Localization at ω-compact types, as sequential colimits.

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Localization is a process

of adding inverses

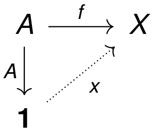
to an algebraic structure

in a universal way.

A type X is said to be ***A*-local** if any map $A \to X$ has a unique extension to a map $\mathbf{1} \to X$.
More precisely, X is A-local if the map

$$\lambda(x : \mathbf{1} \to X).\, x \circ A : (\mathbf{1} \to X) \to (A \to X)$$

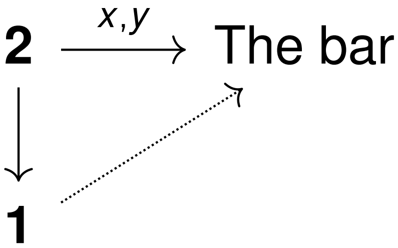
is an equivalence.



Example:

- The unit type $\mathbf{1}$ is A-local for any A.
- The mere propositions are $\mathbf{2}$-local.

Two terms walk into a **2**-local bar...



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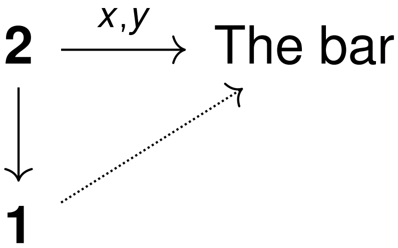

and they order one beer..

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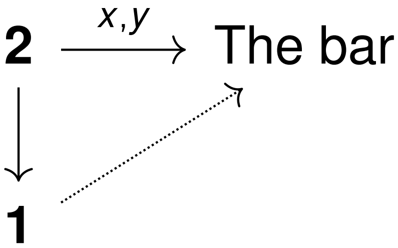

and they order one beer..

...Chuck Norris drinks it.

A **reflective subuniverse** consists of a subuniverse $P : \mathrm{U} \to \mathrm{Prop}$ of ***P*-types** such that for each type X

- there is a type $\bigcirc(X)$ for which $P(\bigcirc(X))$ holds,
- there is a map $\eta_X : X \to \bigcirc(X)$

such that the map

$$\lambda f.\, f \circ \eta_X : (\bigcirc(X) \to Y) \to (X \to Y)$$

is an equivalence for each P-type Y.

The **sequential colimit**, denoted by either $\mathrm{colim}_n(A_n)$ or A_∞, of a type sequence has constructors

$$
\begin{aligned}
i &: \prod_{(n:\mathbb{N})} A_n \to A_\infty \\
j &: \prod_{(n:\mathbb{N})} \prod_{(x:A_n)} i_n(x) = i_{n+1}(a_n(x)).
\end{aligned}
$$

with the expected universal property.

We say that X is **ω-compact** if the canonical function of type

$$\mathrm{colim}_n(X \to A_n) \to (X \to A_\infty)$$

is an equivalence for *every* type sequence $(A_n, a_n)_{n:\mathbb{N}}$.

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Examples:

- Finite types are ω-compact. $\mathbb{N}$ is not.
- If B is ω-compact, and $E(x, y)$ is ω-compact for each $x, y : B$, then the higher inductive type X with constructors

$$b : B \to X$$
$$e : \prod_{(x,y:B)} E(x, y) \to (b(x) = b(y))$$

is ω-compact.

The circle is ω-compact

The **circle** is a higher inductive type $\mathbb{S}^1$ with

$$\text{base} : \mathbb{S}^1$$
$$\text{loop} : \text{base} =_{\mathbb{S}^1} \text{base}$$

recursion principle of the circle: for any type X, a function of type $\mathbb{S}^1 \to X$ is determined by

$$x : X$$
$$p : x =_X x.$$

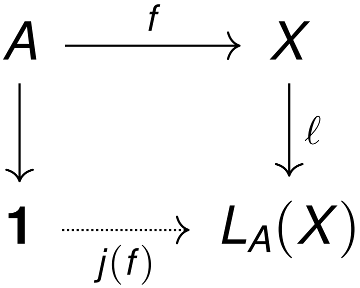
Each n-sphere is ω-compact

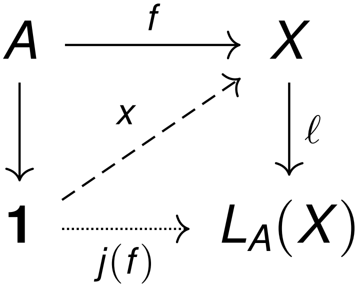
The **$(n+1)$-sphere** is a higher inductive type $\mathbb{S}^{n+1}$ with

$$\begin{aligned} \mathsf{N} &: \mathbb{S}^{n+1} \\ \mathsf{S} &: \mathbb{S}^{n+1} \\ \mathsf{equator} &: \mathbb{S}^{n} \to (\mathsf{N} =_{\mathbb{S}^{n+1}} \mathsf{S}). \end{aligned}$$

recursion principle of the $(n+1)$-sphere: for any type X, a function of type $\mathbb{S}^{n+1} \to X$ is determined by

$$\begin{aligned} x &: X \\ y &: X \\ e &: \mathbb{S}^{n} \to x =_X y \end{aligned}$$


A
f
X
ℓ
1
j(f)
L_A(X)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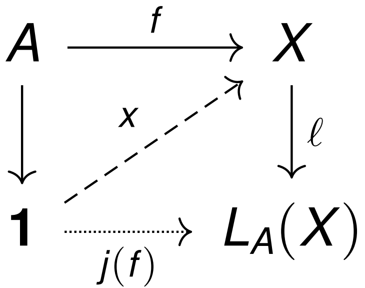


$$\ell : X \to L_A(X)$$

$$j : (A \to X) \to L_A(X)$$

$$J : \prod_{(f:A\to X)} \prod_{(a:A)} j(f) = \ell(f(a))$$

$$k : \prod_{(f:A\to X)} \prod_{(x:X)} \prod_{(H:\prod_{(a:A)}\ x=f(a))} \ell(x) = j(f)$$

$$K : \prod_{(f:A\to X)} \prod_{(x:X)} \prod_{(H:\prod_{(a:A)}\ x=f(a))} k(f,x,H)^{-1} \cdot \ell(H(a)) = J(f,a).$$

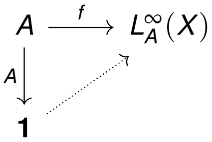
Outline of the formalization project with Floris

We shall give a formal proof (in LEAN) that for any type X and any ω-compact type A, the sequential colimit $L_A^\infty(X)$ is the localization of X at A, i.e. that $L_A^\infty(X)$ is the A-local reflection of X.

Outline of the formalization project with Floris

We shall give a formal proof (in LEAN) that for any type X and any ω-compact type A, the sequential colimit $L_A^\infty(X)$ is the localization of X at A, i.e. that $L_A^\infty(X)$ is the A-local reflection of X.

- In particular, we need to show that $L_A^\infty(X)$ is A-local, i.e. that maps of type $A \to L_A^\infty(X)$ extend uniquely to maps of type $\mathbf{1} \to L_A^\infty(X)$.



- The ω-compactness comes up when one starts manipulating the type $A \to L_A^\infty(X)$.

Fact: Let (P_n, p_n) be a sequence in which each P_n is a mere proposition. Then we have the equivalence

$$\mathrm{colim}(P_n) \simeq \exists_{(n:\mathbb{N})} P_n.$$

Theorem (Independence of premises)

If every mere proposition is ω-compact, then for any type X, and any sequence (P_n, p_n) of mere propositions, one has

$$(X \to \exists_{(n:\mathbb{N})} P_n) \leftrightarrow (\exists_{(n:\mathbb{N})} X \to P_n)$$

Question: The condition that all mere propositions are ω-compact seems to be non-constructive. Is it? How does it relate to other non-constructive principles?