Inverse potential problems for elliptic PDE, with physical applications: magnetic moments recovery from partial field measurements

SIGMA'2016, CIRM, Luminy

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From joint works with: Laurent Baratchart, Sylvain Chevillard, Doug Hardin, Eduardo Lima, Jean-Paul Marmorat, Dmitry Ponomarev

Context

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- Planetary sciences, paleomagnetism, remanent magnetization of ancient rocks $\rightsquigarrow$ history and future of Earth magnetic field
- Magnetization not measurable
  $\rightsquigarrow$ measures of generated magnetic field
  $\rightsquigarrow$ inverse problems, non destructive inspection

How do rocks acquire magnetization?

- Igneous rocks (from Earth volcanoes, lava, magma; basalts)

- Thermoremanent magnetization, ferromagnetic particles (small magnets) follow the magnetic field:

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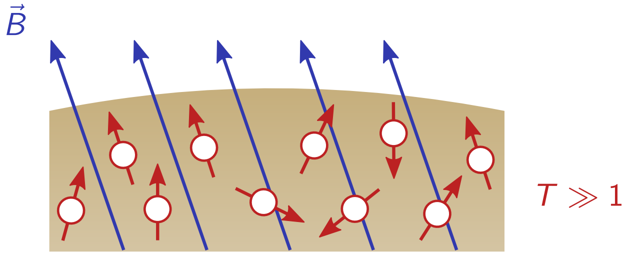

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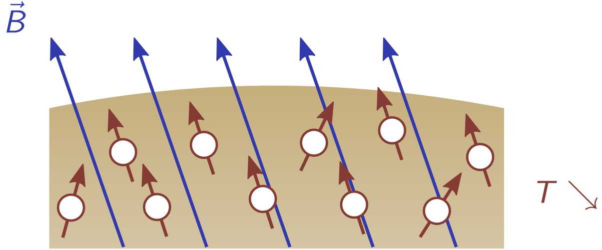

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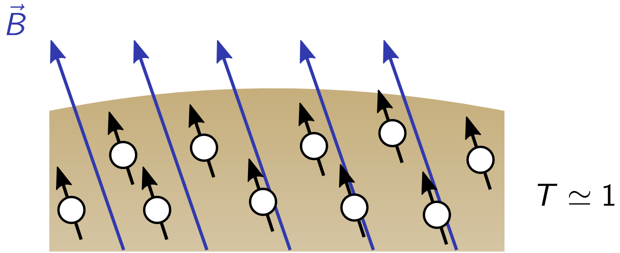

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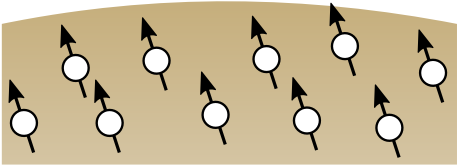

$\vec{M}$

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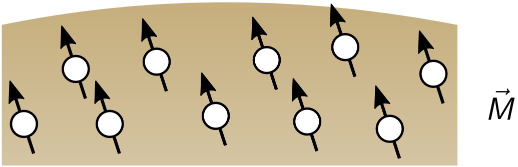


- Can be subsequently altered, under high pressure or temperature

SQUID microscope

Scanning magnetic microscopes: (MIT, EAPS)
for weakly magnetized small samples

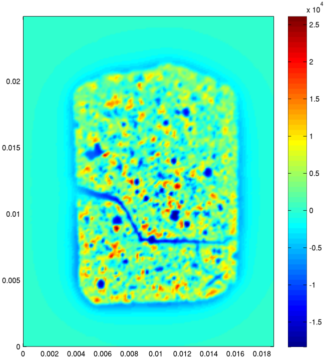
pedestal + sensor

sapphire window

$\rightsquigarrow$ map of the vertical component of the (tiny) magnetic field
on a rectangular region above the sample

Hawaiian Basalt example

picture of thin sample

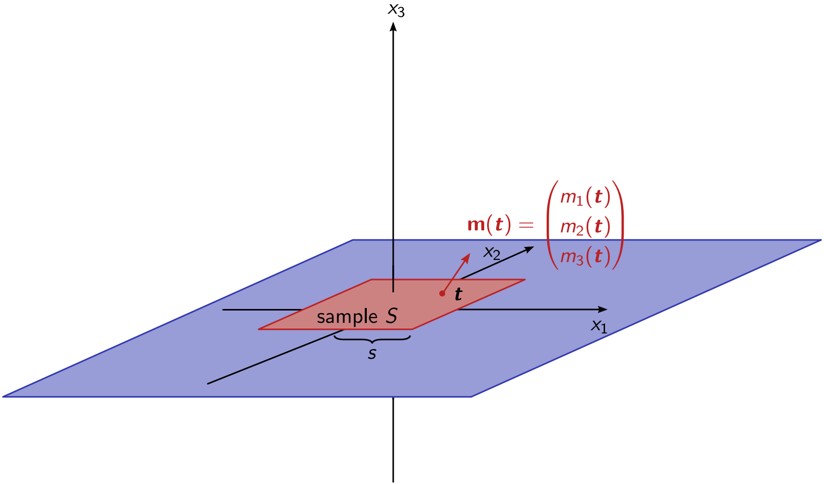


vertical component of the magnetic field (nT)

General scheme

Thin sample $\rightsquigarrow$ planar (rectangle) $S \subset \mathbb{R}^2$

support of unknown magnetization (source term) $\vec{M} = \boldsymbol{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$



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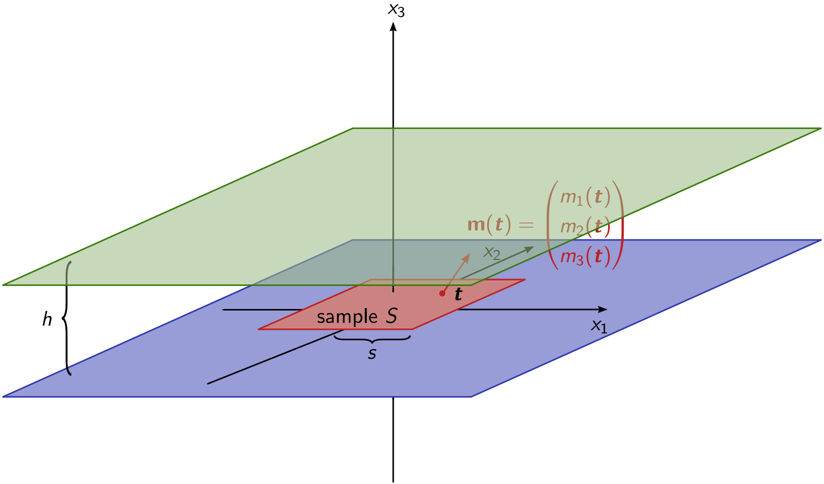

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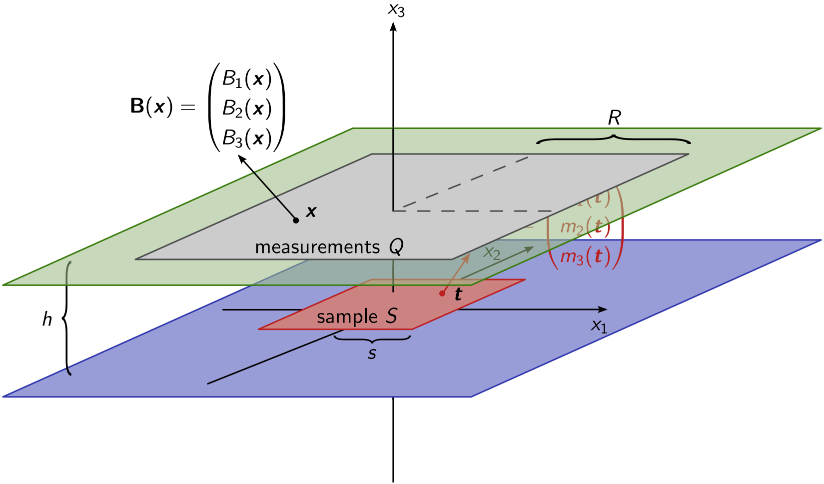

Problem setting

- Generates a magnetic field at $\boldsymbol{x} \notin S$: $\boldsymbol{B}[\boldsymbol{m}](\boldsymbol{x}) = -\mu_0 \nabla U[\boldsymbol{m}](\boldsymbol{x})$

$$\text{where} \quad U[\boldsymbol{m}](\boldsymbol{x}) = \frac{1}{4\pi} \iint_S \frac{\langle \boldsymbol{m}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle}{|\boldsymbol{x} - \boldsymbol{y}|^3} \mathrm{d}\boldsymbol{y}$$

μ_0 magnetic constant, ∇ 3D gradient

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μ_0 magnetic constant, ∇ 3D gradient

- Measurements $b_3[\boldsymbol{m}]$ of vertical component $B_3[\boldsymbol{m}] = -\mu_0 \partial_{x_3} U[\boldsymbol{m}]$ performed on square $Q \subset \mathbb{R}^2$ at height h (incomplete data)

∂_{x_i} partial derivative w.r.t. x_i

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- Inverse problems: from pointwise values of $b_3[\boldsymbol{m}]$ on Q recover magnetization $\boldsymbol{m}$ or net moment $\langle \boldsymbol{m} \rangle$ on S average, $m_i \in L^2(S) \subset L^1(S)$, $i = 1, 2, 3$

$$\langle \boldsymbol{m} \rangle = \begin{pmatrix} \langle m_1 \rangle \\ \langle m_2 \rangle \\ \langle m_3 \rangle \end{pmatrix}, \quad \langle m_i \rangle = \iint_S m_i(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

or suitable extension of $b_3[\boldsymbol{m}]$ to $\mathbb{R}^2$? then Fourier...

Magnetic, harmonic quantities

- Scalar magnetic potential $U[\boldsymbol{m}]$ and magnetic field $\boldsymbol{B}[\boldsymbol{m}]$

  Maxwell, magnetostatics, time-harmonic, macroscopic

  $$\boldsymbol{B}[\boldsymbol{m}] = -\mu_0 \, \nabla U[\boldsymbol{m}] \Rightarrow B_3[\boldsymbol{m}] = -\mu_0 \, \partial_{x_3} U[\boldsymbol{m}]$$

  $$\rightsquigarrow \text{ data: } b_3[\boldsymbol{m}] = B_3[\boldsymbol{m}]_{|Q \times \{h\}} = (\partial_{x_3} U[\boldsymbol{m}])_{|Q \times \{h\}}$$

- Poisson-Laplace equation in divergence form in $\mathbb{R}^3$ $\quad \Delta$ 3D Laplacian

  $$\Delta \, U \, [\boldsymbol{m}] = \operatorname{div} \boldsymbol{m}$$

- $U[\boldsymbol{m}]$ and $B_3[\boldsymbol{m}]$ harmonic functions in $\{x_3 > 0\}$

  $\boldsymbol{m}$ supported on $\overline{S} \subset \mathbb{R}^2$

  $$\Delta \, U[\boldsymbol{m}] = \Delta \, B_3[\boldsymbol{m}] = 0 \text{ in } \{x_3 > 0\}$$

$\rightsquigarrow$ magnetization: $\boldsymbol{m}$ or net moment (average) $\langle \boldsymbol{m} \rangle$ on $S \times \{0\}$?

Operators

Put $X = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\text{grad} \;= \nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} \text{ with } \partial_{x_i} = \frac{\partial}{\partial_{x_i}}$$

$$\text{div} \;= \nabla \cdot, \quad \text{curl} \;= \nabla \times$$

$$\text{Laplacian} \;= \Delta = \text{div}\,(\text{grad}) = \nabla \cdot \nabla = \partial^2_{x_1^2} + \partial^2_{x_2^2} + \partial^2_{x_3^2}$$

Maxwell's equations (magnetostatics)

Quasi-static assumptions **H** magnetic field

Ampère's law, no external current density ($\mathcal{J} = 0$):

$\nabla \times \mathbf{H} = 0 \Rightarrow \mathbf{H} = -\nabla U$ U magnetic potential (scalar)

magnetic flux density (induction) **B**: $\nabla \cdot \mathbf{B} = 0$

with constitutive relation: $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{m})$ for magnetization $\mathbf{m}$

μ_0 magnetic permeability ($\mathbf{B} = \mu_0 \, \mathbf{H}$ outside support of $\mathbf{m}$ whence in $\{x_3 > 0\}$)

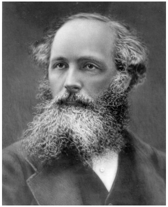
$$\Rightarrow \Delta U = \nabla \cdot \mathbf{m} = \operatorname{div} \mathbf{m}$$

$\rightsquigarrow$ Laplace-Poisson PDE with source term in div form

Interlude: portraits

Laplace
1749-1827

Poisson
1781-1840



Maxwell
1831-1879

Inverse recovery problems

$$\begin{cases} \Delta\, U[\boldsymbol{m}] = \nabla \cdot \boldsymbol{m} \ \text{ in } \mathbb{R}^3\,, \quad \text{supp } \boldsymbol{m} \subset S \times \{0\} \\ b_3[\boldsymbol{m}] = (\partial_{x_3} U[\boldsymbol{m}])_{|Q \times \{h\}} \quad \in L^2(Q) \end{cases}$$

$$b_3[\boldsymbol{m}] \text{ on } Q \times \{h\} \rightsquigarrow \langle \boldsymbol{m} \rangle \text{ or } \boldsymbol{m} \text{ on } S \times \{0\}\,?$$

$$\langle \boldsymbol{m} \rangle = \iint_S \boldsymbol{m}(\boldsymbol{y})\, d\,\boldsymbol{y} \in \mathbb{R}^3\,, \quad \boldsymbol{m} \in [L^2(S)]^3$$

or $b_3[\boldsymbol{m}]$ on $Q \times \{h\} \rightsquigarrow B_3[\boldsymbol{m}]$ on $\mathbb{R}^2 \times \{h\}$?

$B_3[\boldsymbol{m}] \in L^2(\mathbb{R}^2)$

Inverse recovery problems

- Net moment estimation:
  uniqueness, unstability $\rightsquigarrow$ regularization (BEP)

- Preliminary step for magnetization recovery (non-uniqueness):
  $\rightsquigarrow$ mean values $\langle m_i \rangle$ of m_i on S $\rightsquigarrow$ $\langle \boldsymbol{m} \rangle$
  $\rightsquigarrow$ direction and other informations for estimation of $\boldsymbol{m}$

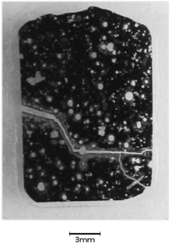
Unidirectional $\begin{pmatrix} m_1(\boldsymbol{y}) \\ m_2(\boldsymbol{y}) \\ m_3(\boldsymbol{y}) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} m(\boldsymbol{y}) \Rightarrow \langle \boldsymbol{m} \rangle = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \langle m \rangle$: moment furnishes direction

(m real valued)

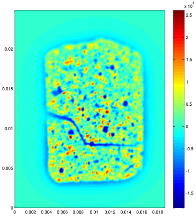
Also for S , $Q \subset \mathbb{R}^2$ open bounded connected Lipschitz-smooth

(whose boundaries are locally graphs of Lipschitz continuous functions)

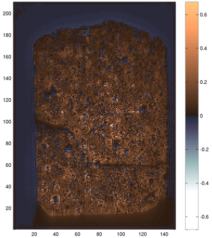
⇝ magnetization recovery



picture of thin sample



vertical component of the magnetic field (nT)



estimated magnetization (Am^2)

knowing direction and after re-magnetization (MIT, EAPS)

Inverse recovery problems

- Linear map:

$$b_3 : \begin{cases} [L^2(S)]^3 \to L^2(Q) \\ \boldsymbol{m} \mapsto B_3[\boldsymbol{m}]_{|Q\times\{h\}} = b_3[\boldsymbol{m}] \end{cases}$$

- Magnetization recovery (full inversion): recover $\boldsymbol{m}$ from $b_3[\boldsymbol{m}]$
  $\rightsquigarrow$ ill-posed (non uniqueness) because Ker $b_3 \neq \{0\}$, silent sources

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- Moment recovery: well-defined $i \in \{1, 2, 3\}$

$$\begin{cases} \text{Ran } b_3 \to \mathbb{R} \\ b_3[\boldsymbol{m}] \mapsto \langle m_i \rangle \end{cases}$$

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- Strategy: linear estimator $\forall \boldsymbol{m}$ of given norm, once and for all
find test functions $\phi_i \in L^2(Q)$ such that $\langle \phi_i, \, b_3[\boldsymbol{m}] \rangle_{L^2(Q)} \simeq \langle m_i \rangle$
$\rightsquigarrow$ best constrained approximation problems (BEP)

and asymptotic formulas, for large measurement area Q

Direct inversion of associated discrete problem: heavy, unstable

Strategy for net moment recovery

Determine $\phi_i \in L^2(Q)$ such that

$$\langle b_3[\boldsymbol{m}], \phi_i \rangle_{L^2(Q)} = \langle \boldsymbol{m}, b_3^*[\phi_i] \rangle_{[L^2(S)]^3} \simeq \langle m_i \rangle = \langle \boldsymbol{m}, \boldsymbol{e}_i \rangle_{[L^2(S)]^3}$$

with adjoint operator $b_3^* : L^2(Q) \to [L^2(S)]^3$ to b_3 and $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0)$, $\boldsymbol{e}_3 = (0, 0, 1)$ on S

($\boldsymbol{e}_1 = (\chi_S, 0, 0)$, $\boldsymbol{e}_2 = (0, \chi_S, 0)$, $\boldsymbol{e}_3 = (0, 0, \chi_S)$ on $\mathbb{R}^2$)

⇝ linear estimator for moment $\langle m_i \rangle$ — given bound on $\|\boldsymbol{m}\|_{[L^2(S)]^3}$

⇝ trade-off between prescribed accuracy $\simeq$ and norm of ϕ_i

in Sobolev space, L^2 norm of gradient of ϕ

⇝ stability w.r.t. measurement errors, robustness

Operator b_3

- Integral expression

also in terms of Poisson and Riesz transforms

with $\boldsymbol{y} = (y_1, y_2)$, for $(x_1, x_2) \in Q$:

$$b_3\left[\boldsymbol{m}\right](x_1, x_2) = -\frac{\mu_0}{4\pi} \times$$

$$\left(\partial_{x_3} \iint_S \frac{m_1(\boldsymbol{y})\,(x_1 - y_1) + m_2(\boldsymbol{y})\,(x_2 - y_2) + m_3(\boldsymbol{y})\,x_3}{\left((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2\right)^{3/2}}\, d\boldsymbol{y} \right)_{|x_3 = h}$$

- $b_3 : [L^2(S)]^3 \to L^2(Q)$ continuous, $\|b_3[\boldsymbol{m}]\|_{L^2(Q)} \lesssim \|\boldsymbol{m}\|_{[L^2(S)]^3}$

Magnetic quantities, harmonic analysis

- Poisson kernel of upper half-space: $(x_1, x_2) \in \mathbb{R}^2$, $x_3 > 0$

$$P_{x_3}(x_1, x_2) = \frac{x_3}{2\,\pi\ ({x_1}^2 + {x_2}^2 + {x_3}^2)^{3/2}}$$

- For $\boldsymbol{m} \in \left[L^2(S)\right]^3$, $h > 0$

$$\widetilde{\boldsymbol{m}} = \begin{cases} \boldsymbol{m} \text{ on } S \\ 0 \text{ outside } S \end{cases}, \ \widetilde{\boldsymbol{m}} = (\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3) \in [L^2(\mathbb{R}^2)]^3$$

$$b_3[\boldsymbol{m}] = -\frac{\mu_0}{2}\left(\partial_{x_1} P_h \star \widetilde{m}_1 + \partial_{x_2} P_h \star \widetilde{m}_2 + [\partial_{x_3} P_{x_3} \star \widetilde{m}_3]_{|x_3=h}\right)_{|Q}$$

$(x_1, x_2) \in Q$, $x_3 = h$

Operator b_3^*

- Adjoint operator: for $\phi \in L^2(Q)$, $\widetilde{\phi} = \phi$ on Q, 0 outside Q

$$b_3^\star[\phi] = \frac{\mu_0}{2} \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ -\partial_{x_3} \end{pmatrix} (P_{x_3} \star \widetilde{\phi})_{|x_3=h}$$

- $b_3^\star$ injective

- Ran $b_3^\star = \mathcal{D}_S^\perp = \nabla_2 \left[W^{1,2}(S) \right] \times L^2(S)$

2D gradient $\nabla_2 = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}$

Bounds on $\|b_3\|$ and $\|b_3^*\|$, Fourier transform of P_h

Operator b_3^*

- Lemma: $\mathbb{R}^2 \supset \Omega \neq \emptyset$ open

If $g \in L^2(\mathbb{R}^2)$ is such that $[\partial_{x_3} P_{x_3} \star g]_{|x_3=h} = 0$ or $\nabla_2 P_h \star g = 0$ on Ω, then $g = 0$

Proof: harmonicity, real analyticity, Poisson

$\rightsquigarrow$ $b_3^\star$ injective

- Silent sources: divergence free, Helmholtz- / Hardy-Hodge decompositions

$$\text{Ker}\, b_3 = \mathcal{D}_S = \left\{ (-\partial_{x_2} \psi, \partial_{x_1} \psi, 0) \,,\ \psi \in W_0^{1,2}(S) \right\}$$

- $[L^2(S)]^3 = \text{Ker } b_3 \oplus_\perp \overline{\text{Ran } b_3^\star}$

$\rightsquigarrow$ $\text{Ran } b_3^\star = \mathcal{D}_S^\perp = \nabla_2 \left[W^{1,2}(S) \right] \times L^2(S)$

Hilbert Sobolev spaces

$\mathbb{R}^2 \supset \Omega \neq \emptyset$ open bounded Lipschitz-smooth

- $W_0^{1,2}(\Omega) \subset L^2(\Omega)$: functions with $L^2(\Omega)$ first derivatives, that vanish on boundary $\partial\Omega$

- $g \in L^2(\Omega)$ belongs to $W_0^{1,2}(\Omega) \Leftrightarrow \nabla_2 \, g \in [L^2(\Omega)]^2$ and $g_{|\partial\Omega} = 0$

- Poincaré inequality: $\|g\|_{W_0^{1,2}(\Omega)} \simeq \|\nabla_2 \, g\|_{[L^2(\Omega)]^2}$

Net moment recovery: uniqueness

Silent sources possess vanishing net moments

$\rightsquigarrow$ uniqueness of $\langle \boldsymbol{m} \rangle$ from $b_3 [\boldsymbol{m}]$

Indeed, recall $\langle m_i \rangle = \langle \boldsymbol{m}, \boldsymbol{e}_i \rangle_{\left[L^2(S)\right]^3}$

with $\boldsymbol{e}_1 = (1,0,0)$, $\boldsymbol{e}_2 = (0,1,0)$, $\boldsymbol{e}_3 = (0,0,1)$ on S

$\boldsymbol{e}_1 = (\chi_S, 0, 0)$, $\boldsymbol{e}_2 = (0, \chi_S, 0)$, $\boldsymbol{e}_3 = (0, 0, \chi_S)$ on $\mathbb{R}^2$

and $\boldsymbol{e}_i \in \nabla_2 \left[W^{1,2}(S)\right] \times L^2(S) = \mathcal{D}_S^{\perp} \subset [L^2(S)]^3$

Hence, $\boldsymbol{m} \in \text{Ker } b_3 = \mathcal{D}_S$ (silent) $\Rightarrow \langle m_i \rangle = 0$

Net moment recovery, strategy

$\boldsymbol{e}_i \in \overline{\text{Ran } b_3^*} = \mathcal{D}_S^\perp$, though $\boldsymbol{e}_i \notin \text{Ran } b_3^*$ yet from Lemma

Hence, no hope to find $\phi \in L^2(Q)$ such that

$$\langle b_3\,[\boldsymbol{m}]\,,\,\phi\rangle_{L^2(Q)} - \langle m_i\rangle$$

vanishes for arbitrary $\boldsymbol{m} \in [L^2(S)]^3$

$$\langle b_3\,[\boldsymbol{m}]\,,\,\phi\rangle_{L^2(Q)} - \langle m_i\rangle = \langle b_3\,[\boldsymbol{m}]\,,\,\phi\rangle_{L^2(Q)} - \langle \boldsymbol{m}\,,\,\boldsymbol{e}_i\rangle_{\left[L^2(S)\right]^3} = \langle \boldsymbol{m}\,,\,b_3^*\,[\phi] - \boldsymbol{e}_i\rangle_{\left[L^2(S)\right]^3}$$

Density result

However

$$\inf_{\phi \in W_0^{1,2}(Q)} \left\| b_3^*[\phi] - \boldsymbol{e}_i \right\|_{[L^2(S)]^3} = 0$$

hence for $\boldsymbol{m} \in [L^2(S)]^3$, $\inf_{\phi \in W_0^{1,2}(Q)} \left| \langle b_3[\boldsymbol{m}], \phi \rangle_{L^2(Q)} - \langle m_i \rangle \right| = 0$

- There exists $\phi_n \in W_0^{1,2}(Q)$ such that $\left\| b_3^*[\phi_n] - \boldsymbol{e}_i \right\|_{[L^2(S)]^3} \to 0$ as $n \to \infty$
- It satisfies $\left\| \nabla_2 \phi_n \right\|_{[L^2(Q)]^2} \to \infty$

Proof: $\boldsymbol{e}_i \in \overline{\text{Ran } b_3^*} \setminus \text{Ran } b_3^*$, $W^{1,2}(Q) \subset L^2(Q)$ compact, $W_0^{1,2}(Q) \subset W^{1,2}(Q)$ closed, Poincaré inequality, b_3^* continuous

Consequence

For $\boldsymbol{m} \in \left[L^2(S)\right]^3$, we can find $\phi \in W_0^{1,2}(Q) \subset L^2(Q)$ such that

$$\left|\langle b_3\,[\boldsymbol{m}]\,,\,\phi\rangle_{L^2(Q)} - \langle m_i\rangle\right| \leq \|b_3^*\,[\phi] - \boldsymbol{e}_i\|_{[L^2(S)]^3}\,\|\boldsymbol{m}\|_{[L^2(S)]^3}$$

arbitrarily small, at the expense of arbitrarily large $\|\nabla_2\,\phi\|_{[L^2(Q)]^2}$

or $\|\phi\|_{W^{1,2}(Q)}$, Poincaré

Unstability $\rightsquigarrow$ regularization, trade-off (error / constraint)

Bounded extremal problem

Best constrained approximation, optimization issue $\qquad M > 0$

- (BEP) Find $\phi_* \in W_0^{1,2}(Q)$, $\|\nabla_2 \phi_*\|_{[L^2(Q)]^2} \leq M$ such that

$$\min_{\phi \in W_0^{1,2}(Q),\, \|\nabla_2 \phi\|_{[L^2(Q)]^2} \leq M} \|b_3^* [\phi] - \boldsymbol{e}_i\|_{[L^2(S)]^3} = \|b_3^* [\phi_*] - \boldsymbol{e}_i\|_{[L^2(S)]^3}$$

$$\min_{\phi \in W_0^{1,2}(Q)} \left[\|b_3^* [\phi] - \boldsymbol{e}_i\|^2_{\left[L^2(S)\right]^3} + \lambda \, \|\nabla_2 \phi\|^2_{\left[L^2(Q)\right]^2} \right] , \ \lambda > 0$$

- $\exists$ unique solution ϕ_* to (BEP) and $\|\nabla_2 \phi_*\|_{[L^2(Q)]^2} = M$

Proof: orthogonal best approximation projection on closed (b_3^* continuous) convex

$$b_3^* \, \{\phi \in W_0^{1,2}(Q) \, , \ \|\nabla_2 \phi_*\|_{[L^2(Q)]^2} \leq M\} \subset \text{Ran } b_3^* \subset [L^2(S)]^3$$

$\boldsymbol{e}_i \in \overline{\text{Ran } b_3^*} \setminus \text{Ran } b_3^* \Rightarrow$ constraint saturated

Critical point equation

- Solution ϕ_* to (BEP) satisfies critical point equation (CPE)

$\exists$ unique $\lambda > 0$ such that $\|\nabla_2\, \phi_*\|_{[L^2(Q)]^2} = M$ and

$$\text{(CPE)} \qquad b_3\, b_3^*\, [\phi_*] - \lambda\, \Delta_2\, \phi_* = b_3\, [\boldsymbol{e}_i] \;\text{ on }\; Q$$

Proof: using $\perp$ best approximation projection, or by differentiation and density of $W_0^{1,2}(Q)$ in $L^2(Q)$; indeed, $\exists$ unique $\lambda \in \mathbb{R}$ such that $\forall \delta \in W_0^{1,2}(Q)$,

$$\langle b_3^*[\phi_*] - \boldsymbol{e}_i\,,\, b_3^*[\delta]\rangle_{[L^2(S)]^3} + \lambda \langle \nabla_2 \phi_*\,,\, \nabla_2 \delta\rangle_{[L^2(Q)]^2} = \langle b_3 b_3^*[\phi_*] - b_3[\boldsymbol{e}_i] - \lambda \Delta_2 \phi_*\,,\, \delta\rangle_{L^2(Q)} = 0$$

argument of minimum $\Rightarrow \lambda \geq 0$; relation below and b_3^* injective $\Rightarrow \lambda > 0$

Critical point equation

- Lagrange parameter λ, constraint M, error in (BEP):

$$\langle b_3^* \, [\phi_*] - \boldsymbol{e}_i \, , \, b_3^* \, [\phi_*] \rangle_{[L^2(S)]^3} = -\lambda \, M^2$$

and $\lambda \to 0$ as $M \to \infty$ by density result

- Variational formulation, $\forall \psi \in W_0^{1,2}(Q)$, $\lambda > 0$:

$$\langle b_3^* \, [\phi_*] \, , \, b_3^* \, [\psi] \rangle_{[L^2(S)]^3} + \lambda \, \langle \nabla_2 \phi_* \, , \, \nabla_2 \psi \rangle_{[L^2(Q)]^2} = \langle \boldsymbol{e}_i \, , \, b_3^* \, [\psi] \rangle_{[L^2(S)]^3}$$

Ongoing, solving (CPE)

$\rightsquigarrow$ numerical magnetometer

- FEM, preliminary numerics, $\phi_* = \phi_* [\boldsymbol{e}_i]$, $i = 1, 2, 3$

  $-\lambda \, \Delta_2 \, \phi_* + b_3 \, b_3^* \; [\phi_*] = b_3 \; [\boldsymbol{e}_i]$ and $\phi_{*|\partial Q} = 0$

- iterative scheme, $\phi_n = \phi_n [\boldsymbol{e}_i] \to \phi_* [\boldsymbol{e}_i]$ $\quad n \to \infty$

  $-\lambda \, \varrho \, \Delta_2 \, \phi_n + \phi_n = -\varrho \, b_3 \, b_3^* \; [\phi_{n-1}] + \phi_{n-1} + \varrho \, b_3 \; [\boldsymbol{e}_i]$ and $\phi_{n|\partial Q} = 0$, $n \geq 1$

- in Fourier basis $\quad$ product of sin of separated variables

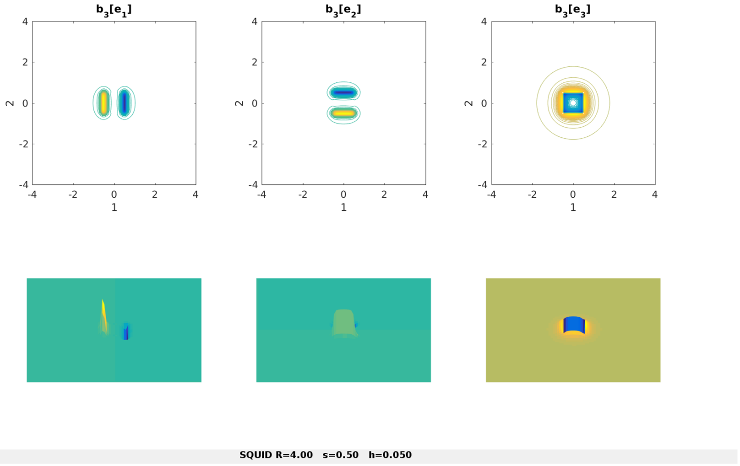
Dirichlet problems, elliptic 2D PDE

Solving (CPE), preliminary numerics

FEM, P1, squares, matlab
right hand sides $b_3[e_i]$

$n_Q = 80 \times 80$

$i = 1, 2, 3$



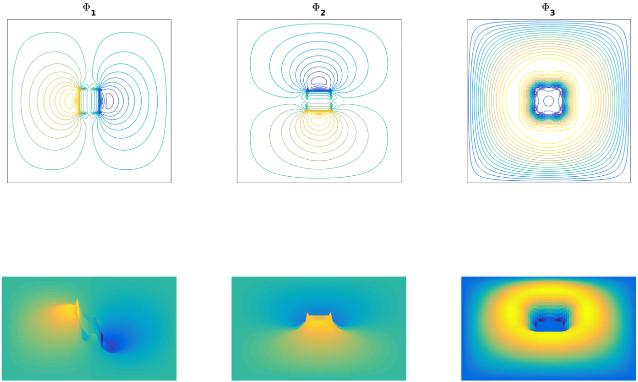
Solving (CPE), preliminary numerics

$\phi_i = \phi_* [\boldsymbol{e}_i], \; i = 1, 2, 3$

$-\lambda \, \Delta_2 \, \phi_i + b_3 \, b_3^* \; [\phi_i] = b_3 \; [\boldsymbol{e}_i]$ and $\phi_{i|\partial Q} = 0$

$\langle \phi_i \, , \, b_3 \; [\boldsymbol{m}_i] \rangle_{L^2(Q)} \simeq \langle \boldsymbol{m}_i \rangle$

$\lambda = 1.5$



SQUID R=4.00 s=0.50 h=0.050 - nQ=80 nS=30 - lambda=1.5

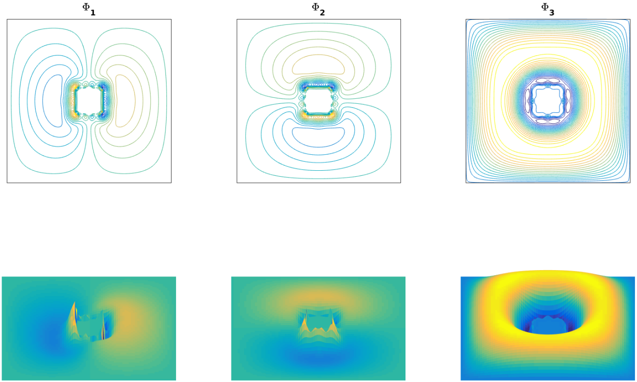
Solving (CPE), preliminary numerics

$$\phi_i = \phi_* \, [\boldsymbol{e}_i], \; i = 1, 2, 3$$

$$-\lambda \, \Delta_2 \, \phi_i + b_3 \, b_3^* \; [\phi_i] = b_3 \; [\boldsymbol{e}_i] \text{ and } \phi_{i|\partial Q} = 0$$

$$\langle \phi_i \, , \, b_3 \; [\boldsymbol{m}_i] \rangle_{L^2(Q)} \simeq \langle \boldsymbol{m}_i \rangle$$

$$\lambda = 0.0001$$

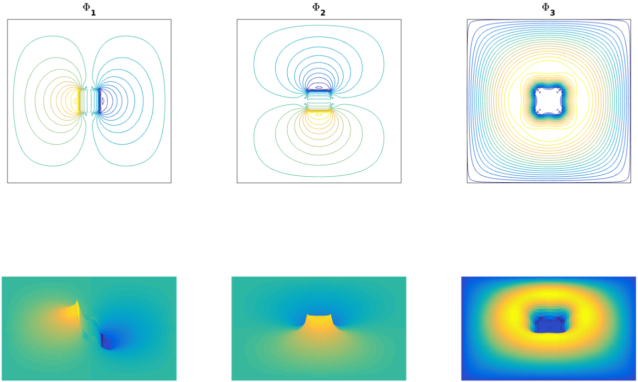


SQUID R=4.00 s=0.50 h=0.050 - nQ=80 nS=30 - lambda=0.0001

Solving (CPE), preliminary numerics

$\phi_i = \phi_* \, [\boldsymbol{e}_i], \; i = 1, 2, 3$ $\qquad$ $-\lambda \, \Delta_2 \, \phi_i + b_3 \, b_3^* \; [\phi_i] = b_3 \; [\boldsymbol{e}_i]$ and $\phi_{i|\partial Q} = 0$

$\langle \phi_i \, , \; b_3 \; [\boldsymbol{m}_i] \rangle_{L^2(Q)} \simeq \langle \boldsymbol{m}_i \rangle$ $\qquad$ $\lambda = 4$

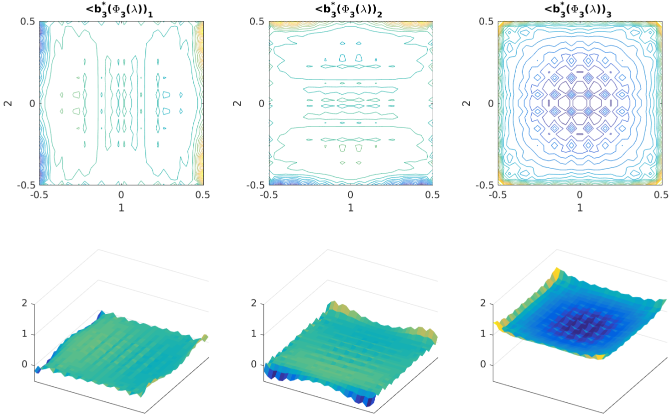


SQUID R=4.00 s=0.50 h=0.050 - nQ=80 nS=30 - lambda=4

Solving (CPE), preliminary numerics

$$\phi_3 = \phi_* [\boldsymbol{e}_3]$$

$\langle b_3^* \ [\phi_3] \ , \ \boldsymbol{e}_i \rangle_{[L^2(S)]^3} \simeq \langle \boldsymbol{e}_3 \rangle = \langle \chi_S \rangle$ if $i = 3$, $\simeq 0$ if $i = 1, 2$ $\qquad = \langle \phi_3 \ , \ b_3 \ [\boldsymbol{e}_i] \rangle_{L^2(Q)}$, $\lambda = 1$



$<b_3^*(\Phi_3(\lambda))$ on S $\qquad \lambda$=1

(CPE), iterative resolution scheme

(CPE): $b_3\, b_3^*\ [\phi_*] - \lambda\, \Delta_2\, \phi_* = b_3\ [\boldsymbol{e}_i]$ on Q

For $n \geq 1$, let $\phi_{n-1} \in W_0^{1,2}(Q)$, $\varrho > 0$. Let ϕ_n on Q s.t.

$$\varrho\, b_3\, b_3^*\ [\phi_{n-1}] - \lambda\, \varrho\, \Delta_2\, \phi_n = \varrho\, b_3\ [\boldsymbol{e}_i] - (\phi_n - \phi_{n-1})$$

then $\phi_n \in W_0^{1,2}(Q)$

$$-\lambda\, \varrho\, \Delta_2\, \phi_n + \phi_n = -\varrho\, b_3\, b_3^*\ [\phi_{n-1}] + \phi_{n-1} + \varrho\, b_3\ [\boldsymbol{e}_i]$$

Convergence result: $\forall \phi_0 \in W_0^{1,2}(Q)$, $n \geq 1$

for ϱ small enough, $\phi_n \to \phi_*$ in $L^2(Q)$ (numerics in progress)

actually, ϕ_n and $\phi_* \in C^\alpha(Q)$ Hölder continuous functions $\in C^\alpha(\overline{Q})$
and $\phi_n \to \phi_*$ in $C^\alpha(Q)$ for $0 \leq \alpha < 1/2$ $n \to \infty$

$$\|\psi\|_{C^\alpha(Q)} = \sup_{\boldsymbol{x},\boldsymbol{y} \in Q, \boldsymbol{x} \neq \boldsymbol{y}} \frac{|\psi(\boldsymbol{x}) - \psi(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\alpha}$$

Also, asymptotic estimations

Specific functions ϕ_i on large $Q = [-R, R]^2$ also other shapes

$$\langle b_3^*[\phi_i], \boldsymbol{m} \rangle_{[L^2(S)]^3} = \langle b_3[\boldsymbol{m}], \phi_i \rangle_{L^2(Q)} \simeq \langle m_i \rangle$$

$$\iint_{Q_R} x_i \, b_3[\boldsymbol{m}](x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\mu_0}{2} \langle m_i \rangle \qquad i = 1, 2$$

$$+ \frac{3\mu_0}{\pi R \sqrt{2}} \langle y_i \, m_3 \rangle + \mathcal{O}\left(\frac{1}{R^3}\right)$$

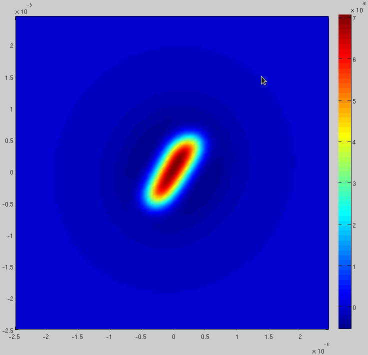
$$\iint_{Q_R} R \, b_3[\boldsymbol{m}](x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 = \frac{2\mu_0}{\pi \sqrt{2}} \langle m_3 \rangle + \mathcal{O}\left(\frac{1}{R^2}\right)$$

Also available for 3D samples

Numerical results (1/2)

$h = 0.27\text{mm}$, $s = 0.53\text{mm}$,

$$\begin{pmatrix} \langle m_1 \rangle \\ \langle m_2 \rangle \\ \langle m_3 \rangle \end{pmatrix} = \begin{pmatrix} -0.06 \\ -0.07 \\ 3.42 \end{pmatrix}$$

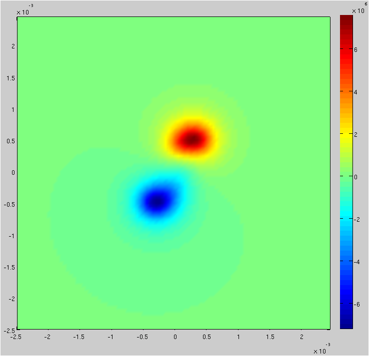


R	R/s	Error on the amplitude	Error on the angle
1.2mm	2.25	-9%	$1.25°$
1.85mm	3.47	-3.7%	$0.17°$
2.45mm	4.6	-0.93%	$0.07°$

Numerical results (2/2)

$h = 0.27$mm, $s = 0.53$mm,

$$\begin{pmatrix} \langle m_1 \rangle \\ \langle m_2 \rangle \\ \langle m_3 \rangle \end{pmatrix} = \begin{pmatrix} 3.53 \\ 6.78 \\ -0.16 \end{pmatrix}$$



R	R/s	Error on the amplitude	Error on the angle
1.2mm	2.25	12.3%	3.17°
1.85mm	3.47	2.36%	0.73°
2.45mm	4.6	1.4%	0.58°

Ongoing, next

- Resolution schemes, numerical analysis (ctn), test actual data
- Local moments determined by $b_3[\boldsymbol{m}]$ if $\boldsymbol{m}$ of minimal $L^2(S)$ norm
- (BEP) for general $\boldsymbol{e} \in \overline{\text{Ran}\, b_3^*} \setminus \text{Ran}\, b_3^* \rightsquigarrow$ higher order moments
- Use $\langle \boldsymbol{m} \rangle$ for magnetization $\boldsymbol{m}$ estimation

References

- L. Baratchart, D. Hardin, E.A. Lima, E.B. Saff, B.P. Weiss, Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions, Inverse Problems, 2013

- L. Baratchart, S. Chevillard, J. Leblond, Silent and equivalent magnetic distributions on thin plates, Theta Series in Advanced Mathematics, to appear, hal-01286117

- L. Baratchart, S. Chevillard, J. Leblond, D. Hardin, E.A. Lima, Magnetic moments estimation and bounded extremal problems, in preparation

Ongoing, next

Planetary sciences, paleomagnetism:

- Magnetizations $\boldsymbol{m}$ in $[L^1(S)]^3$ or distributions

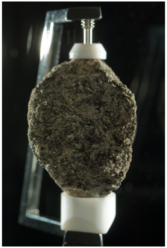
- For 3D samples S, pointwise dipolar sources...

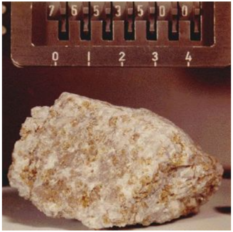
Brain imaging (medical engineering):

- Electroencephalography (EEG), source and conductivity inverse problems, spherical geometry

Lunar paleomagnetism, 3D samples

inverse source recovery problems



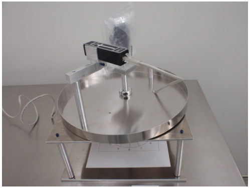


Moon rocks (NASA)

or spherules

Lunar paleomagnetism, 3D samples

CEREGE-CNRS (ANR MagLune), lunometer



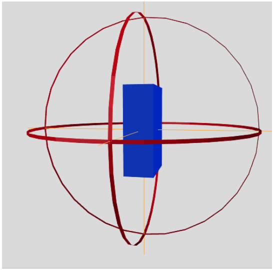
Sparse measures of magnetic field $\boldsymbol{B}[\boldsymbol{m}]$ (magnetometer)
$\rightsquigarrow$ magnetization (in rock), pointwise dipolar source $\boldsymbol{m}$?

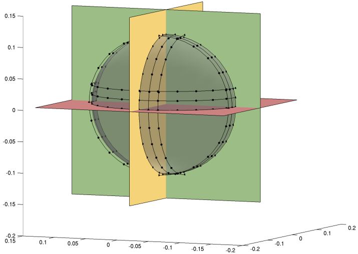
Lunar paleomagnetism, 3D $\leftrightarrow$ 2D

For Moon rocks, dipolar source term $\mathbf{m} = \mathbf{p}\,\delta_{\mathbf{C}}$

with dipole $\mathbf{C} \in S$ (rock sample), moment $\mathbf{p} \in \mathbb{R}^3$

$$\Delta\, U = \nabla \cdot \mathbf{m} = \mathbf{p} \cdot \nabla \delta_{\mathbf{C}}$$





Data: $\mathbf{b} = \mathbf{B}_{|Q}$, $\mathbf{B} \simeq \nabla U = (\partial_\theta U,\, \partial_r U,\, \partial_{x_3} U)$ pointwise (every degree)

on sets Q made of 3 series of 3 (or more) circles, in $\perp$ directions

tangential, radial, vertical components of magnetic field; sparse, spherical / cylindrical

Recover moment $\mathbf{p} = \langle \mathbf{m} \rangle \rightsquigarrow$ first localize source $\mathbf{C}$ ongoing [KM PhD]

Inverse problems, comments

In both situations: $\Delta\, U = \nabla\, .\, \boldsymbol{m}$ in $\mathbb{R}^3$

Data: pointwise values $\mathbf{b} = \mathbf{B}_{|Q}$ of components of $\boldsymbol{B} \simeq \nabla\, U$ (b_3)
on (planar) measurement set $Q \subset \mathbb{R}^3 \setminus S$ (square, circles)
far from magnetization support $S \subset \mathbb{R}^3$ (sample)

Inverse problems: recover
- $\mathbf{m}$ in S or its moment $\langle \mathbf{m} \rangle$ (source $\mathbf{C}$ in S, moment $\mathbf{p}$)

- lacking values of U or / and components of ∇U (outside measurement set Q), or support of $\mathbf{m}$ in S?

Assumptions concerning: $\rightsquigarrow$ existence, uniqueness, stability
- support supp $\mathbf{m} \subset S \subset \mathbb{R}^3$ $\Delta U = 0$ outside $S \subset \mathbb{R}^3$
- models for $\mathbf{m}$ in $L^2(S)$ or pointwise dipolar sources

or more general distributions?

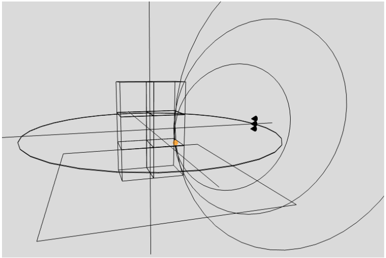
- conductivity (here known, constant) EEG: $\nabla \cdot (\sigma \nabla U) = \nabla \cdot \boldsymbol{m}$

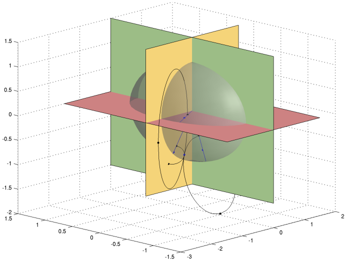
Lunar paleomagnetism, 3D $\leftrightarrow$ 2D

Analysis of denominators of field's components $\quad \boldsymbol{B} \simeq \nabla\, U(X) = \frac{\pi(X)}{|X - C|^5}$

on circles $\subset Q \subset$ planes $\{x_i = \text{ ct}\}$ $\quad$ in the 3 orthogonal directions i

(where data are given)

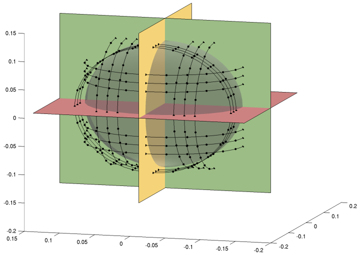


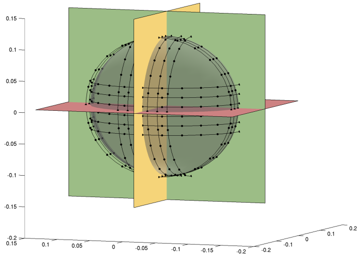


$\rightsquigarrow$ complex variable, best quadratic rational approximation $\quad$ on circles

Lunar paleomagnetism, 3D $\leftrightarrow$ 2D

$\rightsquigarrow$ data on more circles?





$\rightsquigarrow$ 2 or several sources?

(EEG: other data sets, conductivity, several pointwise dipolar sources, ...)