z)}.$$

- Finally, we find that $\varepsilon=-1$ by comparing the following formulas $(\dot{x}>0)$:

$$\widetilde{\mathcal{K}}(\dot{x},0,(0,\eta)) = \cos(2\sqrt{\dot{x}}\,\eta) - i\varepsilon\sin(2\sqrt{\dot{x}}\,\eta) \quad \text{and}$$

$$\widetilde{\mathcal{K}}(\dot{x},0,(0,\eta)) = \sum_{k\in\mathbb{Z}}\mathcal{K}(\dot{x},k,(0,\eta)) = \sum_{\ell_1\in\mathbb{N}}\sum_{|k|\le\ell_1}\frac{i^{\ell_1}}{\ell_1!}F_{\ell_1,0}(k)\eta^{\ell_1}\dot{x}^{\frac{\ell_1}{2}},$$

where $F_{\ell_1,0}(k) = \displaystyle\sum_{k+\ell_1-2\ell_1'=0}\binom{\ell_1}{\ell_1'}$.

Dispersion phenomena for evolution equations on $\mathbb{H}^d$

Dispersive inequalities play a decisive role in the study of nonlinear evolution equations

- Bahouri-Gérard-Xu Heisenberg group 2000

- Del Hierro H-type groups 2005

- Bahouri-Fermanian-Gallagher step 2 stratified Lie groups 2016

For $\mathbb{H}^d$, these inequalities are strikingly different from the $\mathbb{R}^d$ framework :

- No dispersion phenomenon for the Schrödinger equation

- Dispersive estimates for the wave equation with an optimal rate of decay of order $|t|^{-1/2}$ regardless of the dimension d

Evolution equations involving the sublaplacian $\Delta_{\mathbb{H}}$

$$(S_{\mathbb{H}}) \begin{cases} (i\partial_t - \Delta_{\mathbb{H}}) f = 0 \\ f_{|t=0} = f_0 \in \mathcal{S}(\mathbb{H}^d) \end{cases}$$

$$f(t,\cdot) = \mathrm{e}^{-it\Delta_{\mathbb{H}}} f_0$$

- No dispersion for $(S_{\mathbb{H}})$ in the sense that there is f_0 such that

$$\forall\, q \in [1,\infty]\,, \quad \|f(t,\cdot)\|_{L^q(\mathbb{H}^d)} = \|f_0\|_{L^q(\mathbb{H}^d)}\,.$$

- Actually

$$\Delta_{\mathbb{H}} = 4 \sum_{j=1}^{d} (Z_j \overline{Z}_j + i\partial_s)\,,$$

where $Z_j = \partial_{z_j} + i\overline{z}_j \partial_s$, $\overline{Z}_j = \partial_{\overline{z}_j} - i z_j \partial_s$ and $z_j = y_j + i\eta_j$.

Thus for $f_0 \in \mathrm{Ker}\Big(\sum_{j=1}^{d} Z_j \overline{Z}_j\Big)$, $f(t,\cdot) = \mathrm{e}^{4t\partial_s} f_0$: transport equation

$$f(t,z,s) = f_0(z, s + 4dt)\,.$$

Asymptotic behavior on the set of finishing vertical frequencies

Applying the Fourier transform gives

$$\widehat{f}_{\mathbb{H}}(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(f_0)(n,m,\lambda)\,,$$

and thus on the set of finishing vertical frequencies $\widehat{\mathbb{H}}^d_0$

$$\widehat{f}_{\mathbb{H}}(t,\dot{x},k) = e^{4it|\dot{x}|}\mathcal{F}_{\mathbb{H}}(f_0)(\dot{x},k)\,.$$

Thus applying $(\mathcal{G}_{\mathbb{H}})^{-1}$, we get

$$(\mathcal{G}_{\mathbb{H}})^{-1}(\widehat{f}_{\mathbb{H}}(t,\cdot))(Y) = e^{-it\Delta_Y}g_0(Y)\,,$$

with $g_0(Y) = \int_{\mathbb{R}} f_0(Y,s)ds$ the vertical average of f_0, and thus for $t \neq 0$

$$\|(\mathcal{G}_{\mathbb{H}})^{-1}(\widehat{f}_{\mathbb{H}}(t,\cdot))\|_{L^\infty_Y} \lesssim \frac{1}{|t|^d}\|g_0\|_{L^1_Y}\,.$$

Note that in the case when f_0 belongs to Ker $\Big(\sum_{j=1}^{d} Z_j\overline{Z}_j\Big)$, then $g_0 = 0$.