Multilevel quadratic spline quasi-interpolation and application to numerical integration

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Outline

1. Quadratic quasi-interpolant splines
   - Quasi-interpolating spline operators S_1, S_2, W_2
2. Multilevel quasi-interpolant splines
   - Multilevel quasi-interpolating spline operators $S_1^{(p+1)L}$, $S_2^{(p+1)L}$, $W_2^{(p+1)L}$
     - defined by uniform B-splines
     - defined by non uniform B-splines
3. *Matlab* numerical results
4. Application to numerical integration

The spline space $S_2^1(\Lambda_{mn}^{(2)})$

- R=$[a,b]\times[c,d]$ domain in $\mathbb{R}^2$

  m and n integers

  $a = x_0 < x_1 < \ldots < x_m = b$ and $c = y_0 < y_1 < \ldots < y_n = d$

  $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1, \ldots, m$ e $j = 1, \ldots, n$

- $\Lambda_{mn}^{(2)}$ *criss-cross* triangulation of R

- $S_2^1(\Lambda_{mn}^{(2)})$

Let

- $A_{i,j} = (x_i, y_j)\,, \quad -2 \leq i \leq m+2,\ -2 \leq j \leq n+2$
- $h_i := x_i - x_{i-1}, \quad i = -1, ..., m+2,$
  $k_j := y_j - y_{j-1}, \quad j = -1, ..., n+2$
- $\Upsilon = \{M_{i,j} = (s_i, t_j)\,,\}$, with $s_i := \frac{x_{i-1}+x_i}{2}, \quad t_j := \frac{y_{j-1}+y_j}{2},$
  $i = -1, ..., m+2,\ j = -1, ..., n+2$
- $\mathcal{B}_{mn} = \{B_{ij} : (i,j) \in K_{mn}\}$ on extended partitions,

with $K_{mn} := \{(i,j) : 0 \leq i \leq m+1, 0 \leq j \leq n+1\}$

Quasi-interpolating spline operators

$$Q : C(R) \longmapsto S_2^1(\Lambda_{mn}^{(2)})$$

with

$$Qf = \sum_{ij} \lambda_{ij}^{(Q)}(f) B_{ij}(x,y)$$

and

$$\lambda_{ij}^{(Q)}(f) := \sum_{\ell} \nu_{\ell}^{(ij)} f\left(x_{\ell}^{(i)}, y_{\ell}^{(j)}\right),$$

$\nu_{\ell}^{(ij)}$ non zero real numbers s.t. $Qf = f$, $\forall f \in \mathbb{P}_r$, $0 < r \leq 2$.

- S_1 (C.K.Chui, R.-H.Wang, 1984)

$$\lambda_{ij}^{(S_1)}(f) = f\left(M_{ij}\right),$$

$$S_1 f = f, \quad f = 1, x, y, xy$$

- S_2 (P.Sablonniére, 2003)

$$\begin{aligned}\lambda_{ij}^{(S_2)}(f) \quad &:= \quad b_{ij} f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) \\ &+ \quad \bar{a}_j f(M_{i,j-1}) + \bar{c}_j f(M_{i,j+1}),\end{aligned}$$

$$S_2 f = f, \quad f \in \mathbb{P}_2$$

- W_2 (C.K.Chui, R.-H.Wang, 1984)

$$\lambda_{ij}^{(W_2)}(f) := 2f(M_{i,j}) - \frac{1}{4} \sum_{h=-1}^{0} \sum_{k=-1}^{0} f(A_{i+h,j+k}),$$

$$W_2 f = f, \quad f \in \mathbb{P}_2$$

Approximation order

Let

$\Delta = \max\{h, k\}$ with $h = \max_i h_i$, $k = \max_j k_j$;

$||\cdot||_R$ the supremum norm over R;

$\omega(f, \delta) = \max\{|f(x, y) - f(u, v)|, (u, v), (x, y) \in R$
$t.c. \, ||(x, y) - (u, v)|| \leq \delta\}$;

$D^\alpha = D^{(\alpha_1, \alpha_2)} = \partial^{|\alpha|}/(\partial x^{\alpha_1} \partial y^{\alpha_2})$ with $|\alpha| = \alpha_1 + \alpha_2$;

$\omega(D^s f, \delta) = \max\{\omega(D^\alpha f, \delta), |\alpha| = s\}$.

Theorem

Let $Q = S_1$, S_2 e W_2. There exist $C_{0,Q}$, $C_{1,Q}$, $C_{2,Q}$, $C_{3,Q}$, C_{S_1} s.t.

- $f \in C(R) \Rightarrow ||f - Qf||_R \leq C_{0,Q}\omega(f, \Delta)$;
- $f \in C^1(R) \Rightarrow ||f - Qf||_R \leq C_{1,Q}\Delta\omega(Df, \Delta/2)$;
- $f \in C^2(R) \Rightarrow$
  $||f - S_1 f||_R \leq C_{S_1}\Delta^2 \max_{|\alpha|=2} ||D^\alpha f||_R$
  $||f - Qf||_R \leq C_{2,Q}\Delta^2\omega(D^2 f, \Delta/2), \quad Q = S_2, W_2$;
- $f \in C^3(R) \Rightarrow$
  $||f - Qf||_R \leq C_{3,Q}\Delta^3 \max_{|\alpha|=3} ||D^\alpha f||_R \quad Q = S_2, W_2.$

Multilevel quasi-interpolating spline operators $Q^{(p+1)L}$

- $R = [0,1] \times [0,1]$;
- two kinds of extended partitions:
  1. simple knots

$$X_1 : x_{-2} < x_{-1} < a = x_0 < \ldots < x_m = b < x_{m+1} < x_{m+2},$$

$$Y_1 : y_{-2} < y_{-1} < c = y_0 < \ldots < y_n = d < y_{n+1} < y_{n+2},$$

  2. simple knots and triple ones on the boundary of R

$$X_2 : x_{-2} = x_{-1} = a = x_0 < x_1 < \ldots < x_m = b = x_{m+1} = x_{m+2},$$

$$Y_2 : y_{-2} = y_{-1} = c = y_0 < y_1 < \ldots < y_n = d = y_{n+1} = y_{n+2};$$

- 'uniform' criss-cross triangulation $\Lambda_{mn}^{(2)}$ of R

Then for X_1, Y_1 and $\forall i, j$:

- $h_i = h = \frac{1}{m}, \quad k_j = k = \frac{1}{n};$
- $x_i = a + ih = \frac{i}{m}, \quad y_j = c + jk = \frac{j}{n};$
- $M_{ij} = (s_i, t_j) = \left(\frac{2i-1}{2m}, \frac{2j-1}{2n}\right);$
- $B_{ij}(x, y) = B\left(mx - i + \frac{1}{2}, ny - j + \frac{1}{2}\right).$

Similarly for X_2, Y_2.

★ **First step: two-level quasi-interpolation operators**

m, n even

$$\Upsilon^{(0)} = \left\{ M_{ij}^{(0)} = \left(s_i^{(0)}, t_j^{(0)}\right) = (2^0 s_i, 2^0 t_j) = \left(\frac{2i-1}{2m}, \frac{2j-1}{2n}\right), \right.$$
$$\left. i = -\kappa, \ldots, \frac{m}{2^0} + \ell, j = -\kappa, \ldots, \frac{n}{2^0} + \ell \right\},$$

with $\kappa = \kappa(Q, X_i, Y_i, p)$, $\ell = \ell(Q, X_i, Y_i, p)$, $i = 1, 2$, $p = 0, 1$;

$$\Upsilon^{(1)} = \left\{ M_{ij}^{(1)} = \left({s_i}^{(1)}, {t_j}^{(1)}\right) = (2^1 s_i, 2^1 t_j) = \left(\frac{2i-1}{m}, \frac{2j-1}{n}\right), \right.$$
$$\left. i = -\kappa, \ldots, \frac{m}{2^1} + \ell, j = -\kappa, \ldots, \frac{n}{2^1} + \ell \right\},$$

coarser than $\Upsilon^{(0)}$.

Let

- $Q^{(1)}f = \sum_{i=-\kappa}^{\frac{m}{2^1}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^1}+\ell} \lambda_{ij}^{(Q,1)}(f) B_{ij}^{(1)}$,

where $\lambda_{ij}^{(Q,1)}(f)$ is computed at $\Upsilon^{(1)}$ and
$B_{ij}^{(1)} = B\left(\frac{m}{2^1} \cdot -i + \frac{1}{2}, \frac{n}{2^1} \cdot -j + \frac{1}{2}\right)$ with support centre at $M_{ij}^{(1)}$;

- *error function* $\Delta_1 f = f - Q^{(1)}f$

- $Q^{(0)}\Delta_1 f = Q\Delta_1 f = \sum_{i=-\kappa}^{\frac{m}{2^0}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^0}+\ell} \lambda_{ij}^{(Q,0)}(\Delta_1 f) B_{ij}^{(0)}$

with $\lambda_{ij}^{(Q,0)}(f) = \lambda_{ij}^{(Q)}(f), \quad B_{ij}^{(0)} = B_{ij}$

$$\Downarrow$$

$$Q^{(2L)}f = Q^{(1)}f + Q^{(0)}\Delta_1 f$$

Nested scheme with a constraint for p:

Property

Let m and n be s.t.

$$m = u \cdot 2^q, \qquad n = v \cdot 2^r$$

with $u,\ v,\ q,\ r\ \in\ \mathbb{N}$ and u, v odd.
Then it is possible to define at most $(p+1)$-level quasi-interpolating spline operators with

$$p = \min\{q, r\}.$$

★ **Second step: $(p+1)$-level quasi-interpolation operators**

- $\Upsilon^{(p)} = \Phi^{(p)}$ for S_1, S_2, $\quad \Upsilon^{(p)} = \Phi^{(p)} \cup \Theta^{(p)}$ for W_2

where $\Phi^{(p)} = \left\{ M_{ij}^{(p)} = \left(s_i{}^{(p)}, t_j{}^{(p)}\right) = (2^p s_i, 2^p t_j) \right\}$
and $\quad \Theta^{(p)} = \left\{ A_{ij}^{(p)} = \left(x_i{}^{(p)}, y_j{}^{(p)}\right) = (2^p x_i, 2^p y_j) \right\}$,

$i = -\kappa, ..., \frac{m}{2^p} + \ell, j = -\kappa, ..., \frac{n}{2^p} + \ell;$

- $Q^{(p)} f = \sum_{i=-\kappa}^{\frac{m}{2^p}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^p}+\ell} \lambda_{ij}^{(Q,p)}(f) B_{ij}^{(p)}$,

where $\lambda_{ij}^{(Q,p)}(f)$ is computed at $\Upsilon^{(p)}$ and
$B_{ij}^{(p)} = B\left(\frac{m}{2^p} \cdot -i + \frac{1}{2}, \frac{n}{2^p} \cdot -j + \frac{1}{2}\right)$ with support centre at $M_{ij}^{(p)}$;

- $\Delta_p f = f - Q^{(p)} f$ first error function;

- $Q^{(p-1)}\Delta_p f = \sum_{i=-\kappa}^{\frac{m}{2^{p-1}}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^{p-1}}+\ell} \lambda_{ij}^{(Q,p-1)}(\Delta_p f) B_{ij}^{(p-1)}$

on the data set $\Upsilon^{(p-1)}$;

- $\Delta_{p-1}^2 f = \Delta_{p-1}(\Delta_p f) = \Delta_p f - Q^{(p-1)}\Delta_p f$

second error function;

- $Q^{(p-2)}\Delta_{p-1}^2 f = \sum_{i=-\kappa}^{\frac{m}{2^{p-2}}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^{p-2}}+\ell} \lambda_{ij}^{(Q,p-2)}(\Delta_{p-1}^2 f) B_{ij}^{(p-2)}$

on the data set $\Upsilon^{(p-2)}$;

$$\vdots$$

$$\vdots$$

- $\Delta_2^{p-1} f = \Delta_2(\Delta_3^{p-2} f) = \Delta_3^{p-2} f - Q^{(2)} \Delta_3^{p-2} f$ error function;
- $Q^{(1)} \Delta_2^{p-1} f = \sum_{i=-\kappa}^{\frac{m}{2^1}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^1}+\ell} \lambda_{ij}^{(Q,1)} (\Delta_2^{p-1} f) B_{ij}^{(1)}$;
- $\Delta_1^p f = \Delta_1(\Delta_2^{p-1} f) = \Delta_2^{p-1} f - Q^{(1)} \Delta_2^{p-1} f$;
- $Q^{(0)} \Delta_1^p f = \sum_{i=0}^{\frac{m}{2^0}+1} \sum_{j=0}^{\frac{n}{2^0}+1} \lambda_{ij}^{(Q,0)} (\Delta_1^p f) B_{ij}^{(0)}$

$$\Downarrow$$

$$Q^{(p+1)L} f = Q^{(p)} f + Q^{(p-1)} \Delta_p f + Q^{(p-2)} \Delta_{p-1}^2 f + \dots$$

$$\dots + Q^{(2)} \Delta_3^{p-2} f + Q^{(1)} \Delta_2^{p-1} f + Q^{(0)} \Delta_1^p f.$$

General scheme:

$\Upsilon^{(p)} \longrightarrow Q^{(p)}f \longrightarrow \Delta_p f$ — Level p

$\downarrow$

$\Delta^{2}_{p-1} f \longleftarrow Q^{(p-1)}\Delta_p f$ — Level $p-1$

$\downarrow$

$Q^{(p-2)}\Delta^{2}_{p-1} f \longrightarrow \Delta^{3}_{p-2} f$ — Level $p-2$

$\downarrow$

.... — $\vdots$

$\vdots$ — $\vdots$

.... $\longrightarrow \Delta^{p-1}_{2} f$ — Level 2

$\downarrow$

$\Delta^{p}_{1} f \longleftarrow Q^{(1)}\Delta^{p-1}_{2} f$ — Level 1

$\downarrow$

$Q^{(0)}\Delta^{p}_{1} f$ — Level 0

Summing up $Q^{(p-r)}\Delta^{r}_{p-r+1} f$, $r = 0, \ldots, p$ with $\Delta^{0}_{p+1} f = f$, at all levels, we get $Q^{(p+1)L} f$.

In case of partitions X_2, Y_2:

Theorem

Let $Q^{(p+1)L}$ be a $(p+1)$-level quasi-interpolating operator with $Q = S_1,\ S_2,\ W_2$. Then

- $S_1^{(p+1)L} f = f \quad \text{if} \quad f(x,y) = 1, x, y, xy$
- $Q^{(p+1)L} f = f \quad \text{if} \quad f \in \mathbb{P}_2, \quad \text{and} \quad Q = S_2, W_2$

Theorem

Let $Q^{(p+1)L}$ be a $(p+1)$-level quasi-interpolating operator with $Q = S_1,\ S_2,\ W_2$. Then

- *if* $f \in C^\mu(R),\ \mu = 1, 2$
  $S_1^{(p+1)L} f$ *at least has the error estimate* $||f - S_1^{(p+1)L} f||_R = O(\Delta^\mu)$;
- *if* $f \in C^\mu(R),\ \ \mu = 1, 2, 3,$
  $Q^{(p+1)L} f(x,y)$ *at least has the error estimate* $||f - Q^{(p+1)L} f||_R = O(\Delta^\mu)$
  for $Q = S_2, W_2$.

However in case of partitions X_1, Y_1:

Theorem

Let $Q^{(p+1)L}$ be a $(p+1)$-level quasi-interpolating operator with $Q = S_1, S_2, W_2$. Then

$$Q^{(p+1)L} f = f \quad \text{if} \quad f \in \mathbb{P}_2.$$

Theorem

Let $Q^{(p+1)L}$ be a $(p+1)$-level quasi-interpolating operator with $Q = S_1, S_2, W_2$.
Then, if $f \in C^{\mu}(R)$, $\mu = 1, 2, 3$, $Q^{(p+1)L} f$ at least has the error estimate

$$||f - Q^{(p+1)L} f||_R = O(\Delta^{\mu}).$$

with $\Delta = \max\{h, k\}$.

Comments:

- starting from $\Upsilon^{(0)}$, then $\Upsilon^{(k)}$, $k = 1, 2, ..., p$ are coarser and coarser sets;
- in case of partitions X_1, Y_1, data points outside the domain R are needed.

  Problems:

  - data points outside R could be not known;
  - f could be not defined outside R;
- in case of partitions X_2, Y_2, all evaluation points are either inside R or on its boundary.

Matlab numerical results

- $f_1(x,y) = x^2 + 2y$
- $f_2(x,y) = \begin{cases} (\sqrt{x^2+y^2} - 0.6)^4 & \text{se } \sqrt{x^2+y^2} > 0.6 \\ 0 & \text{elsewhere} \end{cases}$
- $f_3(x,y) = \begin{cases} e^{-1/(\sqrt{x^2+y^2}-0.6)^2} & \text{se } \sqrt{x^2+y^2} > 0.6 \\ 0 & \text{elsewhere} \end{cases}$
- $f_4(x,y) = \frac{1}{9}(\tanh(9y - 9x) + 1)$
- $f_5(x,y) = \frac{1.25+\cos(5.4y)}{6(1+(3x-1)^2)}$
- $f_6(x,y) = \frac{1}{3} e^{-\frac{81}{16}((x-\frac{1}{2})^2+(y-\frac{1}{2})^2)}$
- $f_7(x,y) = \frac{1}{3} e^{-\frac{81}{4}((x-\frac{1}{2})^2+(y-\frac{1}{2})^2)}$
- $f_8(x,y) = \frac{1}{9}\sqrt{(64 - 81((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2))} - \frac{1}{2}$
- $f_9(x,y) = \left(1 + 2e^{-3(10\sqrt{x^2+y^2}-6.7)}\right)^{-\frac{1}{2}}$
- $f_{10}(x,y) = e^{-\frac{(5-10x)^2}{2}} + 0.75e^{-\frac{(5-10y)^2}{2}} + 0.75e^{-\frac{(5-10x)^2}{2}} e^{-\frac{(5-10y)^2}{2}}$
- $f_{11}(x,y) = 2\cos(10x)\sin(10y) + \sin(10xy)$
- $f_{12}(x,y) = e^{-0.04\sqrt{(80x-40)^2+(90y-45)^2}} \cos(0.15\sqrt{(80x-40)^2 + (90y-45)^2})$
- $f_{13}(x,y) = \begin{cases} (y-x)^6 & \text{se } y > x \\ 0 & \text{elsewhere} \end{cases}$
- $f_{14}(x,y) = \begin{cases} |x|y & \text{se } xy > 0 \\ 0 & \text{elsewhere} \end{cases}$

	$m=n$	$ES_1(X_1, Y_1)$	$ES_1^{2L}(X_1, Y_1)$	$ES_1(X_2, Y_2)$	$ES_1^{2L}(X_2, Y_2)$
f_1	8	3.90(-03)	1.11(-15)	3.91(-03)	9.71(-04)
	16	9.76(-04)	8.88(-16)	9.77(-04)	2.43(-04)
	32	2.44(-04)	8.88(-16)	2.44(-04)	6.07(-05)
	64	6.10(-05)	8.88(-16)	6.10(-05)	4.80(-06)
	128	1.52(-05)	6.66(-16)	1.53(-05)	8.88(-16)
f_2	8	1.19(-01)	2.80(-02)	1.16(-02)	3.62(-03)
	16	2.94(-02)	2.54(-03)	3.58(-03)	1.06(-03)
	32	7.33(-03)	1.90(-04)	1.02(-03)	2.78(-04)
	64	1.83(-03)	1.89(-05)	2.55(-04)	2.41(-05)
	128	7.23(-05)	2.48(-07)	6.37(-05)	2.48(-07)
f_4	8	1.90(-02)	1.58(-02)	1.90(-02)	1.63(-02)
	16	5.95(-03)	3.72(-03)	5.95(-03)	4.06(-03)
	32	1.63(-03)	4.81(-04)	1.63(-03)	6.08(-04)
	64	4.17(-04)	3.15(-05)	4.17(-04)	4.56(-05)
	128	1.05(-04)	1.74(-06)	1.05(-04)	1.74(-06)
f_6	8	1.28(-02)	4.48(-03)	1.28(-02)	4.48(-03)
	16	3.25(-03)	3.22(-04)	3.25(-03)	3.22(-04)
	32	8.17(-04)	1.99(-05)	8.17(-04)	4.59(-05)
	64	2.05(-04)	2.94(-06)	2.05(-04)	4.23(-06)
	128	5.12(-05)	3.81(-07)	5.12(-05)	3.81(-07)

$S_1^{(p+1)L} f = f$ if $f(x,y) = 1, x, y, xy$ for partitions X_2, Y_2.
However ...

m	n	$ES1^{2L} f_1$	m	n	$ES1^{2L} f_1$
8	8	9.71(-04)	98	98	1.33(-15)
16	16	2.43(-04)	106	106	1.33(-15)
32	32	6.07(-05)	114	114	1.33(-15)
64	64	4.80(-06)	128	128	8.88(-16)
70	70	2.19(-06)	132	132	8.88(-16)
76	76	7.78(-07)	152	152	8.88(-16)
80	80	2.85(-07)	196	196	8.88(-16)
86	86	5.02(-09)	200	200	8.88(-16)
86	32	5.02(-09)	200	204	8.88(-16)
86	64	5.02(-09)	204	204	8.88(-16)
86	80	5.02(-09)	208	208	8.88(-16)
88	80	1.33(-15)	216	216	8.88(-16)
88	88	8.88(-16)	232	232	8.88(-16)
90	80	1.33(-15)	256	256	8.88(-16)
90	90	1.33(-15)	512	512	8.88(-16)
94	94	1.33(-15)	1024	1024	8.88(-16)

	$m=n$	$ES_1(X_1, Y_1)$	$ES_1^{2L}(X_1, Y_1)$	$ES_1(X_2, Y_2)$	$ES_1^{2L}(X_2, Y_2)$
f_8	8	1.43(-02)	1.67(-02)	7.25(-03)	2.00(-03)
	16	3.42(-03)	2.66(-03)	2.21(-03)	6.13(-04)
	32	8.46(-04)	5.55(-05)	6.62(-04)	1.82(-04)
	64	2.11(-04)	3.12(-06)	1.65(-04)	1.61(-05)
	128	5.27(-05)	2.19(-07)	4.11(-05)	1.73(-07)
f_{10}	8	5.24(-01)	4.89(-01)	5.24(-01)	4.89(-01)
	16	1.43(-01)	8.20(-02)	1.43(-01)	8.20(-02)
	32	3.67(-02)	6.15(-03)	3.67(-02)	6.15(-03)
	64	9.43(-03)	4.76(-04)	9.43(-03)	4.76(-04)
	128	2.36(-03)	5.54(-05)	2.36(-03)	5.54(-05)
f_{11}	8	7.20(-01)	6.19(-01)	7.20(-01)	6.19(-01)
	16	2.02(-01)	7.27(-02)	2.02(-01)	7.27(-02)
	32	5.19(-02)	5.55(-03)	5.20(-02)	9.94(-03)
	64	1.31(-02)	4.44(-04)	1.31(-02)	7.30(-04)
	128	3.28(-03)	6.37(-05)	3.28(-03)	6.29(-05)
f_{12}	8	5.73(-01)	6.00(-01)	5.73(-01)	6.00(-01)
	16	1.88(-01)	1.42(-01)	1.88(-01)	1.42(-01)
	32	4.85(-02)	1.67(-02)	4.85(-02)	1.67(-02)
	64	1.17(-02)	3.49(-04)	1.17(-02)	3.49(-04)
	128	2.94(-03)	1.10(-04)	2.94(-03)	1.10(-04)

Let $f \in C(R)$.

$$I(f) = \int_R f(x,y)\,dx\,dy$$

$$\Downarrow$$

$$I(Q^{(p+1)L}f) = \sum_r \sum_i \sum_j w_{ij}^{(Q,r)} f(P_{ij}^{(r)})$$

with

- $Q^{(p+1)L} = S_1^{(p+1)L},\ S_2^{(p+1)L},\ W_2^{(p+1)L}$;
- $P_{ij}^{(r)} \in \Lambda_{mn}^{(2)}$, $r = 0, \ldots, p$, with $P_{ij}^{(0)} = A_{ij}, M_{ij}$;
- $w_{ij}^{(Q,r)}$ as follows:

1. partitions X_1, Y_1:

$$w_{ij}^{(Q,r)} = \int_{\Xi_{ij} \cap R} N_{ij}^{(r)}(x,y)dxdy;$$

2. partitions X_2, Y_2:

$$w_{ij}^{(Q,r)} = \int_{\Xi_{ij}} N_{ij}^{(r)}(x,y)dxdy,$$

with Ξ_{ij} support of ij-th either B-spline or fundamental function and

$$\begin{aligned} N_{ij} &= B_{ij} && \text{if} \quad Q = S_1, \\ N_{ij} &= \widetilde{B}_{ij} = b_{ij} B_{ij} + a_{i+1} B_{i+1,j} + c_{i-1} B_{i-1,j} \\ &\quad + \overline{a}_{j+1} B_{i,j+1} + \overline{c}_{j-1} B_{i,j-1} && \text{if} \quad Q = S_2, \\ N_{ij} &= \overline{B}_{ij}, \widehat{B}_{ij} && \text{if} \quad Q = W_2. \end{aligned}$$

Let $w_{ij}^{(Q,r)} = w_{ij}^{(r)}$.

$$
\begin{aligned}
I(Q^{(p+1)L}f) &= I(Q^{(p)}f) + I(Q^{(p-1)}\Delta_p f) + I(Q^{(p-2)}\Delta_{p-1}^2 f) + \ldots \\
&+ I(Q^{(2)}\Delta_3^{p-2} f) + I(Q^{(1)}\Delta_2^{p-1} f) + I(Q^{(0)}\Delta_1^p f) \\
&= \sum_{i=-\kappa}^{\frac{m}{2^p}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^p}+\ell} w_{ij}^{(p)} f(P_{ij}^{(p)}) + \sum_{i=-\kappa}^{\frac{m}{2^{(p-1)}}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^{(p-1)}}+\ell} w_{ij}^{(p-1)} \Delta_p f(P_{ij}^{(p-1)}) \\
&+ \sum_{i=-\kappa}^{\frac{m}{2^{(p-2)}}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^{(p-2)}}+\ell} w_{ij}^{(p-2)} \Delta_{p-1}^2 f(P_{ij}^{(p-2)}) + \ldots + \sum_{i=-\kappa}^{\frac{m}{2^2}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^2}+\ell} w_{ij}^{(2)} \Delta_3^{p-2} f(P_{ij}^{(2)}) \\
&+ \sum_{i=-\kappa}^{\frac{m}{2^1}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^1}+\ell} w_{ij}^{(1)} \Delta_2^{p-1} f(P_{ij}^{(1)}) + \sum_{i=-\kappa}^{\frac{m}{2^0}+\ell} \sum_{j=-\kappa}^{\frac{n}{2^0}+\ell} w_{ij}^{(0)} \Delta_1^p f(P_{ij}^{(0)}).
\end{aligned}
$$

By simmetry property of $w_{ij}^{(r)}$, $r = 0, \ldots, p$
and by $w_{ij}^{(r)} = c_{ij} h^{(r)} k^{(r)} = 4^r c_{ij} hk = 4^r w_{ij}^{(0)}$:

$$I(Q^{(p+1)L} f) = \sum_{i=\sigma}^{\tau} \sum_{j=\sigma}^{i} w_{ij}^{(0)} \Big(4^p z_{ij}(f^{(p)}) + 4^{p-1} z_{ij}(\Delta_p f^{(p-1)}) + 4^{p-2} z_{ij}(\Delta_{p-1}^2 f^{(p-2)}) + \ldots + 4^2 z_{ij}(\Delta_3^{p-2} f^{(2)}) + 4^1 z_{ij}(\Delta_2^{p-1} f^{(1)}) + 4^0 z_{ij}(\Delta_1^p f^{(0)}) \Big)$$

where $\sigma = \sigma(Q, X_i, Y_i)$, $\tau = \tau(Q, X_i, Y_i)$, $i = 1, 2$ and

(S_1, X_1, Y_1)

$\mathbf{w}_{00}^{(0)}$	$\mathbf{w}_{10}^{(0)}$	$\mathbf{w}_{11}^{(0)}$	$\mathbf{w}_{20}^{(0)}$	$\mathbf{w}_{21}^{(0)}$	$\mathbf{w}_{22}^{(0)}$
$\frac{1}{48}hk$	$\frac{7}{48}hk$	$\frac{33}{48}hk$	$\frac{1}{6}hk$	$\frac{5}{6}hk$	hk

(S_1, X_2, Y_2)

$\mathbf{w}_{00}^{(0)}$	$\mathbf{w}_{10}^{(0)}$	$\mathbf{w}_{11}^{(0)}$	$\mathbf{w}_{20}^{(0)}$	$\mathbf{w}_{21}^{(0)}$	$\mathbf{w}_{22}^{(0)}$
$\frac{1}{12}hk$	$\frac{1}{4}hk$	$\frac{5}{12}hk$	$\frac{1}{3}hk$	$\frac{2}{3}hk$	hk

(S_2, X_1, Y_1)

$\widetilde{\mathbf{w}}^{(0)}_{-1,-1}$	$\widetilde{\mathbf{w}}^{(0)}_{0,-1}$	$\widetilde{\mathbf{w}}^{(0)}_{00}$	$\widetilde{\mathbf{w}}^{(0)}_{1,-1}$	$\widetilde{\mathbf{w}}^{(0)}_{10}$	$\widetilde{\mathbf{w}}^{(0)}_{11}$	$\widetilde{\mathbf{w}}^{(0)}_{2,-1}$
0	$\frac{-1}{384}hk$	$\frac{-1}{192}hk$	$\frac{-7}{384}hk$	$\frac{7}{64}hk$	$\frac{151}{192}hk$	$\frac{-1}{48}hk$
$\widetilde{\mathbf{w}}^{(0)}_{20}$	$\widetilde{\mathbf{w}}^{(0)}_{21}$	$\widetilde{\mathbf{w}}^{(0)}_{22}$	$\widetilde{\mathbf{w}}^{(0)}_{30}$	$\widetilde{\mathbf{w}}^{(0)}_{31}$	$\widetilde{\mathbf{w}}^{(0)}_{32}$	$\widetilde{\mathbf{w}}^{(0)}_{33}$
$\frac{41}{384}hk$	$\frac{117}{128}hk$	$\frac{25}{24}hk$	$\frac{5}{48}hk$	$\frac{43}{48}hk$	$\frac{49}{48}hk$	hk

(S_2, X_2, Y_2)

$\widetilde{\mathbf{w}}_{00}^{(0)}$	$\widetilde{\mathbf{w}}_{10}^{(0)}$	$\widetilde{\mathbf{w}}_{11}^{(0)}$	$\widetilde{\mathbf{w}}_{20}^{(0)}$	$\widetilde{\mathbf{w}}_{21}^{(0)}$	$\widetilde{\mathbf{w}}_{22}^{(0)}$	$\widetilde{\mathbf{w}}_{30}^{(0)}$	$\widetilde{\mathbf{w}}_{31}^{(0)}$	$\widetilde{\mathbf{w}}_{32}^{(0)}$	$\widetilde{\mathbf{w}}_{33}^{(0)}$
$-\frac{1}{12}hk$	$\frac{7}{36}hk$	$\frac{2}{3}hk$	$\frac{1}{9}hk$	$\frac{8}{9}hk$	$\frac{37}{36}hk$	$\frac{1}{9}hk$	$\frac{7}{8}hk$	$\frac{73}{72}hk$	hk

(W_2, X_1, Y_1)

$\widehat{\mathbf{w}}^{(0)}_{-1,-1}$	$\widehat{\mathbf{w}}^{(0)}_{0,-1}$	$\widehat{\mathbf{w}}^{(0)}_{00}$	$\widehat{\mathbf{w}}^{(0)}_{1,-1}$	$\widehat{\mathbf{w}}^{(0)}_{10}$	$\widehat{\mathbf{w}}^{(0)}_{11}$	$\widehat{\mathbf{w}}^{(0)}_{2,-1}$	$\widehat{\mathbf{w}}^{(0)}_{20}$	$\widehat{\mathbf{w}}^{(0)}_{21}$	$\widehat{\mathbf{w}}^{(0)}_{22}$
$-\frac{1}{192}hk$	$-\frac{1}{24}hk$	$-\frac{1}{4}hk$	$-\frac{15}{192}hk$	$-\frac{11}{24}hk$	$-\frac{161}{192}hk$	$-\frac{1}{12}hk$	$-\frac{1}{2}hk$	$-\frac{11}{12}hk$	$-hk$

(W_2, X_2, Y_2)

$\widehat{\mathbf{w}}_{00}^{(0)}$	$\widehat{\mathbf{w}}_{10}^{(0)}$	$\widehat{\mathbf{w}}_{11}^{(0)}$	$\widehat{\mathbf{w}}_{20}^{(0)}$	$\widehat{\mathbf{w}}_{21}^{(0)}$	$\widehat{\mathbf{w}}_{22}^{(0)}$
$-\frac{7}{16}hk$	$-\frac{9}{16}hk$	$-\frac{11}{16}hk$	$-\frac{2}{3}hk$	$-\frac{5}{6}hk$	$-hk$

Comments

- in case of partitions X_1, Y_1, the degree of precision is 3
- Matlab numerical results show that numerical convergence order is consistent with the theoretical one, obtained by results on approximation
- $f_1(x,y) = | x^2 + y^2 - 0.25 |$, $f_2(x,y) = y^2 \sin(x)$

	(m,n)	$E(I(S_1 f))$	$E(I(W_1 f))$	$E(I(W_1^{2L} f))$	$E(I(S_1^{2L} f))$
f_1	(8,6)	1.11(-2)	1.36(-2)	1.09(-4)	2.29(-4)
	(10,10)	5.12(-3)	6.23(-3)	3.05(-4)	1.26(-4)
	(20,20)	3.18(-3)	3.96(-3)	9.22(-5)	1.89(-5)
	(40,30)	4.38(-4)	5.61(-4)	6.35(-5)	4.19(-6)
f_2	(8,6)	2.89(-3)	3.85(-3)	3.16(-4)	5.25(-5)
	(10,10)	9.56(-4)	1.27(-3)	1.20(-4)	1.18(-5)
	(20,20)	2.39(-4)	3.19(-4)	4.13(-5)	7.36(-7)
	(40,30)	1.15(-4)	1.54(-4)	2.31(-5)	8.43(-8)

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