Multi-degree smooth polar splines

a framework for design and analysis

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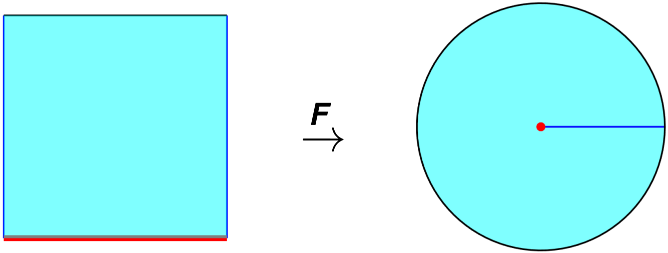
Outline

Outline

Polar splines: Overview

What?

- a periodic surface defined on a rectangular domain with one of its boundaries degenerating to a point
- smooth boundary



Polar splines: Overview

What?

- a periodic surface defined on a rectangular domain with one of its boundaries degenerating to a point
- smooth boundary

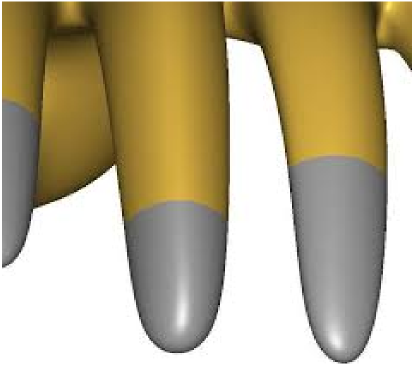
Where?

- surfaces of revolution
- filleting an end-point of a part with large radius

(For, e.g., the head of an airplane, end of a screwdriver)

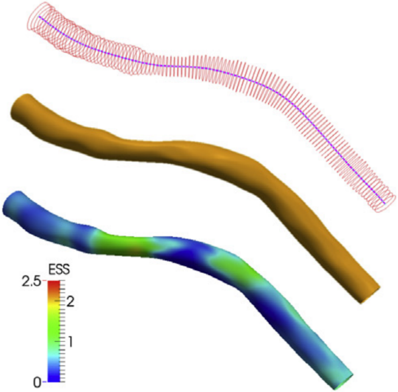
Polar splines: Applications

in computer graphics



Polar splines: Applications

in biomechanics

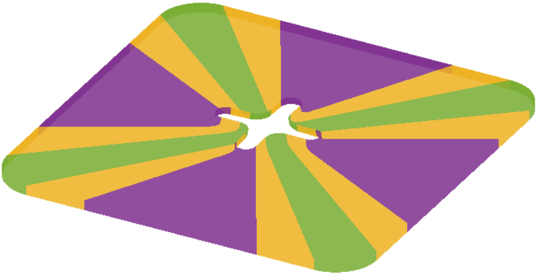


Polar splines: Objective

To use C^k polar spline patches as “standard” tools for design and analysis

- Analysis:
  - C^k basis functions (higher order PDEs)
  - optimal approximation behavior
- Design:
  - convex, partition of unity, “nice” basis functions
  - control net:
    - to be combined with C^k basis functions to construct C^k polar surfaces
    - to be used to manipulate such surfaces in an intuitive manner

Polar splines: Objective



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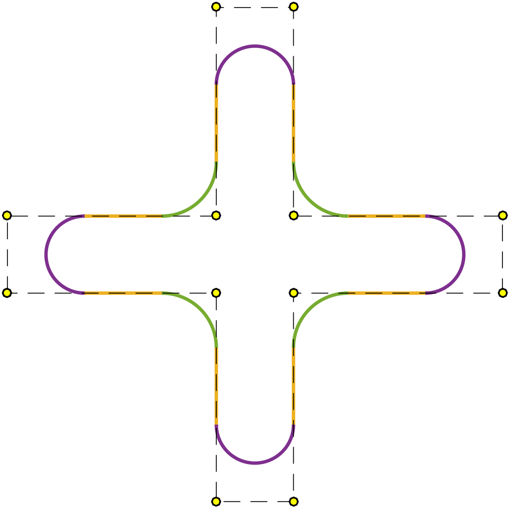

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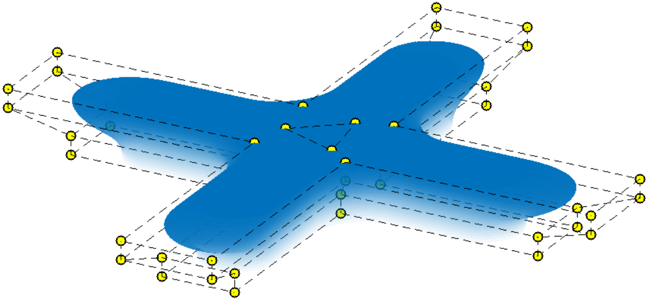
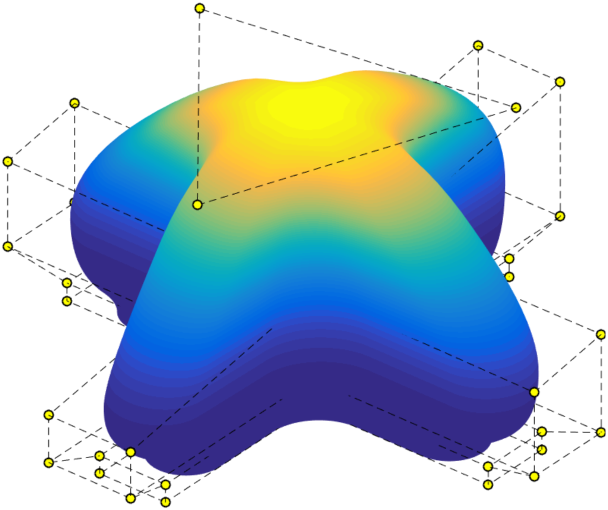


Polar splines: Objective



Existing body of work

Smooth parametrization of conics

- C^k smooth circles of degree $2(k+1)$

  C. Bangert, H. Prautzsch: *Circle and sphere as rational splines.* Neural Parallel Scient. Comput. 5 (1997)

- Smooth circular elements

  J. Lu: *Circular element: Isogeometric elements of smooth boundary.* Comput. Methods Appl. Mech. Eng. 198 (2009)

- ...

Non-uniform degree splines

- Multi-degree splines

  T.W. Sederberg, J. Zheng, X. Song: *Knot intervals and multi-degree splines* Comput. Aided Geom. Des. 20 (2003)

- Changeable degree splines

  W. Shen, G. Wang: *Changeable degree spline basis functions.* J. Comput. Appl. Math. 234 (2010)

- ...

Existing body of work

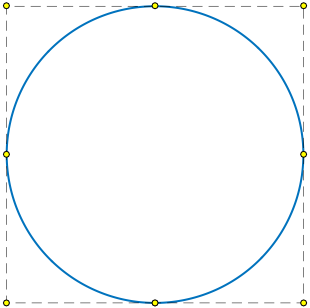
Polar surfaces

- C^1 and C^2 polar subdivision surfaces
  - K. Karčiauskas, J. Peters: *Bicubic polar subdivision.* ACM Trans. Graph. 26 (2007)
  - A. Myles, J. Peters: *Bi-3 C^2 polar subdivision.* ACM Trans. Graph. 28 (2009)
- C^2 polar splines
  - A. Myles, J. Peters: *C^2 splines covering polar configurations* Comput. Aided Des. 43 (2011)
- G^k polar NURBS
  - K.-L. Shi, et al.: *G^n blending multiple surfaces in polar coordinates.* Comput. Aided Des. 42 (2010)
  - K.-L. Shi, et al.: *Polar NURBS surface with curvature continuity.* Comput. Graph. Forum 32 (2013)
- ...

Outline

Point of departure

Usual quadratic 8-point circle



$$\boldsymbol{C}(\xi) = \boldsymbol{Q}^T \hat{\boldsymbol{b}}(\xi)$$

Point of departure

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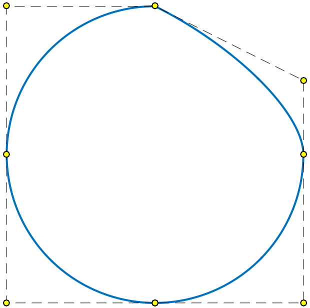

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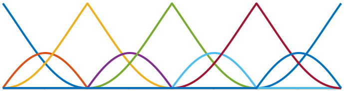
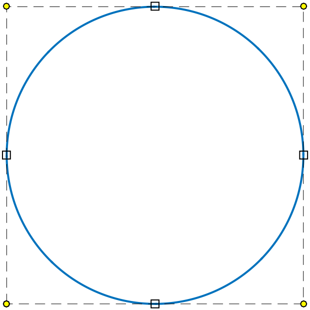


These splines are C^0 NURBS

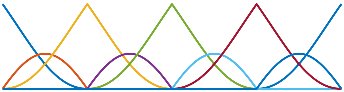
Point of departure

Quadratic 4-point circle



$$C(\xi) = P^T \times \ldots$$

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \hat{b}(\xi)$$



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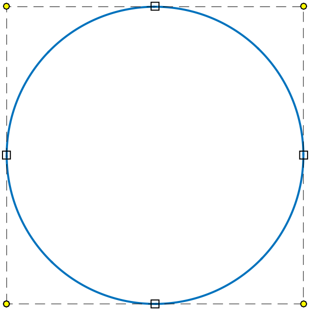

$$\boldsymbol{C}(\xi) = \boldsymbol{P}^T \times \hat{\boldsymbol{B}}(\xi)$$

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \hat{\boldsymbol{b}} =: \hat{\boldsymbol{B}}$$

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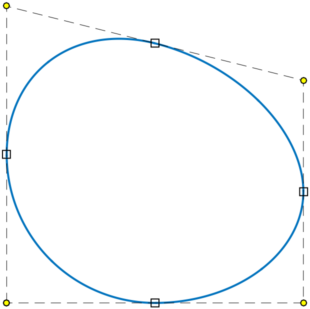

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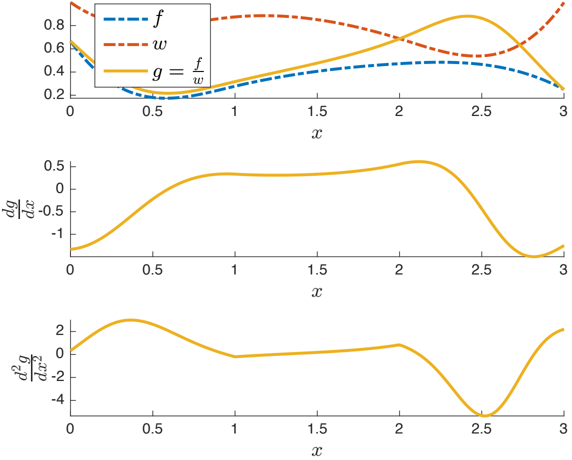
These splines are not C^0 NURBS, but C^1 *piecewise*-NURBS

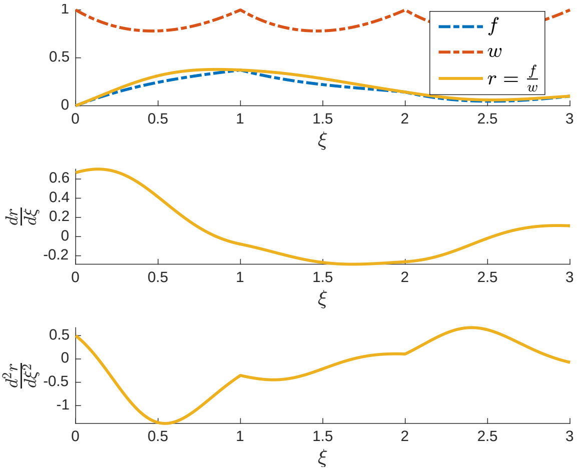
Point of departure

NURBS



Point of departure

Piecewise-NURBS



Point of departure

Spline extraction operator:

$$\hat{\boldsymbol{B}} = \boldsymbol{H}\hat{\boldsymbol{b}}$$

What conditions must ***H*** satisfy?

Point of departure

Spline extraction operator:

$$\hat{\boldsymbol{B}} = \boldsymbol{H}\hat{\boldsymbol{b}}$$

What conditions must $\boldsymbol{H}$ satisfy?

IGA-suitable extraction

(Motivation: Bézier extraction operator)

- Maximally-sparse
- Non-negative entries
- Each column sums up to 1
- Full-rank

Point of departure

Spline extraction operator:

$$\hat{\boldsymbol{B}} = \boldsymbol{H}\hat{\boldsymbol{b}}$$

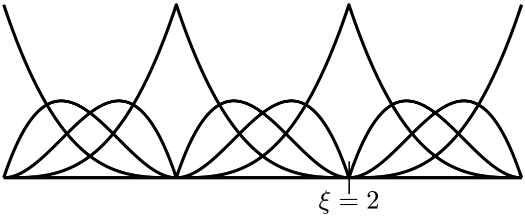
What conditions must $\boldsymbol{H}$ satisfy?

IGA-suitable extraction

(Motivation: Bézier extraction operator)

- Maximally-sparse (B-splines are splines of minimal support)
- Non-negative entries (B-splines are non-negative)
- Each column sums up to 1 (B-splines form a partition of unity)
- Full-rank (B(asis)-splines)

Smoothness constraints



$\hat{\boldsymbol{b}}$ defined on $U = [0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 3]$

Smoothness constraints

We wish to increase smoothness to C^2 at $\xi = 2$:

$$\lim_{\xi \to 2^-} \frac{d^m f}{d\xi^m} = \lim_{\xi \to 2^+} \frac{d^m f}{d\xi^m}, \quad m = 1,\ 2$$

where $f(\xi) = \sum_{i=1}^{10} f_i \, \hat{b}_i(\xi)$

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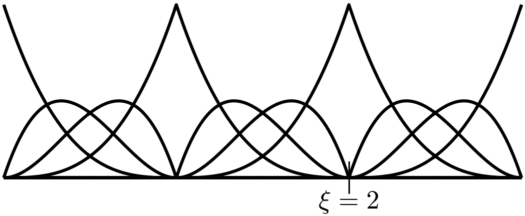
where $f(\xi) = \sum_{i=1}^{10} f_i \, \hat{b}_i(\xi)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -3 & 6 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & -12 & 0 & 12 & -6 & 0 \end{bmatrix} \times [f_i] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

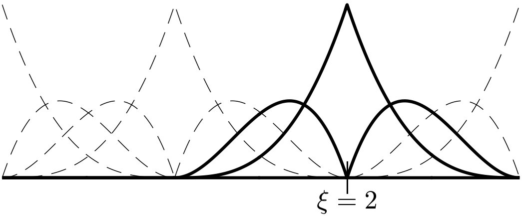
$\boldsymbol{H}^T \leftarrow$Null-space

Maximally sparse null-space: C^1



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -3 & 6 & -3 & 0 & 0 \end{bmatrix}$$

Maximally sparse null-space: C^1



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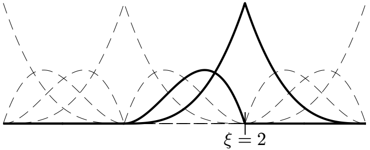
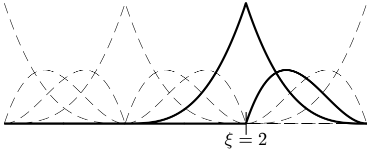
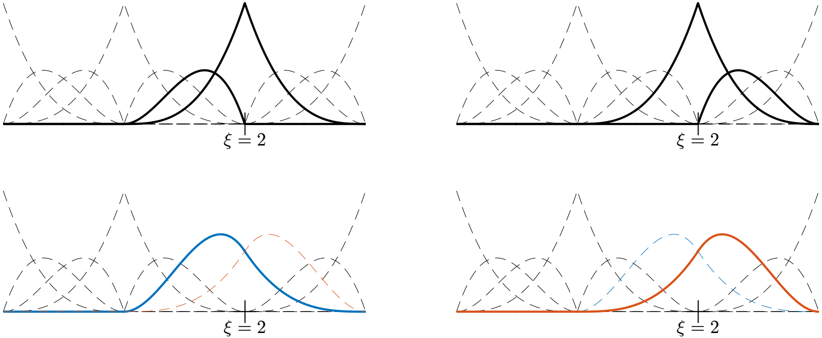


$$\begin{bmatrix} -3 & 6 & -3 \end{bmatrix}$$



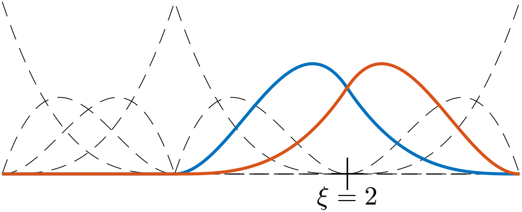
Maximally sparse null-space: C^1

$$\begin{bmatrix} -3 & 6 & -3 \end{bmatrix} \qquad \begin{bmatrix} -3 & 6 & -3 \end{bmatrix}$$



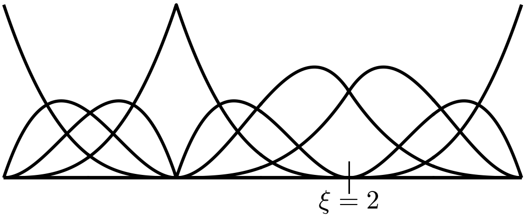
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}^T \qquad \begin{bmatrix} 0 & \frac{1}{2} & 1 \end{bmatrix}^T$$

Maximally sparse null-space: C^1



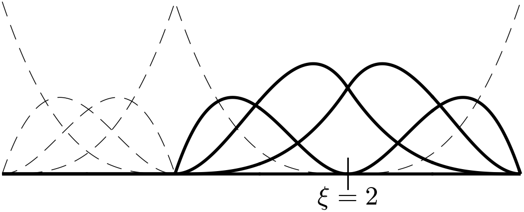
$\hat{\boldsymbol{b}}$ defined on $U = [0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 3]$

Maximally sparse null-space: C^2



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 6 & -12 & 12 & -6 & 0 \end{bmatrix}$$

Maximally sparse null-space: C^2



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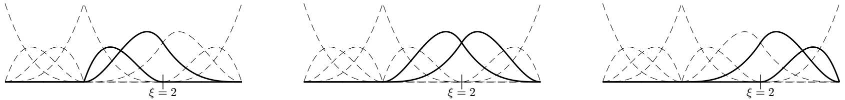
Maximally sparse null-space: C^2

$$\begin{bmatrix} 6 & -12 & 12 & -6 \end{bmatrix} \quad \begin{bmatrix} 6 & -12 & 12 & -6 \end{bmatrix} \quad \begin{bmatrix} 6 & -12 & 12 & -6 \end{bmatrix}$$



Maximally sparse null-space: C^2

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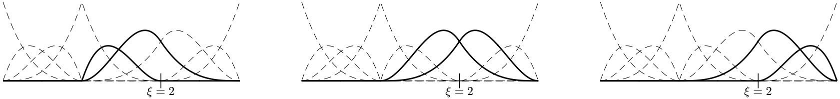
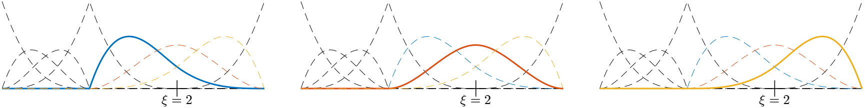
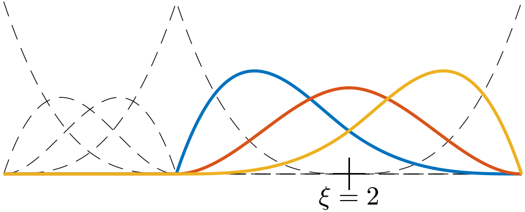




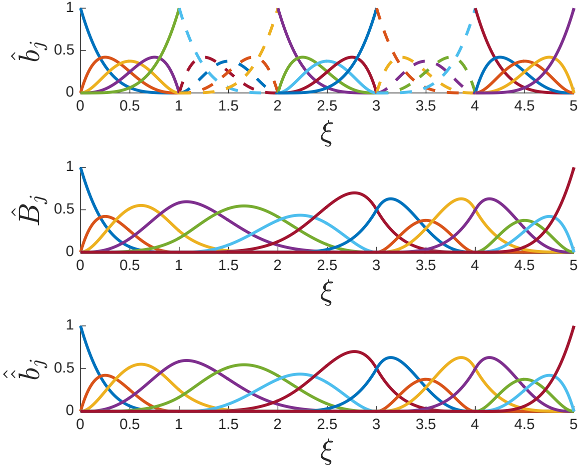
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}^T \quad \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^T \quad \begin{bmatrix} 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}^T$$

Maximally sparse null-space: C^2



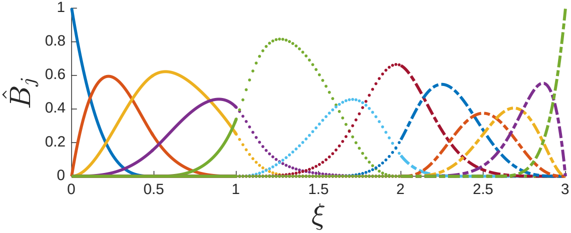
$\hat{\boldsymbol{b}}$ defined on $U = [0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 2,\ 3,\ 3,\ 3,\ 3]$

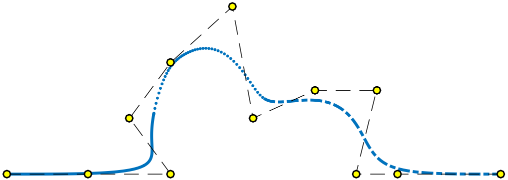
Uniform degree B-splines



Smoothness $\kappa = (2,\ 3,\ 1,\ 1)$

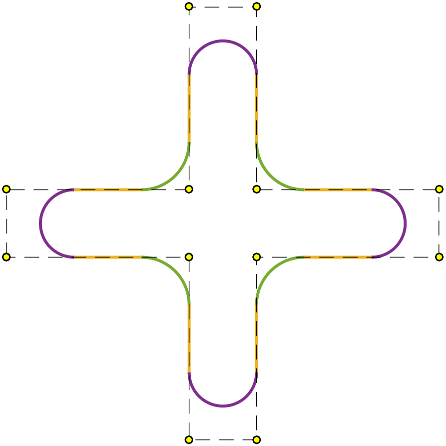
Non-uniform degree piecewise-NURBS





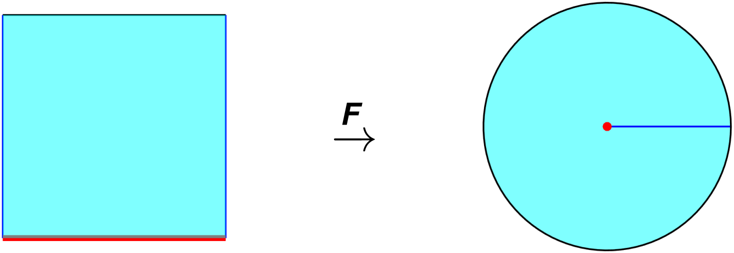
Smoothness $\kappa = (2,\ 2)$

Non-uniform degree piecewise-NURBS



Outline

Setting of C^k polar splines



$$\boldsymbol{F} : (\xi, \eta) \mapsto (u, v)$$

Setting of C^k polar splines

Ingredients

- Smooth tensor-product spline space $\mathcal{R}^{\xi\eta} := \mathcal{R}^{\xi} \otimes \mathcal{R}^{\eta}$
- A suitable map $\boldsymbol{F}$
  - polar point: $\boldsymbol{F}(\xi, 0) = (0, 0)$ for all ξ
  - mapped splines remain smooth everywhere except at $(0, 0)$

Setting of C^k polar splines

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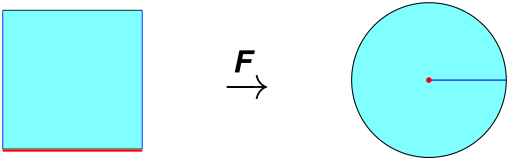
  - mapped splines remain smooth everywhere except at $(0, 0)$

C^k polar spline recipe

1. Map tensor-product basis functions using $\boldsymbol{F}$
2. Impose smoothness constraints at the polar point:
   - require reproduction of a linearly independent Hermite data set
   - obtain an extraction operator as the null-space of constraints

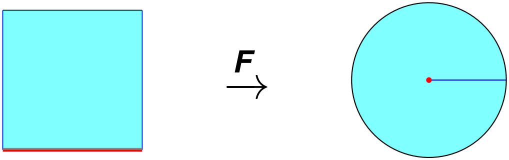
$$\boldsymbol{N} = \boldsymbol{E}^k \boldsymbol{B}$$

Setting of C^k polar splines



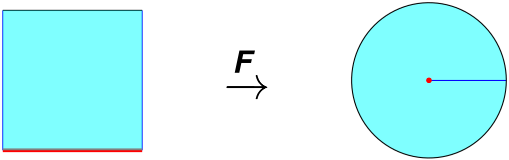
$$s(u,v) = \sum_{i=1}^{n^\xi} \sum_{j=1}^{n^\eta} s_{ij} B_{ij}(u,v)$$

Setting of C^k polar splines



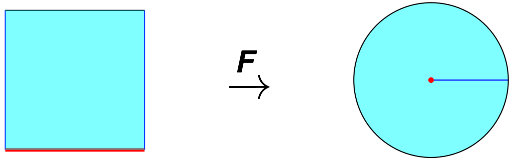
$$\hat{s}(\xi,\eta) := s(\boldsymbol{F}(\xi,\eta)) = \sum_{i=1}^{n^\xi}\sum_{j=1}^{n^\eta} s_{ij} B_{ij}\left(\boldsymbol{F}(\xi,\eta)\right) = \sum_{i=1}^{n^\xi}\sum_{j=1}^{n^\eta} s_{ij}\hat{B}_{ij}(\xi,\eta)$$

Setting of C^k polar splines



$$\left.\frac{\partial \hat{s}}{\partial \eta}\right|_{\eta=0} = \left.\begin{bmatrix} \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} \end{bmatrix}\right|_{\eta=0} \left.\begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial v} \end{bmatrix}\right|_{(u,v)=(0,0)}$$

Setting of C^k polar splines



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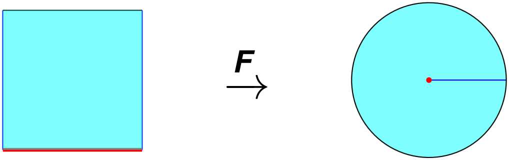
$\hat{s} \in \mathcal{R}^{\xi\eta}$ (say, bi-cubics), $u, v \in \mathcal{R}_F^{\xi\eta}$ (say, bi-cubics)

Setting of C^k polar splines



$$\left.\frac{\partial \hat{s}}{\partial \eta}\right|_{\eta=0} = \left.\left[\begin{array}{cc} \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} \end{array}\right]\right|_{\eta=0} \left.\left[\begin{array}{c} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial v} \end{array}\right]\right|_{(u,v)=(0,0)}$$

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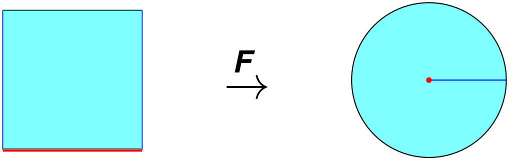
Setting of C^k polar splines

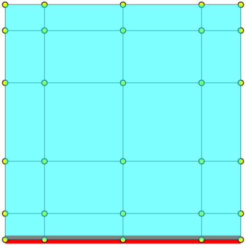


$$\left.\frac{\partial \hat{s}}{\partial \eta}\right|_{\eta=0} = \left.\begin{bmatrix} \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} \end{bmatrix}\right|_{\eta=0} \left.\begin{bmatrix} \frac{\partial s}{\partial u} \\ \frac{\partial s}{\partial v} \end{bmatrix}\right|_{(u,v)=(0,0)}$$

- In general, for $C^{\geq 2}$, products of u and v must belong to $\mathcal{R}^{\xi\eta}$.

Polar spline extraction operator

$$\boldsymbol{N} = \boldsymbol{E}^k \boldsymbol{B}$$



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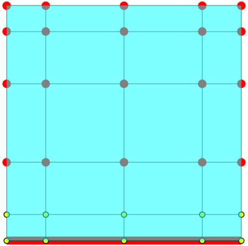

Basis functions B_{ij} for $j > k + 1$ are already C^k at the polar point (first k derivatives are zero).

Polar spline extraction operator

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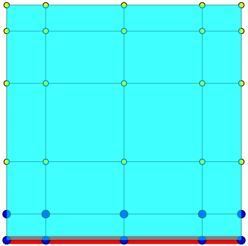

Basis functions B_{ij} for $j <= k + 1$ have non-zero k^{th} derivatives at $\eta = 0$.

Polar spline extraction operator

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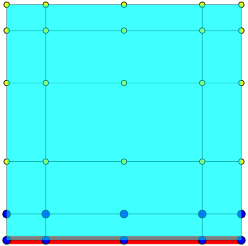

Basis functions B_{ij} for $j <= k + 1$ have non-zero k^{th} derivatives at $\eta = 0$.

For a flexible C^k space, we require at least $n_k = \frac{(k+1)(k+2)}{2}$ basis functions non-zero at the polar point.

Polar spline extraction operator

Computation of n_k new basis functions is done as follows:

Reproduction of Hermite data at $(0,0)$

For all $m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $m_1 + m_2 \leq k$,

$$\lim_{(u,v)\to(0,0)} \frac{\partial^{m_1+m_2} N_l}{\partial u^{m_1} \partial v^{m_2}}(u,v) = \frac{\partial^{m_1+m_2} T_l}{\partial u^{m_1} \partial v^{m_2}}(0,0) \ ,$$

where $\{T_l\}_{l=1}^{n_k}$ are Bernstein polynomials of degree k defined with respect to particular triangles $\mathcal{T}_k$.

Polar spline extraction operator

Triangular Bernstein polynomials

Let $\mathcal{T}_k = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$, we have $n_k = \frac{(k+1)(k+2)}{2}$ triangular Bernstein polynomials of degree k:

$$T_{i_1 i_2 i_3}(u, v) := \binom{k}{i_1\ i_2\ i_3} \prod_{j=1}^{3} (\lambda_j)^{i_j}, \quad i_1 + i_2 + i_3 = k$$

where $\{\lambda_j\}_{j=1}^{3}$ are barycentric coordinates of (u, v) with respect to $\mathcal{T}_k$:

$$\sum_{j=1}^{3} \lambda_j \boldsymbol{v}_j = (u, v) \, , \quad \sum_{j=1}^{3} \lambda_j = 1$$

Polar spline extraction operator

Polar extraction operator $\boldsymbol{N} = \boldsymbol{E}^k \boldsymbol{B}$

$$\boldsymbol{E}^k = \begin{bmatrix} \bar{\boldsymbol{E}}^k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}^k \end{bmatrix}$$

where $\boldsymbol{I}^k$ is an identity matrix of size $(n - n_k) \times (n - n_k)$ and $\bar{\boldsymbol{E}}^k$ is a matrix of size $n_k \times n^\xi(k+1)$, where $n := n^\xi n^\eta + n_k - n^\xi(k+1)$

Polar spline extraction operator

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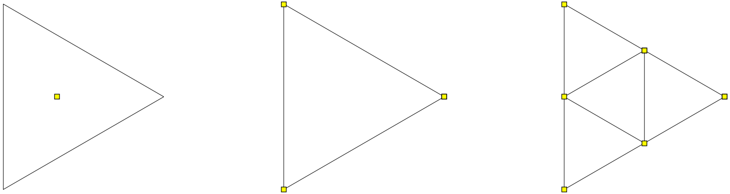
Theorem

The $\{N_l\}_{i=1}^{n}$ form a partition of unity. Moreover, for $\mathcal{T}_k$ sufficiently large, we are guaranteed an IGA-suitable $\boldsymbol{E}^k$

Polar spline extraction operator

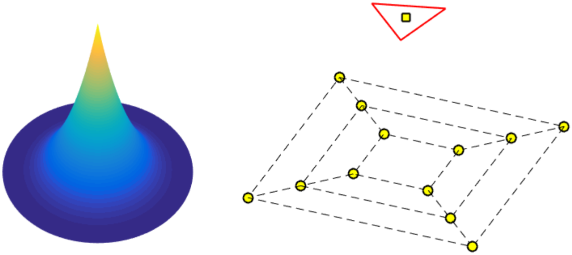
Control-net at polar point



(a) $\mathcal{T}_0$ (b) $\mathcal{T}_1$ (c) $\mathcal{T}_2$

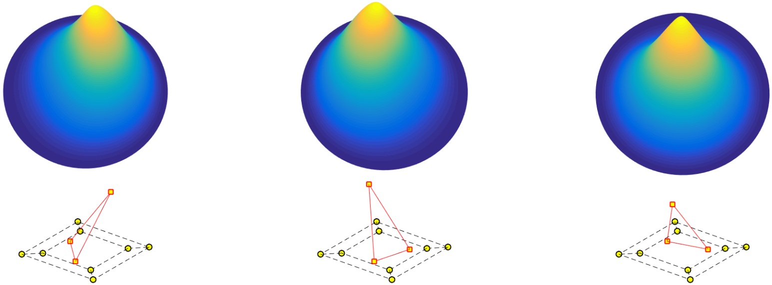
C^0 polar splines

An example of a C^0 polar basis function



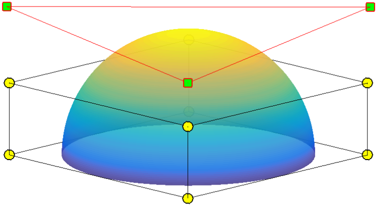
C^1 polar splines

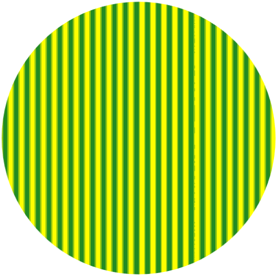
An example of C^1 polar basis functions



C^1 polar splines

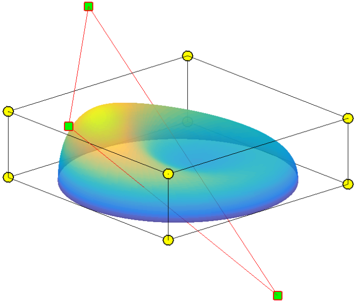
C^1 hemisphere (original)

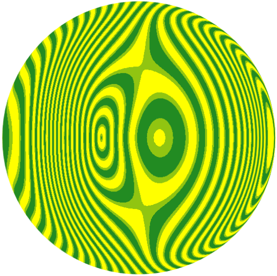




C^1 polar splines

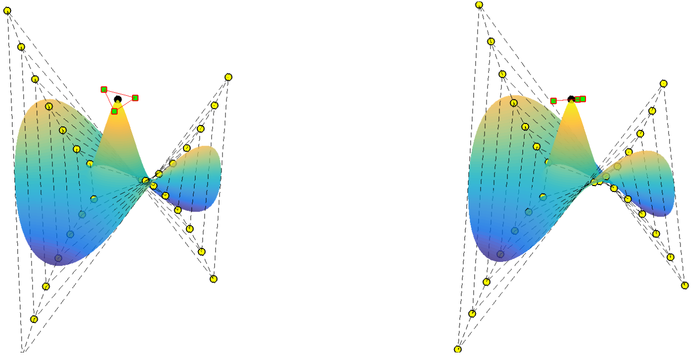
C^1 hemisphere (deformed)





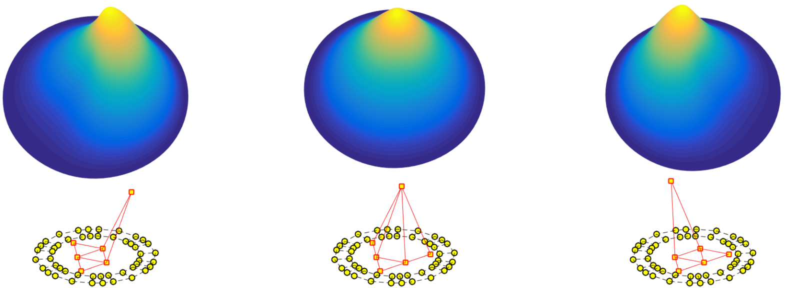
C^1 polar splines

$\mathcal{T}_1$ tangent to the surface



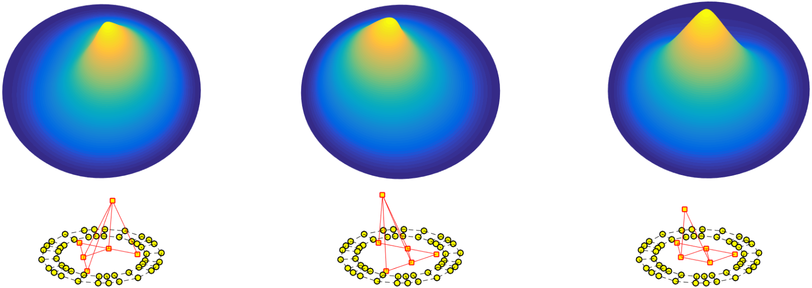
C^2 polar splines

An example of C^2 polar basis functions (first three)



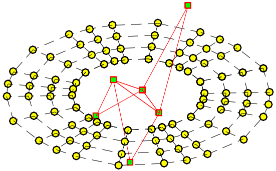
C^2 polar splines

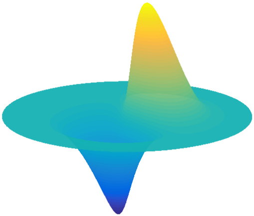
An example of C^2 polar basis functions (last three)



C^2 polar splines

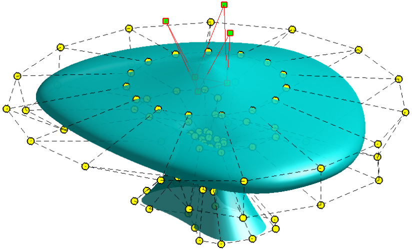
C^2 surface

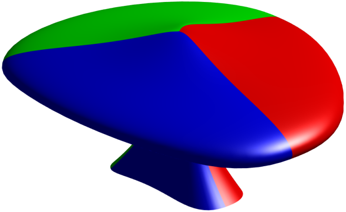




Outline

Freeform design: Mushroom





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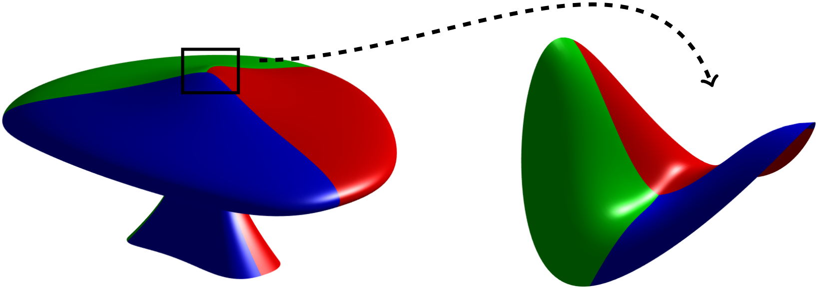


Numerical results for analysis suitability

Convergence behaviour

Problems solved:

- Function approximation
- Poisson equation

Spaces used:

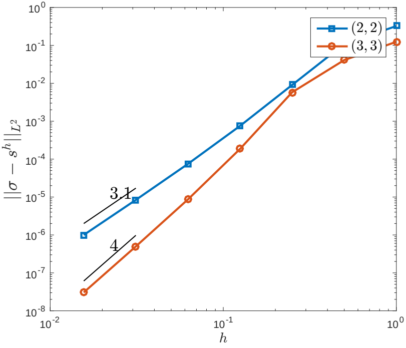
- C^1: $(2,2)$ and $(3,3)$
- C^2: $(6,5)$ and $(6,6)$

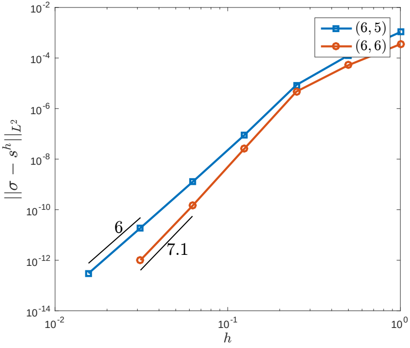
Cahn–Hilliard on a circular disk

C^1 space used: $(2,2)$

Function approximation

$$s(u,v) = \sin\left(\pi u + \frac{\pi}{3}\right)\cos\left(\pi v + \frac{\pi}{4}\right)$$





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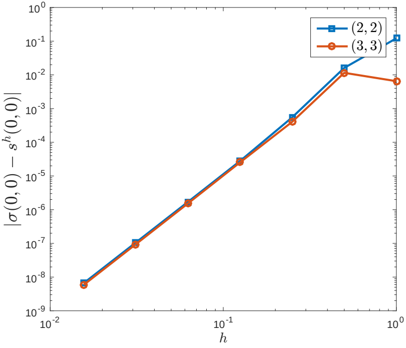
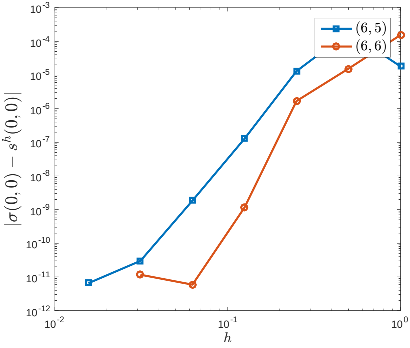
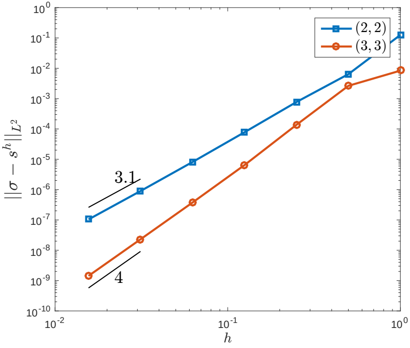
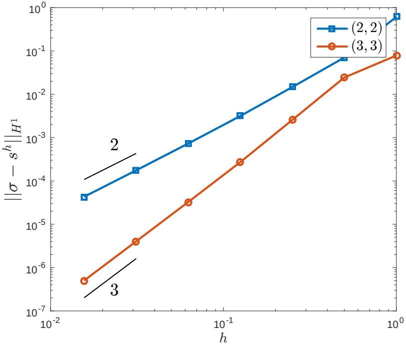




Poisson equation

$$-\Delta s(u,v) = 2\sin(u)\sin(v) \quad \text{on } \Omega$$
$$s(u,v) = \sin(u)\sin(v) \quad \text{on } \partial\Omega$$





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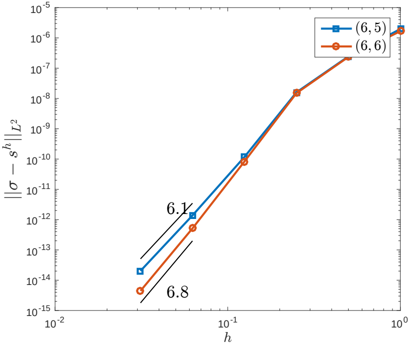


Cahn–Hilliard on a circular disk

Model for phase separation

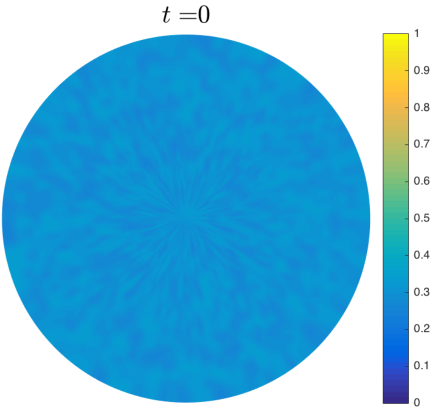
$$\frac{\partial c}{\partial t} = \nabla.\left(c(1-c)\nabla(\mathbb{N}_2\mu_c - \Delta c)\right) \quad \text{on } \Omega \times [0,\ T]\ ,$$
$$c(1-c)\nabla\mu_c.\boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times [0,\ T]\ ,$$
$$c(1-c)\nabla c.\boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times [0,\ T]\ ,$$
$$c(\boldsymbol{x}, 0) = c_0 \quad \text{on } \Omega,$$

where

$$\mu_c := \frac{1}{3}\log\left(\frac{c}{1-c}\right) + 1 - 2c$$

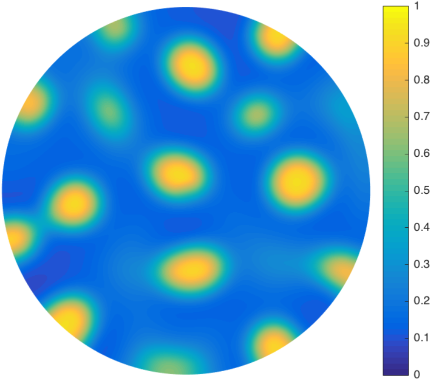
initial volume-fraction $\bar{c} = 0.3 +$ noise , $\mathbb{N}_2$: 753.08

Cahn–Hilliard on a circular disk



Cahn–Hilliard on a circular disk

$t = 0.00084435$



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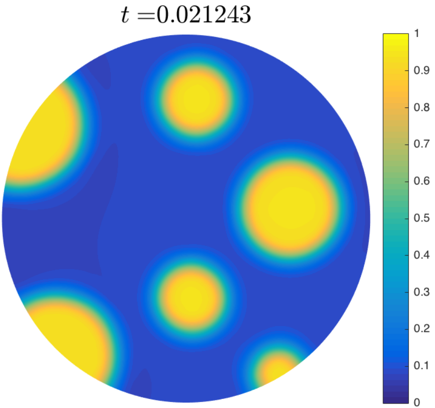

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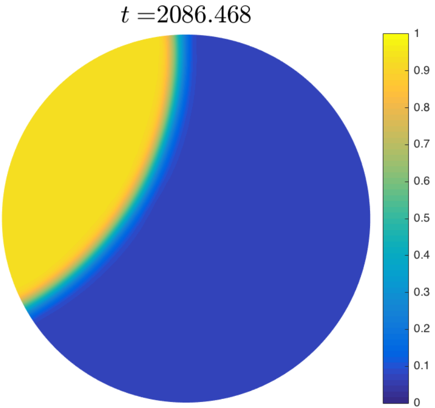

Summary

- Smooth (*piecewise*-NURBS) splines of non-uniform degree
- A unified theoretical framework for construction of C^k polar splines with applications in both design and analysis
- Numerical results demonstrating:
  - applications in design
  - best possible approximation properties of the polar spline spaces
  - applications to higher order PDEs

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D. Toshniwal, H. Speleers, R.R. Hiemstra, T.J.R. Hughes: *Multi-degree smooth polar splines: a framework for geometric modeling and isogeometric analysis*, Comput. Methods Appl. Mech. Engrg., in press