Best Polynomial Approximation on the Unit Ball

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Introduction

Introduction

- Our purpose is to study the best approximation by polynomials of degree at most n on the unit ball $\mathbb{B}^d$ in $\mathbb{R}^d$.
- For

$$\varpi_\mu(x) = (1 - \|x\|^2)^\mu, \qquad \mu > -1, \quad x \in \mathbb{B}^d,$$

we let $\|\cdot\|_\mu$ be the norm of $L^2(\varpi_\mu; \mathbb{B}^d)$, defined by

$$\|f\|_\mu := \left(b_\mu \int_{\mathbb{B}^d} |f(x)|^2 \varpi_\mu(x) dx \right)^{1/2},$$

where $b_\mu = 1 / \int_{\mathbb{B}^d} \varpi_\mu(x) dx$.
- Let Π_n^d the space of polynomials of degree at most n in d variables.
- We consider the error, $E_n(f)_\mu$, of best approximation by polynomials in Π_n^d in the space $L^2(\varpi_\mu; \mathbb{B}^d)$, defined by

$$E_n(f)_\mu := \inf_{p_n \in \Pi_n^d} \|f - p_n\|_\mu.$$

Our main goal

- For $d = 1$ and $f' \in L^2(\varpi_{\mu+1}; [-1,1])$ there exists a nice estimate

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$$E_n(f)_\mu \leq \frac{c}{n} \sum_{i=1}^{d} E_{n-1}(\partial_i f)_{\mu+1},$$

For $d \geqslant 2$, this inequality does not hold on the unit ball!.

Differential operators

- Let Δ denote the usual Laplace operator $\Delta = \partial_1^2 + \cdots + \partial_d^2$.
- In spherical-polar coordinates $x = r\xi$, $r \geq 0$ and $\xi \in \mathbb{S}^{d-1}$,

$$\Delta = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_0.$$

  Δ_0 denotes the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$.
- We will also use the angular derivatives, $D_{i,j}$, defined by

$$D_{i,j} := x_i\partial_j - x_j\partial_i, \qquad 1 \leq i < j \leq d.$$

- The angular derivatives $D_{i,j}$ and the Laplace-Beltrami operator Δ_0 are related by

$$\Delta_0 = \sum_{1\leq i<j\leq d} D_{i,j}^2.$$

Spherical harmonics

Harmonic polynomials

- $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n.
- Harmonic polynomials of d-variables are homogeneous polynomials in $\mathcal{P}_n^d$ that satisfy the Laplace equation $\Delta Y = 0$
- $\mathcal{H}_n^d$ denotes the space of harmonic polynomials of degree n.
- We will denote

$$a_n^d := \dim \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n}.$$

Spherical harmonics

- Spherical harmonics are the restriction of harmonic polynomials on the unit sphere.
- They are eigenfunctions of the Laplace-Beltrami operator,

$$\Delta_0 Y(\xi) = -n(n+d-2)Y(\xi), \quad \forall\, Y \in \mathcal{H}_n^d, \quad \xi \in \mathbb{S}^{d-1}.$$

- Spherical harmonics of different degrees are orthogonal with respect to the inner product

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} := \frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\sigma(\xi).$$

where $d\sigma$ denote the surface measure and σ_{d-1} denote the surface area,

$$\sigma_{d-1} := \int_{\mathbb{S}^{d-1}} d\sigma = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}.$$

A basis of spherical harmonics

Let $T_n(t)$ and $U_n(t)$ denote the Chebyshev polynomials of the first and the second kind, respectively. Define

$$g_{0,n}(x_1,x_2) = (x_1^2+x_2^2)^{n/2}\, T_n\left(x_2(x_1^2+x_2^2)^{-1/2}\right),$$

$$g_{1,n-1}(x_1,x_2) = x_1(x_1^2+x_2^2)^{(n-1)/2}\, U_{n-1}\left(x_2(x_1^2+x_2^2)^{-1/2}\right).$$

For $d>2$ and $\mathbf{n}=(n_1,n_2,\ldots,n_d)\in\mathbb{N}_0^d$ with $n_1=0$ or 1, define

$$Y_{\mathbf{n}}(x) = g_{n_1,n_2}(x_1,x_2)\prod_{j=3}^{d}(x_1^2+\ldots+x_j^2)^{n_j/2}\, C_{n_j}^{\lambda_j}\left(x_j(x_1^2+\ldots+x_j^2)^{-1/2}\right),$$

where $\lambda_j=\lambda_j(n_1,\ldots,n_{j-1}) := \sum_{i=1}^{j-1} n_i + \frac{j-2}{2}$.

Proposition

$\{Y_{\mathbf{n}};|\mathbf{n}|=n \text{ with } n_1=0 \text{ or } 1\}$ is a mutually orthogonal basis of $\mathcal{H}_n^d$.

A basis of spherical harmonics

We need information on two operations on this basis,

- partial derivatives ∂_i
- multiplication by x_i.

They are related by the orthogonal projection operator

$$\mathrm{proj}^d_{n,\mathbb{S}} : \mathcal{P}^d_n \mapsto \mathcal{H}^d_n$$

It is known that

$$\mathrm{proj}^d_{n,\mathbb{S}} P = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{4^j j! (-n+2-d/2)_j} \|x\|^{2j} \Delta^j P,$$

which implies, for a spherical harmonic $Y_{\mathbf{n}} \in \mathcal{H}^d_n$,

$$\mathrm{proj}^d_{n+1,\mathbb{S}}(x_i Y_{\mathbf{n}}(x)) = x_i Y_{\mathbf{n}}(x) - \frac{1}{2n+d-2} \|x\|^2 \partial_i Y_{\mathbf{n}}(x).$$

A basis of spherical harmonics

The basis satisfies the following property

Theorem

Let $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$ with $n_1 = 0$ or 1 and $|\mathbf{n}| = n$. Then

1. $\partial_i Y_{\mathbf{n}}(x)$ is an spherical harmonic of degree $n-1$ and

$$\langle \partial_i Y_{\mathbf{n}}, Y_{\mathbf{m}} \rangle_{\mathbb{S}^{d-1}} \neq 0, \qquad |\mathbf{m}| = n-1$$

   for at most 2^{d-2} many $\mathbf{m} \in \mathbb{N}_0^d$ with $m_1 = 0$ or 1.

2. $\mathrm{proj}^d_{n+1,\mathbb{S}}(x_i Y_{\mathbf{n}})$ is an spherical harmonic of degree $n+1$ and

$$\langle \mathrm{proj}^d_{n+1,\mathbb{S}}(x_i Y_{\mathbf{n}}), Y_{\mathbf{m}} \rangle_{\mathbb{S}^{d-1}} \neq 0, \qquad |\mathbf{m}| = n+1,$$

   for at most 2^{d-2} many $\mathbf{m} \in \mathbb{N}_0^d$ with $m_1 = 0$ or 1.

Orthogonal polynomials on the unit ball

Theorem (Dunkl, Xu)

For $n \in \mathbb{N}_0$ and $0 \le j \le n/2$, let $\{Y_\nu^{n-2j} : 1 \le \nu \le a_{n-2j}^d\}$ denote an orthonormal basis for $\mathcal{H}_{n-2j}^d$. For $\mu > -1$, define

$$P_{j,\nu}^{n,\mu}(x) := P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\,\|x\|^2 - 1)\, Y_\nu^{n-2j}(x).$$

Then the set $\{P_{j,\nu}^{n,\mu} : 1 \le j \le n/2,\, 1 \le \nu \le a_{n-2j}^d\}$ consists of a mutually orthogonal basis of $\mathcal{V}_n^d(\varpi_\mu)$; more precisely,

$$\langle P_{j,\nu}^{n,\mu}, P_{k,\eta}^{m,\mu} \rangle_\mu = h_{j,n}^\mu \delta_{n,m}\, \delta_{j,k}\, \delta_{\nu,\eta},$$

where $h_{j,n}^\mu$ is given by

$$h_{j,n}^\mu := \frac{(\mu+1)_j (\frac{d}{2})_{n-j} (n - j + \mu + \frac{d}{2})}{j! (\mu + \frac{d+2}{2})_{n-j} (n + \mu + \frac{d}{2})}.$$

The orthogonal basis $\{P_{j,\nu}^{n,\mu}\}$ satisfies two other orthogonal relations in the Sobolev space.

Lemma

Let $\mu > -1$. Then the basis $\{P_{j,\nu}^{n,\mu}\}$ satisfies

$$b_\mu \int_{\mathbb{B}^d} \nabla P_{j,\nu}^{n,\mu}(x) \cdot \nabla P_{j',\nu'}^{m,\mu}(x) \varpi_{\mu+1}(x) dx = h_{j,n}^{\mu}(\nabla) \delta_{\nu,\nu'} \delta_{j,j'} \delta_{n,m},$$

where $h_{j,n}^{\mu}(\nabla) = (4j(n-j+\mu+d/2) + 2(n-2j)(\mu+1)) h_{j,n}^{\mu}$.

For the angular derivatives we have

Lemma

Let $\mu > -1$. Then the basis $\{P_{j,\nu}^{n,\mu}\}$ satisfies

$$b_\mu \int_{\mathbb{B}^d} \sum_{1 \le i < j \le d} D_{i,j} P_{\ell,\nu}^{n,\mu}(x) D_{i,j} P_{\ell',\nu'}^{m,\mu}(x) \varpi_\mu(x) dx = h_{\ell,n}^{\mu}(D) \delta_{\nu,\nu'} \delta_{\ell,\ell'} \delta_{n,m},$$

where $h_{\ell,n}^{\mu}(D) = (m - 2\ell)(m - 2\ell + d - 2) h_{\ell,n}^{\mu}$.

Orthogonal polynomials on the unit ball

Define $P^{n,\mu}_{j,\nu} = 0$ if $j < 0$. The polynomial $P^{n,\mu}_{j,\nu}$ enjoys a simple form under both Δ and Δ_0.

Lemma

Let $\mu > -1$, for $P^{n,\mu}_{j,\nu}$ we have

$$\Delta P^{n,\mu}_{j,\nu}(x) = \kappa^{\mu}_{n-j} P^{n-2,\mu+2}_{j-1,\nu}(x) \quad \text{and} \quad \Delta_0 P^{n,\mu}_{j,\nu}(x) = \lambda_{n-2j} P^{n,\mu}_{j,\nu}(x),$$

where $\kappa^{\mu}_{n} := 4(n+\mu+\frac{d}{2})(n+\frac{d-2}{2})$ and $\lambda_n := -n(n+d-2)$.

Derivatives of orthogonal polynomials on the unit ball

Lemma

For $1 \le i \le d$, let $\widehat{Y}_{\eta,i}^{m+1} := \mathrm{proj}_{m+1,\mathbb{S}}^{d}(x_i Y_\eta^m)$. Let $\beta_\ell = m - 2\ell + \frac{d-2}{2}$. Then,

$$\partial_i P_{\ell,\eta}^{m,\mu}(x) = \frac{\beta_\ell + \ell}{\beta_\ell} P_\ell^{(\mu+1,\beta_\ell-1)}(2r^2-1)\partial_i Y_\eta^{m-2\ell}(x) + 2(\ell+\mu+\beta_\ell+1) P_{\ell-1}^{(\mu+1,\beta_\ell+1)}(2r^2-1)\widehat{Y}_{\eta,i}^{m-2\ell+1}(x).$$

Derivatives of orthogonal polynomials on the unit ball

Up to this point, we assumed that the spherical harmonics Y_ν^{n-2j} in the basis $P_{j,\nu}^{n,\mu}$ form an orthonormal basis of $\mathcal{H}_{n-2j}^d$ but did not specify this basis. In our next proposition, however, we need to specify this basis as the one previously defined.

Proposition

Let $P_{\ell,\nu}^{n,\mu}$ be an orthogonal polynomial with Y_ν^{n-2j} being the orthonormal basis previously defined. Let $\eta \in \mathbb{N}_0^d$ with $|\eta| = n - 2k$ and $\eta_1 = 0$ or 1. Then

1. for $1 \leq i \leq d$, $\langle \partial_i P_{\ell,\nu}^{n,\mu}, P_{k,\eta}^{n-1,\mu+1} \rangle_{\mu+1} \neq 0$ only if $k = \ell$ or $\ell - 1$ and, in each case, for at most 2^{d-1} many $\nu \in \mathbb{N}_0^d$ with $\nu_1 = 0$ or 1;
2. for $1 \leq i < j \leq d$, $\langle D_{i,j} P_{\ell,\nu}^{n,\mu}, P_{k,\eta}^{n,\mu} \rangle_{\mu+1} \neq 0$ only if $k = \ell$ and for at most 2^{2d-1} many $\nu \in \mathbb{N}_0^d$ with $\nu_1 = 0$ or 1.

Fourier orthogonal expansions and approximation

Fourier orthogonal expansions and approximation

The Fourier orthogonal expansion of $f \in L^2(\varpi_\mu, \mathbb{B}^d)$ is defined by

$$f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu} \widehat{f}_{j,\nu}^{n,\mu} P_{j,\nu}^{n,\mu}(x), \quad \text{with} \quad \widehat{f}_{j,\nu}^{n,\mu} := \frac{1}{h_{j,n}^{\mu}} \langle f, P_{j,\nu}^{n,\mu} \rangle_{\mu}.$$

Let $\operatorname{proj}_n^\mu : L^2(\varpi_\mu, \mathbb{B}^d) \mapsto \mathcal{V}_n^d(\varpi_\mu)$ and $S_n^\mu : L^2(\varpi_\mu, \mathbb{B}^d) \mapsto \Pi_n^d$ denote the projection operator and the n-th partial sum operator, respectively. Then

$$S_n^\mu f(x) = \sum_{m=0}^{n} \operatorname{proj}_m^\mu f(x) \quad \text{and} \quad \operatorname{proj}_m^\mu f(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \widehat{f}_{j,\nu}^{m,\mu} P_{j,\nu}^{m,\mu}(x).$$

By definition, $S_n^\mu f = f$ if $f \in \Pi_n^d$ and $\langle f - S_n^\mu f, v \rangle_\mu = 0$ for all $v \in \Pi_n^d$.

Fourier orthogonal expansions and approximation

It turns out that the partial derivatives commute with the partial sum operators, a fact that plays an essential role.

Lemma

Let $\mu > -1$. Then

$$\partial_i S_n^\mu f = S_{n-1}^{\mu+1}(\partial_i f), \qquad 1 \le i \le d,$$

and

$$D_{i,j} S_n^\mu f = S_n^\mu (D_{i,j} f), \qquad 1 \le i < j \le d.$$

Obviously, it also holds

$$\Delta S_n^\mu f = S_{n-2}^{\mu+2}(\Delta f), \quad \text{and} \quad \Delta_0 S_n^\mu f = S_n^\mu(\Delta_0 f)$$

Fourier orthogonal expansions and approximation

Theorem 1

Let $f \in W_2^2(\varpi_\mu, \mathbb{B}^d)$. Then, for $n \geq 2$,

$$E_n(f)_\mu \leq \frac{c}{n^2}\left[E_{n-2}(\Delta f)_{\mu+2} + E_n(\Delta_0 f)_\mu\right].$$

It is easy to see that both terms in the right hand side are necessary.

- If f is a harmonic function, then $\Delta f = 0$ and we need $E_n(\Delta_0 f)_\mu$.
- If f is a radial function, $f(x) = f_0(\|x\|)$, then $\Delta_0 f = 0$ and we need $E_{n-2}(\Delta f)_{\mu+2}$.

Sketch of the proof of Theorem 1

1. Let $f \in \mathcal{W}_2^2(\varpi_\mu, \mathbb{B}^d)$, from

   $$\Delta P_{j,\nu}^{n,\mu}(x) = \kappa_{n-j}^{\mu} P_{j-1,\nu}^{n-2,\mu+2}(x) \quad \text{and} \quad \Delta_0 P_{j,\nu}^{n,\mu}(x) = \lambda_{n-2j} P_{j,\nu}^{n,\mu}(x),$$

   and the conmutativity of the operators, we deduce

   $$\widehat{\Delta f}_{j,\nu}^{n-2,\mu+2} = \kappa_{n-j-1}^{\mu} \widehat{f}_{j+1,\nu}^{n,\mu} \quad \text{and} \quad \widehat{\Delta_0 f}_{j,\nu}^{n,\mu} = \lambda_{n-2j} \widehat{f}_{j,\nu}^{n,\mu}.$$

2. From Parseval's identity,

   $$E_n(f)_\mu^2 = \|f - S_n^\mu f\|_\mu^2 = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \left| \widehat{f}_{j,\nu}^{m,\mu} \right|^2 h_{j,m}^{\mu} = \Sigma_1 + \Sigma_2,$$

   where we split the sum as

   $$\Sigma_1 = \sum_{m=n+1}^{\infty} \sum_{j=\lfloor \frac{m}{4} \rfloor}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \left| \widehat{f}_{j,\nu}^{m,\mu} \right|^2 h_{j,m}^{\mu} \quad \Sigma_2 = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{4} \rfloor - 1} \sum_{\nu} \left| \widehat{f}_{j,\nu}^{m,\mu} \right|^2 h_{j,}^{\mu}$$

Sketch of the proof of Theorem 1

3. From the definition of $h^{\mu}_{j,m}$ we get

$$\frac{h^{\mu}_{j,m}}{h^{\mu+2}_{j-1,m-2}} = \frac{(\mu+1)_2(m-j-1+\frac{d}{2})(m-j+\mu+\frac{d}{2})}{(\mu+1+\frac{d}{2})_2 j(\mu+j+1)},$$

which is bounded by a constant, independent of m, when $\lfloor \frac{m}{4} \rfloor \leq j \leq \lfloor \frac{m}{2} \rfloor$. Consequently, it follows that

$$\Sigma_1 \leq c \sum_{m=n+1}^{\infty} \sum_{j=\lfloor \frac{m}{4} \rfloor}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} m^{-4} \left| \widehat{\Delta f}^{m-2,\mu+2}_{j-1,\nu} \right|^2 h^{\mu+2}_{j-1,m-2}$$

$$\leq \frac{c}{n^4} E_{n-2}(\Delta f)^2_{\mu+2}.$$

Sketch of the proof of Theorem 1

4. We have

$$\left|\widehat{f}_{j,\nu}^{m,\mu}\right|^2 = (\lambda_{m-2j})^{-2}\left|\widehat{\Delta_0 f}_{j,\nu}^{m,\mu}\right|^2 \sim m^{-4}\left|\widehat{\Delta_0 f}_{j,\nu}^{m,\mu}\right|^2$$

for $0 \leq j \leq \lfloor \frac{m}{4} \rfloor$. Consequently, it follows that

$$\begin{aligned}\Sigma_2 &\leq c \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{4} \rfloor - 1} \sum_{\nu} m^{-4}\left|\widehat{\Delta_0 f}_{j,\nu}^{m,\mu}\right|^2 h_{j,m}^{\mu} \\ &\leq \frac{c}{n^4} E_n(\Delta_0 f)_{\mu}^2.\end{aligned}$$

Theorem 2

Let $f \in W_2^1(\varpi_\mu, \mathbb{B}^d)$. Then, for $n \geq 1$,

$$E_n(f)_\mu \leq \frac{c}{n}\left[\sum_{i=1}^{d} E_{n-1}(\partial_i f)_{\mu+1} + \sum_{1\leq i<j\leq d} E_n(D_{i,j} f)_\mu\right].$$

Fourier orthogonal expansions and approximation

Theorem 3

Let $s \in \mathbb{N}$ and let $f \in W_2^{2s}(\varpi_\mu, \mathbb{B}^d)$. Then, for $n \geq 2s$,

$$E_n(f)_\mu \leq \frac{c}{n^{2s}} \left[E_{n-2s}(\Delta^s f)_{\mu+2s} + E_n(\Delta_0^s f)_\mu\right].$$

By its definition, $E_n(f)_\mu \leq \|f\|_\mu$, which allows us to state the estimates in terms of norms of the derivatives.

Corollary

Let $s \in \mathbb{N}$ and let $f \in W_2^{2s}(\varpi_\mu, \mathbb{B}^d)$. Then, for $n \geq 2s$,

$$E_n(f)_\mu \leq \frac{c}{n^{2s}} \left(\|\Delta^s f\|_{\mu+2s} + \|\Delta_0^s f\|_\mu\right).$$

Theorem 4

Let $f \in W_2^{2s+1}(\varpi_\mu, \mathbb{B}^d)$. Then, for $n \geq 2s+1$,

$$E_n(f)_\mu \leqslant$$

$$\leqslant \frac{c}{n^{2s+1}} \left[\sum_{i=1}^{d} E_{n-2s-1}(\partial_i \Delta^s f)_{\mu+2s+1} + \sum_{1 \le i < j \le d} E_n(D_{i,j} \Delta_0^s f)_\mu \right].$$

For Further Reading

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Thank you!