Bishop's Stone-Weierstrass theorem for compact metric spaces revisited

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SWchts

Theorem

Suppose that $(X, \mathcal{T})$ is a compact Hausdorff topological space and $\Phi \subseteq C(X)$ satisfying:

(i) Φ is a subalgebra of $C(X)$.

(ii) Φ separates the points of X: $\forall_{x,y \in X}(x \neq y \rightarrow \exists_{f \in \Phi}(f(x) \neq f(y)))$.

(iii) $\mathrm{Const}(X) \subseteq \Phi$.

Then the uniform closure of Φ is $C(X)$.

Constructive versions of **SWchts**

Banaschewski and **Mulvey** (1997) considered a compact, completely regular locale instead of a compact Hausdorff topological space.

Coquand (2001) gave a simple, constructive localic proof of it, replacing the ring structure of $C(X)$ by its lattice structure.

Coquand (2005) studied the usual formulation of **SWchts** in this point-free topological framework.

Bishop (1967) formulated **BSWcms** a theorem of Stone-Weierstrass type for compact metric spaces (complete and totally bounded metric spaces) using the notion of a Bishop-separating set of uniformly continuous functions.

His result holds for **totally bounded** metric spaces (for every $\epsilon > 0$ there exists a finite ϵ-approximation of X).

Total boundedness is maybe more fundamental than compactness.

Throughout this talk:

(X, d) is a totally bounded metric space,

$C_u(X)$ denotes the uniformly continuous real-valued functions on X,

$\Phi \subseteq C_u(X)$.

Definition

Φ is called **Bishop-separating**, if there is

$$\delta : \mathbb{R}^+ \to \mathbb{R}^+$$

such that:

(Bsep_1) For all $\epsilon > 0$ and $x_0, y_0 \in X$, if $d(x_0, y_0) \geq \epsilon$, there exists $g_{\epsilon,x_0,y_0} \in \Phi$ such that

$$\forall_{z \in X}(d_{x_0}(z) \leq \delta(\epsilon) \to |g_{\epsilon,x_0,y_0}(z)| \leq \epsilon) \text{ and}$$
$$\forall_{z \in X}(d_{y_0}(z) \leq \delta(\epsilon) \to |g_{\epsilon,x_0,y_0}(z) - 1| \leq \epsilon).$$

(Bsep_2) For all $\epsilon > 0$ and $x_0 \in X$ there exists $g_{\epsilon,x_0} \in \Phi$ such that

$$\forall_{z \in X}(d_{x_0}(z) \leq \delta(\epsilon) \to |g_{\epsilon,x_0}(z) - 1| \leq \epsilon).$$

For every $x_0 \in X$ the map

$$d_{x_0} : X \to \mathbb{R},$$
$$x \mapsto d(x_0, x)$$

is in $C_u(X)$ with $\omega_{d_{x_0}} = \mathrm{id}_{\mathbb{R}^+}$.

$$U_0(X) := \{d_{x_0} \mid x_0 \in X\}.$$
$$U_0^*(X) := U_0(X) \cup \{\overline{1}\}.$$

We call Φ **positively separating**, if

$$\forall_{x,y \in X}(x \bowtie_d y \to \exists_{g \in \Phi}(g(x) \bowtie_{\mathbb{R}} g(y))),$$

where

$$x \bowtie_d y \leftrightarrow d(x, y) > 0,$$

for every $x, y \in X$, and

$$a \bowtie_{\mathbb{R}} b \leftrightarrow |a - b| > 0 \leftrightarrow a < b \vee b < a,$$

for every $a, b \in \mathbb{R}$. Clearly, $U_0(X)$ is positively separating.

Remark

If Φ is Bishop-separating, then Φ is positively separating.

BSWtbms

$\mathcal{A}(\Phi)$ is the least subset of $C_u(X)$ that includes Φ and it is closed with respect to addition, multiplication, and multiplication by reals.

Bishop didn't define $\mathcal{A}(\Phi)$ inductively but explicitly, as the set of compositions of strict real polynomials in several variables with vectors of elements of Φ.

Theorem (Bishop's Stone-Weierstrass theorem for totally bounded metric spaces (BSWtbms))

If Φ is Bishop-separating, then $\mathcal{A}(\Phi)$ is dense in $C_u(X)$.

The condition of Φ being Bishop-separating implies that the constant map $\overline{1}$ is in the closure of $\mathcal{A}(\Phi)$.

Its proof is non-trivial!

Bishop's formulation of **BSWtbms** represents a non-trivial technical achievement, namely to find a formulation of a theorem of Stone-Weierstrass type in the constructive theory of metric spaces that **resembles** the formulation of the classical **SWchts**.

Coquand and **Spitters** 2009: constructive proofs using a concrete presentation of topological notions (e.g., the Gelfand spectrum as a lattice) are "more direct than proofs via an encoding of topology in metric spaces, as is common in Bishop's constructive mathematics".

We present a Stone-Weierstrass theorem for metric spaces which

(i) avoids the concept of a Bishop-separating set,

(ii) it has an informative and direct proof,

(iii) it implies **BSWtbms**,

(iv) it proves directly all corollaries of **BSWtbms**.

Definition

If $f, g \in C_u(X)$ and $\epsilon > 0$, then

$$f \wedge g := \min\{f, g\}, f \vee g := \max\{f, g\},$$

and the **uniform closure** $\mathcal{U}(\Phi)$ of Φ is defined by

$$U(g, f, \epsilon) :\leftrightarrow \forall_{x \in X}(|g(x) - f(x)| \leq \epsilon),$$

$$U(\Phi, f) :\leftrightarrow \forall_{\epsilon > 0}\exists_{g \in \Phi}(U(g, f, \epsilon)),$$

$$\mathcal{U}(\Phi) := \{f \in C_u(X) \mid U(\Phi, f)\}.$$

Remark

If Φ is closed under addition, multiplication by reals and multiplication, then $\mathcal{U}(\Phi)$ is closed under addition, multiplication by reals and multiplication.

If Φ is closed under $|.|$, then $\mathcal{U}(\Phi)$ is closed under $|.|$.

Lemma

If $\mathrm{Const}(X) \subseteq \Phi$*, and* Φ *is closed under addition and multiplication, then* $\mathcal{U}(\Phi)$ *is closed under* $|.|, \vee$ *and* $\wedge$*.*

Lemma

If Φ *is closed under addition, multiplication by reals, and multiplication, and* $f \in \mathcal{U}(\Phi)$ *such that for some* $c > 0$

$$\forall_{x \in X}(|f(x)| \geq c),$$

then

$$\frac{1}{f} \in \mathcal{U}(\Phi).$$

Corollary

If $x_0, y_0 \in X$ *such that* $d(x_0, y_0) > 0$*, then* $\overline{1} \in \mathcal{U}(\mathcal{A}(U_0(X)))$*.*

Proof.

If $x \in X$, then $0 < d(x_0, y_0) \leq d(x_0, x) + d(x, y_0) = d_{x_0}(x) + d_{y_0}(x)$ i.e., $d(x_0, y_0) \leq d_{x_0} + d_{y_0} \in \mathcal{A}(U_0(X))$. By the second lemma $\frac{1}{d_{x_0} + d_{y_0}} \in \mathcal{U}(\mathcal{A}(U_0(X)))$, therefore $\overline{1} \in \mathcal{U}(\mathcal{A}(U_0(X)))$. $\square$

Definition

If $\mathbb{F}(X)$ denotes the set of real-valued functions on X, the set of **Lipschitz functions** $\mathrm{Lip}(X)$ on (X, d) is defined by

$$\mathrm{Lip}(X, k) := \{f \in \mathbb{F}(X) \mid \forall_{x,y \in X}(|f(x) - f(y)| \leq kd(x, y))\},$$

$$\mathrm{Lip}(X) := \bigcup_{k \geq 0} \mathrm{Lip}(X, k).$$

Remark

The set $\mathrm{Lip}(X) \subseteq C_u(X)$ *includes* $U_0(X)$, $\mathrm{Const}(X)$ *and it is closed under addition, multiplication by reals, and multiplication.*

Proof.

If $x_0 \in X$, then $|d(x_0, x) - d(x_0, y)| \leq d(x, y)$, for every $x, y \in X$, therefore $U_0(X) \subseteq \mathrm{Lip}(X, 1)$. Clearly, $\mathrm{Const}(X) \subseteq \mathrm{Lip}(X, k)$, for every $k \geq 0$. Recall that $f \cdot g = \frac{1}{2}((f + g)^2 - f^2 - g^2)$, and if $M_f > 0$ is a bound of f, it is immediate to see that

$$f \in \mathrm{Lip}(X, k_1) \to g \in \mathrm{Lip}(X, k_2) \to f + g \in \mathrm{Lip}(X, k_1 + k_2),$$

$$f \in \mathrm{Lip}(X, k) \to \lambda \in \mathbb{R} \to \lambda f \in \mathrm{Lip}(X, |\lambda| k),$$

$$f \in \mathrm{Lip}(X, k) \to f^2 \in \mathrm{Lip}(X, 2M_f k).$$

□

Lemma

If $\Phi = \mathcal{A}(U_0^*(X))$, *then* $\mathrm{Lip}(X) \subseteq \mathcal{U}(\Phi)$.

Proof: It suffices to show that $\mathrm{Lip}(X, 1) \subseteq \mathcal{U}(\Phi)$, since if $f \in \mathrm{Lip}(X, k)$, for some $k > 0$, then $\frac{1}{k} f \in \mathrm{Lip}(X, 1)$ and we have, for every $\epsilon > 0$ and $\theta \in \Phi$,

$$U(\theta, \frac{1}{k} f, \frac{\epsilon}{k}) \to U(k\theta, f, \epsilon).$$

Suppose next that $f \in \mathrm{Lip}(X, 1)$ and $\epsilon > 0$. We find $g \in \mathcal{U}(\Phi)$ such that $U(g, f, \epsilon)$, therefore $f \in \mathcal{U}(\mathcal{U}(\Phi)) = \mathcal{U}(\Phi)$.
If $\{z_1, \ldots, z_m\}$ is an $\frac{\epsilon}{2}$-approximation of X, we find $g \in \mathcal{U}(\Phi)$ such that

$$g(z_i) = f(z_i), \text{ for every } i \in \{1, \ldots, m\},$$

$$|g(x) - g(z_i)| = |g(x) - f(z_i)| \leq \frac{\epsilon}{2},$$

for every $x \in X$ and z_i such that $d(x, z_i) \leq \frac{\epsilon}{2}$. Consequently,

$$\begin{aligned} |g(x) - f(x)| &\leq |g(x) - g(z_i)| + |g(z_i) - f(z_i)| + |f(z_i) - f(x)| \\ &\leq \frac{\epsilon}{2} + 0 + d(z_i, x) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

$$g := \bigwedge_{k=1}^{m} (\overline{f(z_k)} + d_{z_k}).$$

Since $\overline{f(z_k)} + d_{z_k} \in \Phi$ and since by the lemma $\mathcal{U}(\Phi)$ is closed under $\wedge$ we get $g \in \mathcal{U}(\Phi)$. Moreover,

$$g(z_i) = \bigwedge_{k=1}^{m} (f(z_k) + d_{z_k}(z_i)) \leq f(z_i) + d_{z_i}(z_i) = f(z_i).$$

For the converse inequality we suppose that $g(z_i) < f(z_i)$ and reach a contradiction (here we use $\neg(a < b) \to a \geq b$, for every $a, b \in \mathbb{R}$).
If $a, b, c \in \mathbb{R}$, then $a \wedge b < c \to a < c \vee b < c$.

$$\bigwedge_{k=1}^{m} (f(z_k) + d_{z_k}(z_i)) < f(z_i) \to \exists_{j \in \{1,\ldots,m\}} (f(z_j) + d(z_j, z_i) < f(z_i))$$

$$\to d(z_j, z_i) < f(z_i) - f(z_j) \leq |f(z_i) - f(z_j)| \leq d(z_j, z_i),$$

which is a contradiction. Using the equality $g(z_i) = f(z_i)$ we have that

$$g(x) = \bigwedge_{k=1}^{m} (f(z_k) + d_{z_k}(x)) \leq f(z_i) + d_{z_i}(x) \to$$

$$g(x) - g(z_i) \leq d_{z_i}(x) = d(x, z_i) \leq \frac{\epsilon}{2}.$$

If $k \in \{1, \ldots, m\}$, then

$$f(z_i) - f(z_k) \leq |f(z_i) - f(z_k)| \leq d(z_i, z_k) \leq d(z_i, x) + d(x, z_k),$$

therefore

$$\forall_{k \in \{1,\ldots,m\}}(f(z_i) - d(z_i, x) \leq f(z_k) + d(z_k, x)) \rightarrow$$
$$f(z_i) - d(z_i, x) \leq \bigwedge_{k=1}^{m}(f(z_k) + d(z_k, x)) \leftrightarrow$$
$$f(z_i) - \bigwedge_{k=1}^{m}(f(z_k) + d(z_k, x)) \leq d(z_i, x) \rightarrow$$
$$g(z_i) - g(x) \leq d(z_i, x) \rightarrow$$
$$g(z_i) - g(x) \leq \frac{\epsilon}{2}.$$

From $g(x) - g(z_i) \leq \frac{\epsilon}{2}$ and $g(z_i) - g(x) \leq \frac{\epsilon}{2}$ we get

$$|g(x) - g(z_i)| \leq \frac{\epsilon}{2}.$$

Lemma

If $f \in C_u(X)$ *and* $\epsilon > 0$*, there exist* $\sigma > 0$ *and* $g, g^* \in \mathrm{Lip}(X, \sigma)$ *such that*

(i) $\forall_{x \in X}(f(x) - \epsilon \leq g(x) \leq f(x) \leq g^*(x) \leq f(x) + \epsilon)$.
(ii) *For every* $e \in \mathrm{Lip}(X, \sigma)$*, if* $e \leq f$*, then* $e \leq g$.
(iii) *For every* $e^* \in \mathrm{Lip}(X, \sigma)$*, if* $f \leq e^*$*, then* $g^* \leq e^*$.

Proof: (i) Let ω_f be a modulus of continuity of f and $M_f > 0$ a bound of f. We define the functions

$$h_x : X \to \mathbb{R}$$

$$g : X \to \mathbb{R}$$

$$h_x := f + \sigma d_x,$$

$$\sigma := \frac{2M_f}{\omega_f(\epsilon)} > 0,$$

$$g(x) := \inf\{h_x(y) \mid y \in X\} = \inf\{f(y) + \sigma d(x, y) \mid y \in X\},$$

for every $x \in X$. Note that $g(x)$ is well-defined, since $h_x \in C_u(X)$ and the infimum of h_x exists.

First we show that $g \in \mathrm{Lip}(X, \sigma)$. If $x_1, x_2, y \in X$ then

$$\begin{aligned}
& d(x_1, y) \leq d(x_2, y) + d(x_1, x_2) \rightarrow \\
& f(y) + \sigma d(x_1, y) \leq (f(y) + \sigma d(x_2, y)) + \sigma d(x_1, x_2) \rightarrow \\
& g(x_1) \leq (f(y) + \sigma d(x_2, y)) + \sigma d(x_1, x_2) \rightarrow \\
& g(x_1) \leq g(x_2) + \sigma d(x_1, x_2) \rightarrow \\
& g(x_1) - g(x_2) \leq \sigma d(x_1, x_2).
\end{aligned}$$

Starting with the inequality $d(x_2, y) \leq d(x_1, y) + d(x_1, x_2)$ and working similarly we get that

$$g(x_2) - g(x_1) \leq \sigma d(x_1, x_2),$$

therefore

$$|g(x_1) - g(x_2)| \leq \sigma d(x_1, x_2).$$

Next we show that

$$\forall_{x \in X}(f(x) - \epsilon \leq g(x) \leq f(x)).$$

Since

$$f(x) = f(x) + \sigma d(x, x) = h_x(x) \geq \inf\{h_x(y) \mid y \in X\} = g(x),$$

for every $x \in X$, we have that $g \leq f$. Next we show that

$$\forall_{x \in X}(f(x) - \epsilon \leq g(x)).$$

For that we fix $x \in X$ and we show that $\neg(f(x) - \epsilon > g(x))$. Note that if $A \subseteq \mathbb{R}, b \in \mathbb{R}$, then $b > \inf A \to \exists_{a \in A}(a < b)$. Therefore,

$$\begin{aligned}
&f(x) - \epsilon > g(x) \leftrightarrow \\
&f(x) - \epsilon > \inf\{f(y) + \sigma d(x, y) \mid y \in X\} \to \\
&\exists_{y \in X}(f(x) - \epsilon > f(y) + \sigma d(x, y)) \leftrightarrow \\
&\exists_{y \in X}(f(x) - f(y) > \epsilon + \sigma d(x, y)).
\end{aligned}$$

For this y we show that $d(x, y) \leq \omega_f(\epsilon)$. If $d(x, y) > \omega_f(\epsilon)$, we have that

$$2M_f \geq f(x) + M_f \geq f(x) - f(y) > \epsilon + 2M_f \frac{d(x, y)}{\omega_f(\epsilon)} > \epsilon + 2M_f > 2M_f,$$

which is a contradiction. Hence, by the uniform continuity of f we get that $|f(x) - f(y)| \leq \epsilon$, therefore the contradiction $\epsilon > \epsilon$ is reached, since

$$\epsilon \geq |f(x) - f(y)| \geq f(x) - f(y) > \epsilon + \sigma d(x, y) \geq \epsilon.$$

Next we define the functions

$$h_x^* : X \to \mathbb{R}$$

$$g^* : X \to \mathbb{R}$$

$$h_x^* := f - \sigma d_x,$$

$$\sigma = \frac{2M_f}{\omega_f(\epsilon)},$$

$$g^*(x) := \sup\{h_x^*(y) \mid y \in X\} = \sup\{f(y) - \sigma d(x, y) \mid y \in X\},$$

for every $x \in X$. Similarly to g we get that $g^* \in \mathrm{Lip}(X, \sigma)$ and

$$\forall_{x \in X}(f(x) \leq g^*(x) \leq f(x) + \epsilon).$$

(ii) Let $e \in \mathrm{Lip}(X, \sigma)$ such that $e \leq f$. If we fix some $x \in X$, then for every $y \in X$ we have that

$$e(x) - e(y) \leq |e(x) - e(y)| \leq \sigma d(x, y),$$

hence

$$e(x) \leq e(y) + \sigma d(x, y) \leq f(y) + \sigma d(x, y),$$

therefore $e(x) \leq g(x)$.

(iii) Let $e^* \in \mathrm{Lip}(X, \sigma)$ such that $f \leq e^*$. If we fix some $x \in X$, then for every $y \in X$ we have that

$$e^*(y) - e^*(x) \leq |e^*(y) - e^*(x)| \leq \sigma d(x, y),$$

hence

$$f(y) - \sigma d(x, y) \leq e^*(y) - \sigma d(x, y) \leq e^*(x),$$

therefore $g^*(x) \leq e^*(x)$.

Hence g is the largest function in $\mathrm{Lip}(X, \sigma)$ which is smaller than f, and g^* is the smallest function in $\mathrm{Lip}(X, \sigma)$ which is larger than f.

This is in complete analogy to the McShane-Kirszbraun theorem. $A \subseteq X$ is **located**, if the distance

$$d(x, A) := \inf\{d(x, y) \mid y \in Y\}$$

exists for every $x \in X$, and a located subset of a totally bounded metric space is totally bounded.

Proposition (McShane-Kirszbraun theorem for totally bounded metric spaces)

If $\sigma > 0$, $A \subseteq X$ is located, and $f : A \to \mathbb{R} \in \mathrm{Lip}(A, \sigma)$, then there exist $g, g^* \in \mathrm{Lip}(X, \sigma)$ such that $g_{|A} = g^*_{|A} = f$ and for every $e \in \mathrm{Lip}(X, \sigma)$ such that $e_{|A} = f$,

$$g^* \leq e \leq g.$$

Proof.

The functions g, g^* defined by

$$g(x) := \inf\{f(a) + \sigma d(x, a) \mid a \in A\},$$

$$g^*(x) := \sup\{f(a) - \sigma d(x, a) \mid a \in A\},$$

for every $x \in X$, are well-defined and satisfy the required properties. □

Corollary

$\mathcal{U}(\mathrm{Lip}(X)) = C_u(X)$.

Proof.

If $\epsilon > 0$, then the functions $g, g^* \in \mathrm{Lip}(X, \sigma)$ of the lemma satisfy $U(g, f, \epsilon)$, $U(g^*, f, \epsilon)$, respectively.

□

Theorem (Stone-Weierstrass theorem for totally bounded metric spaces (**SWtbms**))

If $\Phi = \mathcal{A}(U_0^*(X))$*, then* $C_u(X) = \mathcal{U}(\Phi)$.

Proof.

We show that $C_u(X) \subseteq \mathcal{U}(\Phi)$. If $f \in C_u(X)$ and $\epsilon > 0$, then by the Corollary there is $h \in \mathrm{Lip}(X)$:

$$U(h, f, \frac{\epsilon}{2}),$$

while by our lemma there exists $g \in \Phi$ such that

$$U(g, h, \frac{\epsilon}{2}),$$

hence $U(g, f, \epsilon)$.

Proposition

SWtbms *implies* **BSWtbms**

Proof.

The proof follows immediately by inspection of Bishop's proof of **BSWtbms**.

Bishop shows there that if Φ is Bishop-separating, then $\overline{1} \in \mathcal{U}(\mathcal{A}(\Phi))$,

and by his lemma 5.14.1 one shows that

$$U_0(X) \subseteq \mathcal{U}(\mathcal{A}(\Phi)),$$

this is is a slight simplification of the final part of Bishop's proof that $C_u(X) \subseteq \mathcal{U}(\mathcal{A}(\Phi))$. Since

$$U_0^*(X) \subseteq \mathcal{U}(\mathcal{A}(\Phi)),$$

then

$$\mathcal{A}(U_0^*(X)) \subseteq \mathcal{U}(\mathcal{A}(\Phi)),$$

therefore

$$C_u(X) = \mathcal{U}(\mathcal{A}(U_0^*(X))) \subseteq \mathcal{U}(\mathcal{U}(\mathcal{A}(\Phi))) = \mathcal{U}(\mathcal{A}(\Phi)).$$

□

Bishop's Corollary 5.16: if (X, d) has positive diameter, then $\mathcal{A}(U_0(X))$ is a Bishop-separating set, therefore by **BSWtbms** we get that

$$\mathcal{U}(\mathcal{A}(U_0(X))) = C_u(X).$$

Hence **SWtbms is only "slightly" stronger than BSWtbms**. If we use **SWtbms**, we get immediately the same result.

Corollary

If (X, d) has positive diameter, then $\mathcal{U}(\mathcal{A}(U_0(X))) = C_u(X)$.

Proof.

The hypothesis of positive diameter implies the existence of $x_1, x_2 \in X$ such that $d(x_1, x_2) > 0$, therefore

$$\overline{1} \in \mathcal{U}(\mathcal{A}(U_0(X))) \subseteq C_u(X),$$

$$U_0^*(X) \subseteq \mathcal{U}(\mathcal{A}(U_0(X))),$$

$$\mathcal{A}(U_0^*(X)) \subseteq \mathcal{A}(\mathcal{U}(\mathcal{A}(U_0(X)))) = \mathcal{U}(\mathcal{A}(U_0(X))),$$

$$C_u(X) = \mathcal{U}(\mathcal{A}(U_0^*(X))) \subseteq \mathcal{U}(\mathcal{U}(\mathcal{A}(U_0(X)))) = \mathcal{U}(\mathcal{A}(U_0(X))).$$

□

If $(X, d), (Y, \rho)$ are totally bounded, then $(X \times Y, \sigma)$ is totally bounded, where

$$\sigma((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2).$$

If $A = \{x_1, \ldots, x_n\}$ is an $\frac{\epsilon}{2}$-approximation of X and $B = \{y_1, \ldots, y_m\}$ is an $\frac{\epsilon}{2}$-approximation of Y, then $A \times B$ is an ϵ-approximation of $X \times Y$.
We denote by π_1 the projection of $X \times Y$ onto X and by π_2 its projection onto Y.

Corollary

If $(X, d), (Y, \rho)$ are totally bounded metric spaces and

$$\Phi := \{\sum_{i=1}^{n}(f_i \circ \pi_1)(g_i \circ \pi_2) \mid f_i \in C_u(X), g_i \in C_u(Y), 1 \leq i \leq n, n \in \mathbb{N}\},$$

then

$$\mathcal{U}(\Phi) = C_u(X \times Y).$$

Proof: Clearly, $\Phi \subseteq C_u(X \times Y)$, Φ is an algebra, actually,

$$\Phi = \mathcal{A}((C_u(X) \circ \pi_1) \cup (C_u(Y) \circ \pi_2)).$$

and

$$\mathcal{U}(\Phi) \subseteq C_u(X \times Y).$$

The constant $\overline{1}$ on $X \times Y$ is equal to $(\overline{1} \circ \pi_1)(\overline{1} \circ \pi_2)$.
If $x_0, x \in X$ and $y_0, y \in Y$, then

$$\sigma_{(x_0,y_0)}((x,y)) = \sigma((x_0,y_0),(x,y)) = d(x_0,x) + \rho(y_0,y) =$$

$$d_{x_0}(x) + \rho_{y_0}(y) = (d_{x_0} \circ \pi_1)((x,y)) + (\rho_{y_0} \circ \pi_2)((x,y)),$$

therefore

$$\sigma_{(x_0,y_0)} = (d_{x_0} \circ \pi_1) + (\rho_{y_0} \circ \pi_2) =$$

$$(d_{x_0} \circ \pi_1)(\overline{1} \circ \pi_2) + (\overline{1} \circ \pi_1)(\rho_{y_0} \circ \pi_2) \in \Phi.$$

Since $U_0^*(X \times Y) \subseteq \mathcal{U}(\Phi)$, by **SWtbms** we get that

$$C_u(X \times Y) \subseteq \mathcal{U}(\Phi).$$

If (X_n, d_n) is totally bounded, where w.l.g.$d_n \leq \overline{1}$, for every $n \in \mathbb{N}$, then (X, σ_∞), where

$$X = \prod_{n=1}^{\infty} X_n,$$

$$\sigma_\infty((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n},$$

is totally bounded; if $A(X_n, \epsilon)$ is an ϵ-approximation of X_n and $x_{0,n}$ inhabits X_n, then

$$A(X, \epsilon) = \prod_{k=1}^{n_0} A(X_k, \frac{2^{k-1}\epsilon}{n_0}) \times \prod_{k=n_0+1}^{\infty} \{x_{0,k}\}$$

is an ϵ-approximation of X, where $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0+1}^{\infty} \frac{1}{2^k} \leq \frac{\epsilon}{2}$.

Corollary

If (X, σ_∞) is the product of the totally bounded metric spaces $(X_n, d_n)_{n=1}^{\infty}$, then

$$\mathcal{U}(\Phi) = C_u(X),$$

$$\Phi_0 := \{\prod_{i=1}^{n}(f_i \circ \pi_i) \mid f_i \in C_u(X_i), 1 \leq i \leq n, n \in \mathbb{N}\},$$

$$\Phi := \{\sum_{k=1}^{n} h_k \mid h_k \in \Phi_0, 1 \leq k \leq n, n \in \mathbb{N}\}.$$

Proof: Without loss of generality let $d_n \leq \overline{1}$, for every $n \in \mathbb{N}$. The only difference with the proof of the finite case is treated as follows.

If $(x_n^0)_{n=1}^{\infty} \in X$ and $\epsilon > 0$, let

$$g := \sum_{k=1}^{n_0} \frac{d_{k,x_k^0} \circ \pi_k}{2^k} = \sum_{k=1}^{n_0} (\frac{d_{k,x_k^0}}{2^k}) \circ \pi_k \in \Phi,$$

where $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0+1}^{\infty} \frac{1}{2^k} \leq \epsilon.$$

We get

$$U(g, \sigma_{\infty,(x_n^0)_{n=1}^{\infty}}, \epsilon),$$

since

$$|g((y_n)_{n=1}^{\infty}) - \sigma_{\infty,(x_n^0)_{n=1}^{\infty}}((y_n)_{n=1}^{\infty})| = |\sum_{k=n_0+1}^{\infty} \frac{d_{k,x_k^0}(y_k)}{2^k}| \leq \sum_{k=n_0+1}^{\infty} |\frac{d_k(x_k^0, y_k)}{2^k}| \leq \epsilon.$$

A totally bounded metric space is separable. $C_u(X)$ is also separable.

Corollary

If $Q = \{q_n \mid n \in \mathbb{N}\}$ *is dense in* (X, d)*, then*

$$\mathcal{U}(\Phi_0^*) = C_u(X),$$

$$\Phi_0^* = \mathcal{A}(U_0(Q) \cup \{\overline{1}\}),$$

$$U_0(Q) := \{d_{q_n} \mid n \in \mathbb{N}\}.$$

Lemma: If $(x_n)_{n=1}^{\infty} \in X^{\mathbb{N}}$ converges pointwise to x, then $(d_{x_n})_{n=1}^{\infty}$ converges uniformly to d_x;

$$\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}(d(x_n, x) \leq \epsilon) \rightarrow$$

$$\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall_{y\in X}(|d(x_n, y) - d(x, y)| \leq \epsilon).$$

If $\epsilon > 0$ and $n \geq n_0$, then

$$d(x_n, y) \leq d(x_n, x) + d(x, y) \rightarrow$$

$$d(x_n, y) - d(x, y) \leq d(x_n, x) \leq \epsilon,$$

$$d(x, y) - d(x_n, y) \leq d(x_n, x) \leq \epsilon.$$

By **SWtbms** it suffices to show that

$$U_0(X) \subseteq U(\mathcal{A}(U_0(Q))).$$

If $d_x \in U_0(X)$, for some $x \in X$, and $(q_{k_n})_{n=1}^{\infty}$ is a subsequence of Q that converges pointwise to x, then $(d_{q_{k_n}})_{n=1}^{\infty}$ converges uniformly to d_x, therefore

$$d_x \in U(\mathcal{A}(U_0(Q))).$$

Conclusions

We presented a direct constructive proof of **SWtbms** with a clear computational content.

Its translation to Type Theory and its implementation to a proof assistant like Coq are expected to be straightforward.

Although **SWtbms** does not look like a theorem of Stone-Weierstrass type, as **BSWtbms** does, it has certain advantages over it. Its proof is "natural", in comparison to Bishop's technical proof and his difficult to motivate concept of a Bishop-separating set. We know of no application of **BSWtbms** which cannot be derived directly by **SWtbms**.

For the case of locally compact metric spaces Bishop just shows that the functions with compact support is a uniformly dense subset of the set of all functions which vanish at infinity.

There are many questions relating our **SWtbms** to **Lipschitz Analysis** (which is underdeveloped constructively).

E.g., if (X, d) totally bounded and (Y, ρ) complete metric space, is the set of Lipschitz functions $\mathrm{Lip}(X, Y)$ a dense subset of $C_u(X, Y)$?

A similar classical result requires a Lipschitz extension property, which indicates that the correlation of our lemma to the McShane-Kirszbraun theorem may not be accidental.

Related literature

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