Rational interpolation of analytic functions

Direct and inverse results

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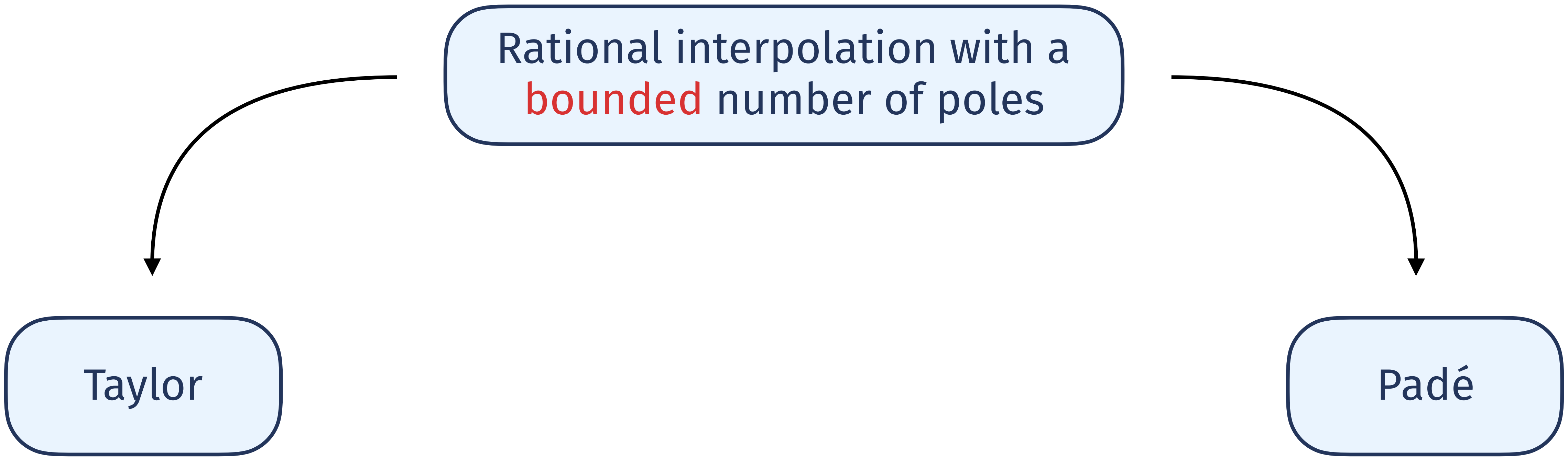
What is this talk about?



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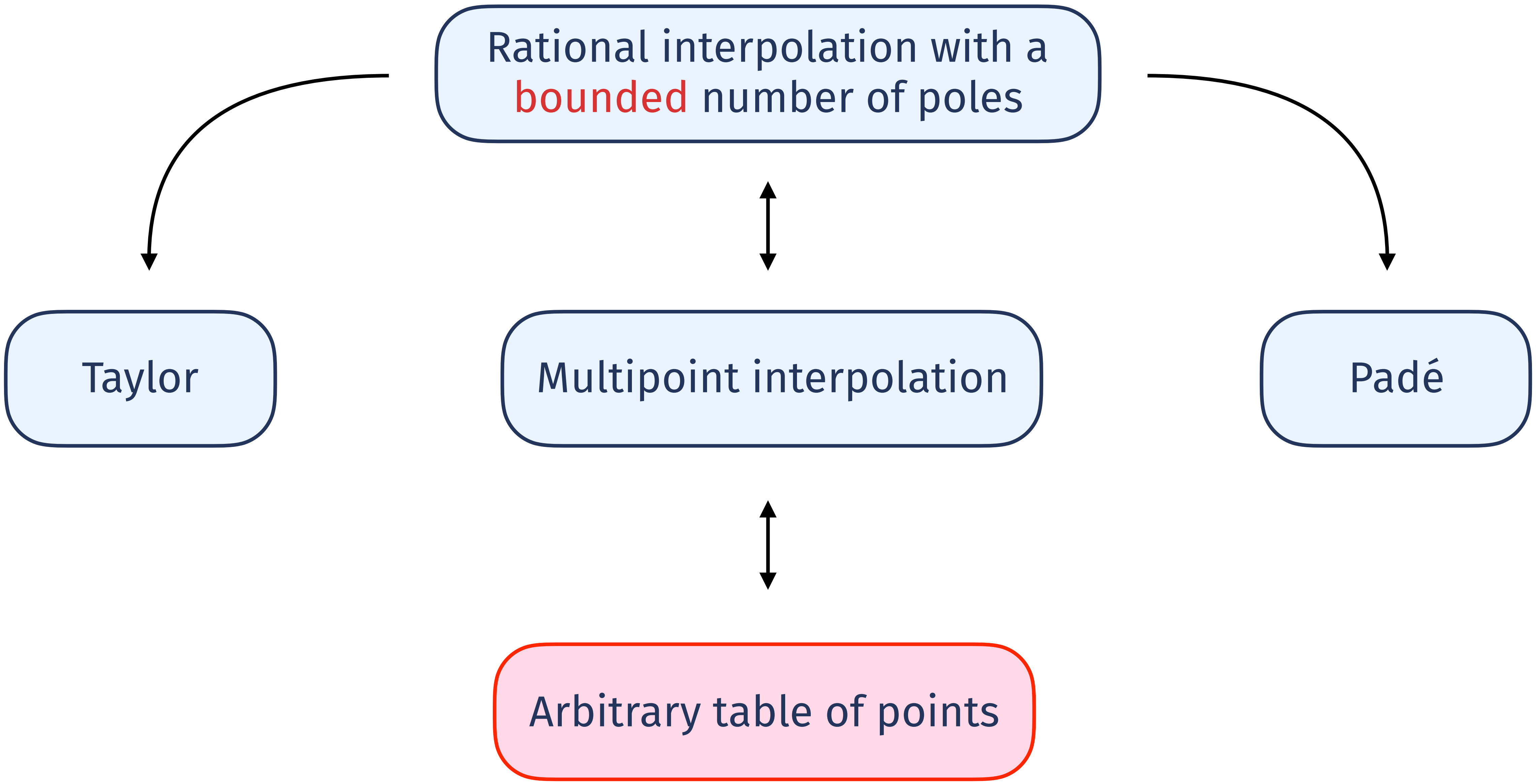

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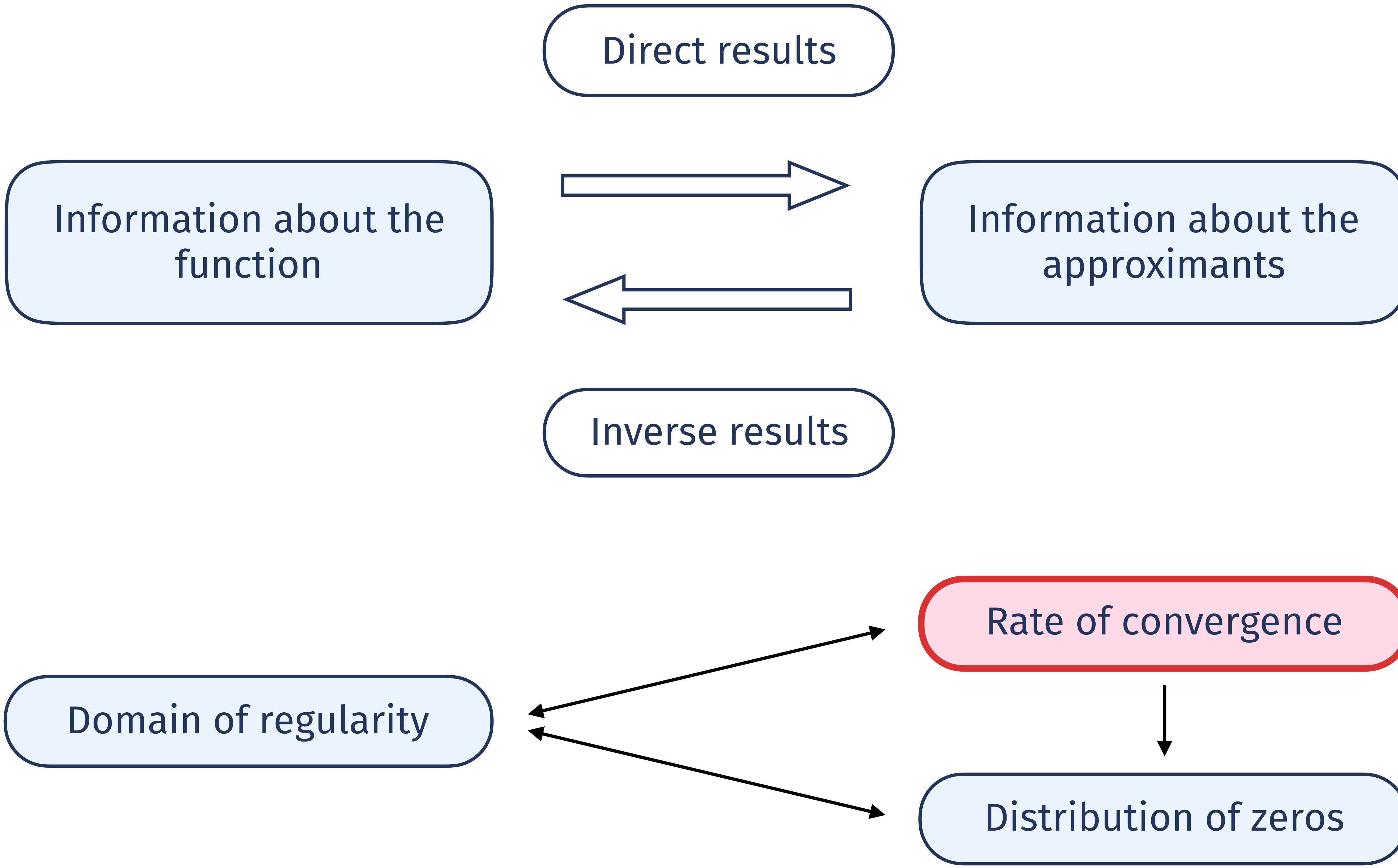

Convergence

Multipoint interpolation

- Let $\Sigma \subset \mathbb{C}$ be a simply connected compact set.
- Let f be an analytic function on an open set $V \supset \Sigma$.
- Let us fix a table of interpolation points at the zeros of the polynomials

$$w_n(z) = \prod_{i=1}^{n}(z - a_{n,i}) \qquad \boxed{a_{n,i} \in \Sigma} \qquad n \in \mathbb{N}.$$

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- Given $n, m \in \mathbb{Z}^+$, there exist polynomials $P_{n,m}, Q_{n,m}$ with $\deg P_{n,m} \leq n$, $\deg Q_{n,m} \leq m$, $Q_{n,m} \not\equiv 0$, such that the function

$$\frac{Q_{n,m}(z)f(z) - P_{n,m}(z)}{w_{n+m+1}(z)} \quad \text{is analytic on } V.$$

- The rational function $\boxed{\Pi_{n,m} = P_{n,m}/Q_{n,m}}$ is the multipoint Padé approximant of type (n, m) of the function f associated with w_{n+m+1}.

Multipoint interpolation

• To describe the convergence of the interpolation process as $n \to \infty$, the interpolation points need to have a limit distribution:

$$dw_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{a_{n,i}} \xrightarrow{*} \alpha$$

Weak convergence of measures

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• The domain of m-meromorphy $D_m(\alpha)$ is the largest set determined by the equipotential curves of α to which f can be continued as a meromorphic function with $\leq m$ poles.

Radius of $D_m(\alpha)$

$$D_m(\alpha) = \{z \in \mathbb{C} \,:\, \exp\{-P(\alpha; z)\} < R_m(\alpha)\}$$

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The convergence of the interpolants $\Pi_{n,m}$ is governed by these equipotential curves of α

(As for Taylor polynomials)

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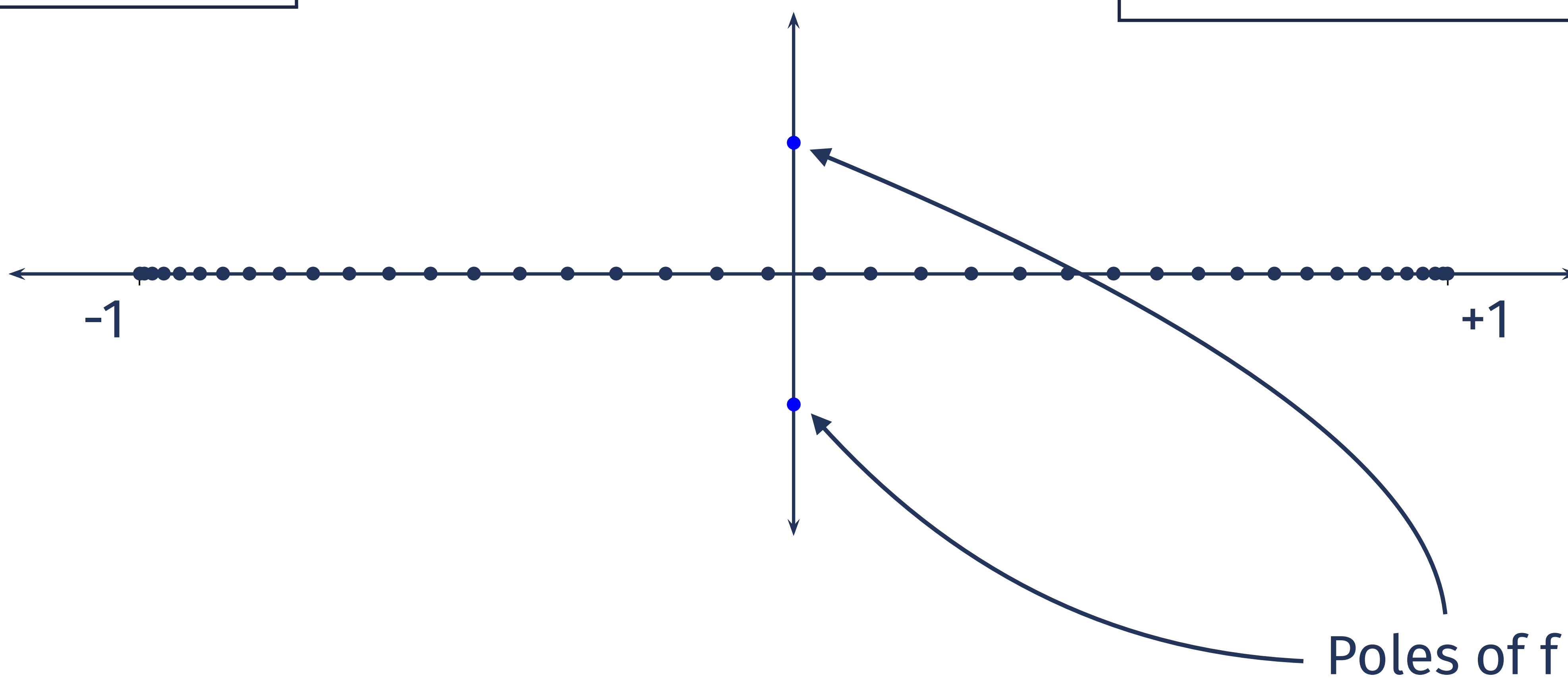
Polynomial examples

- Let P_n be the polynomial of degree n interpolating f at the Chebyshev nodes.

$$f(z) = \frac{1}{1+25z^2}$$

$$dw_n \xrightarrow{*} \alpha = \frac{dx}{\pi\sqrt{1-x^2}}$$



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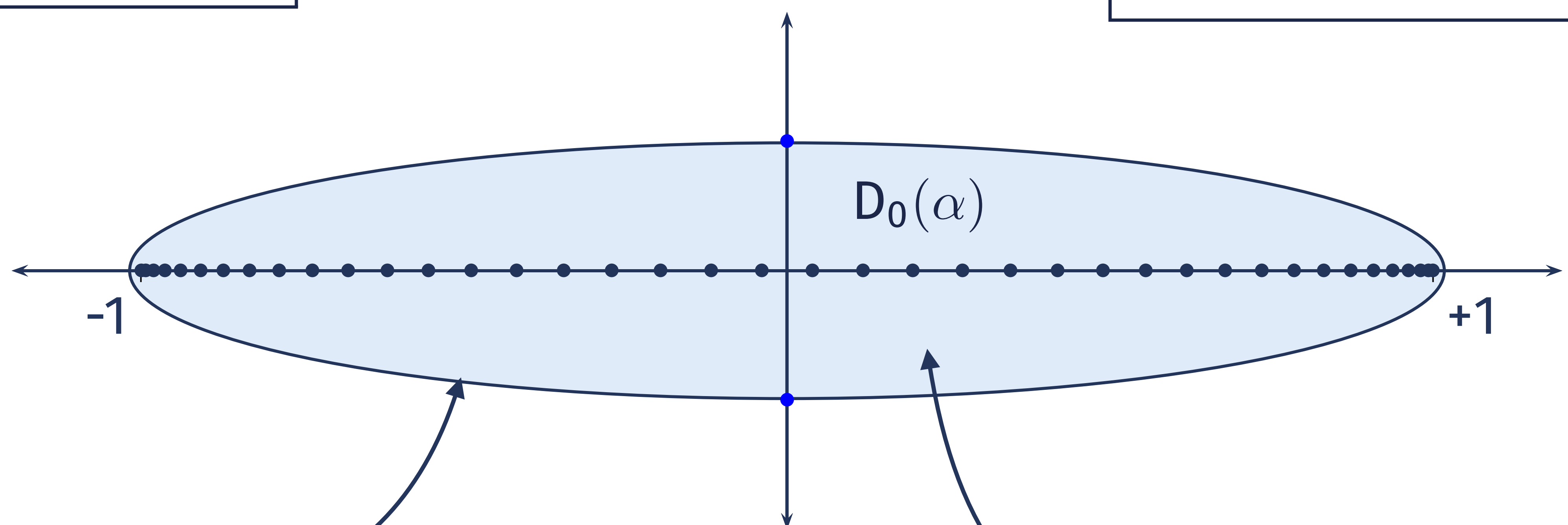

Largest equipotential curve of α inside of which f is analytic

Uniform convergence on compact subsets

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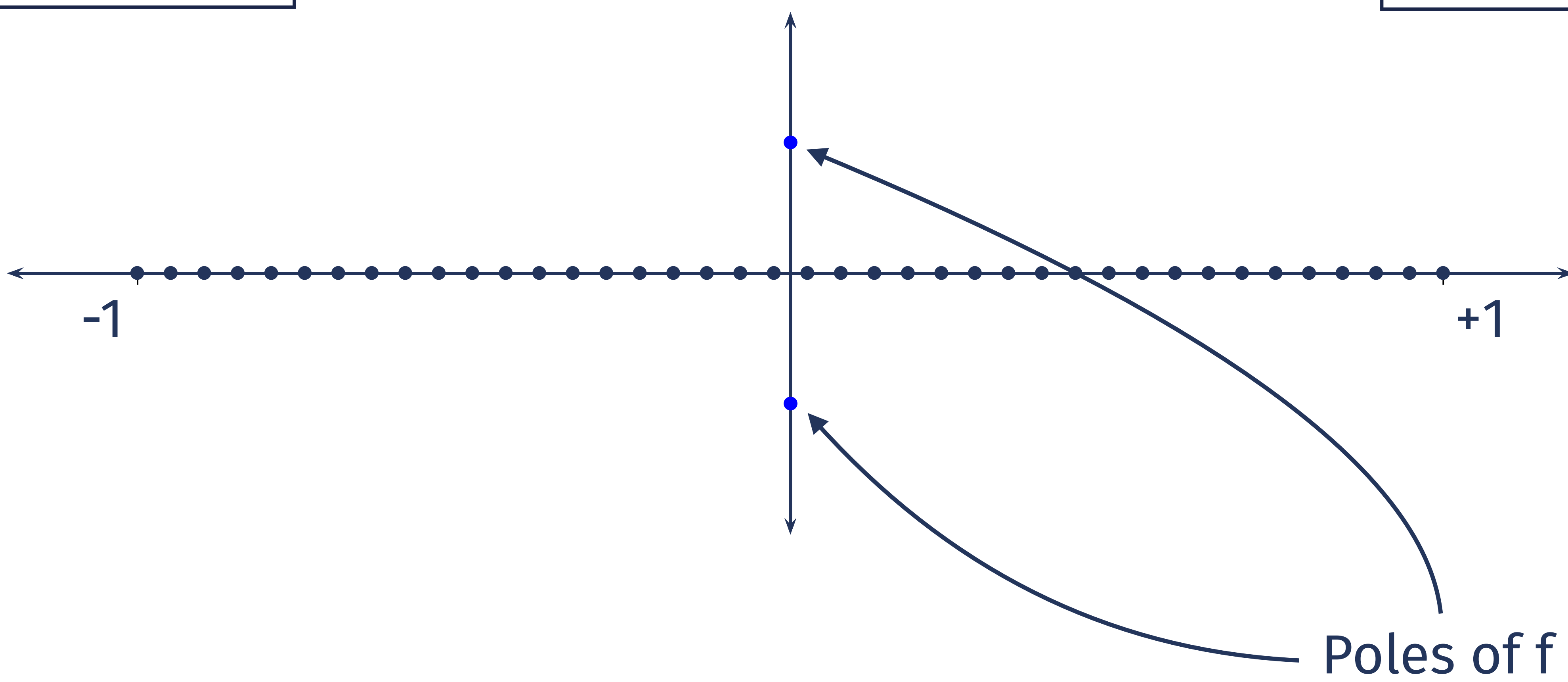

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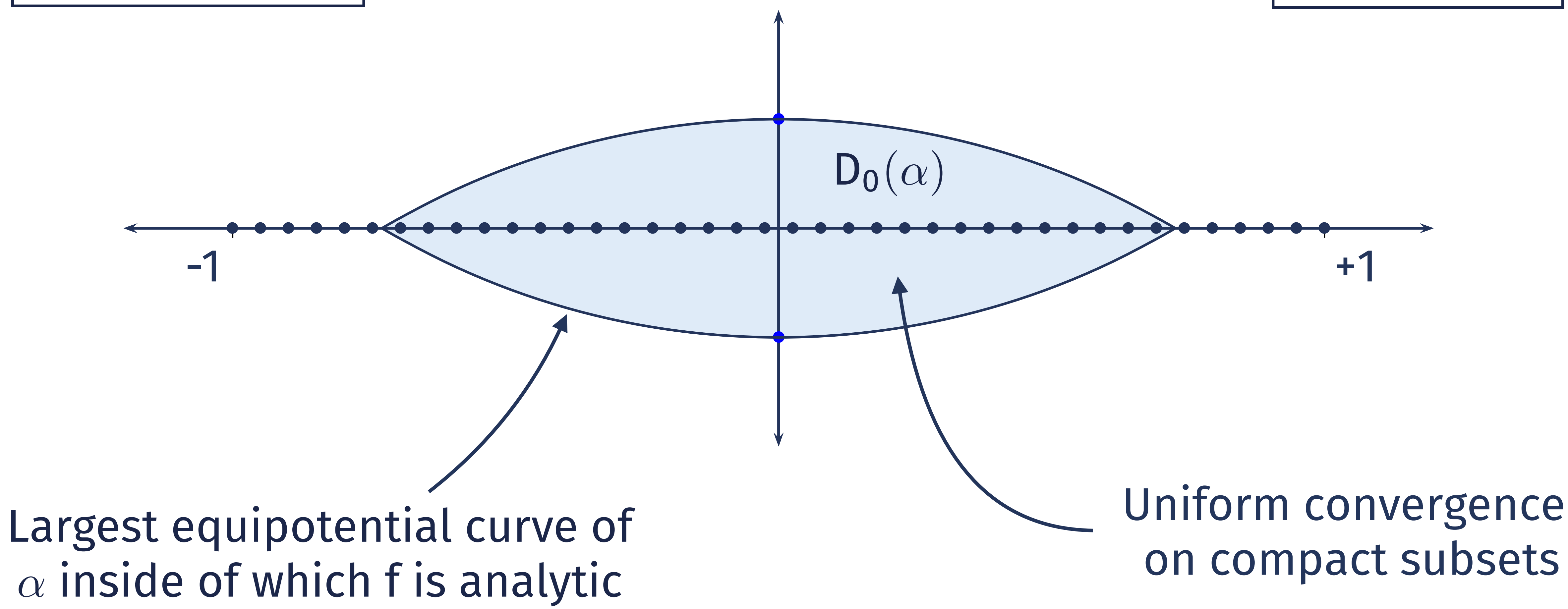

Rational case

If the function f has exactly m poles in $D_m(\alpha)$, then

(1) $\{\Pi_{n,m}\}$ converge uniformly to f in compact subsets of $D_m(\alpha) \setminus \{\text{Poles of } f\}$

(2) The poles of f attract the poles of $\Pi_{n,m}$ according to their multiplicity

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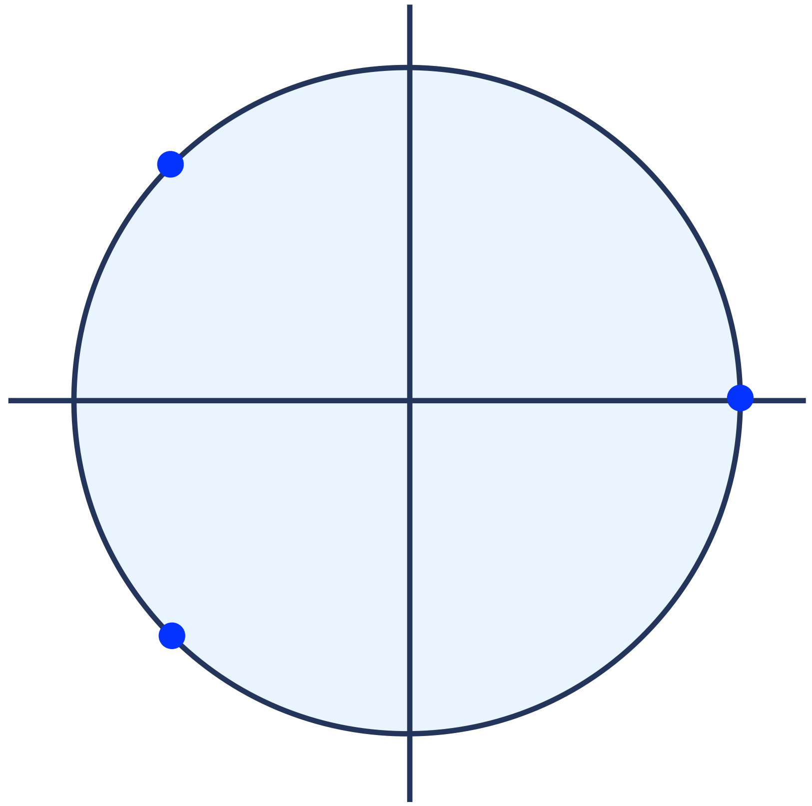

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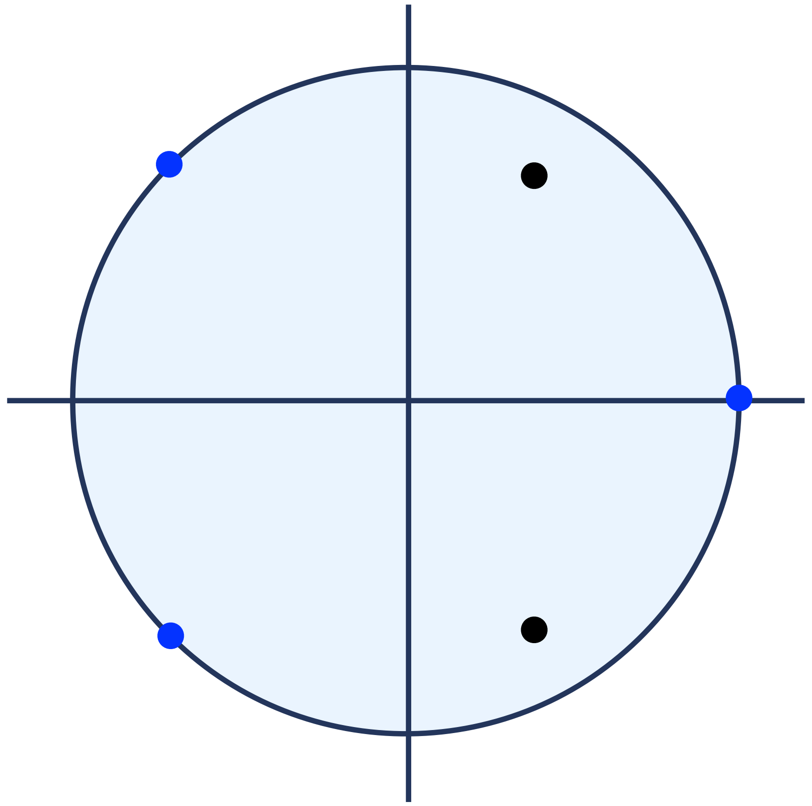

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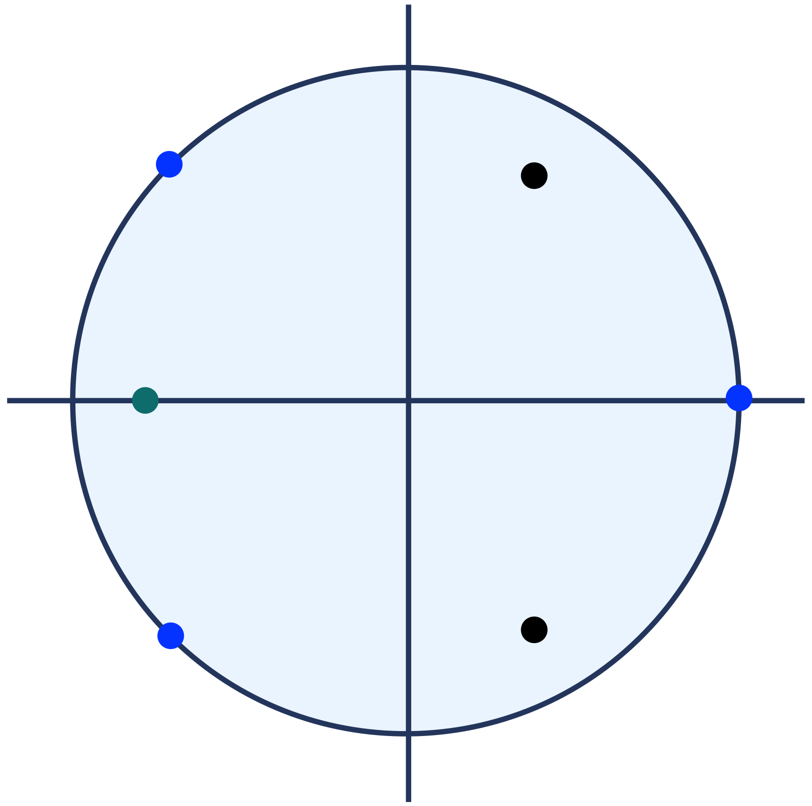

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- Poles of $\Pi_{n,2}$:
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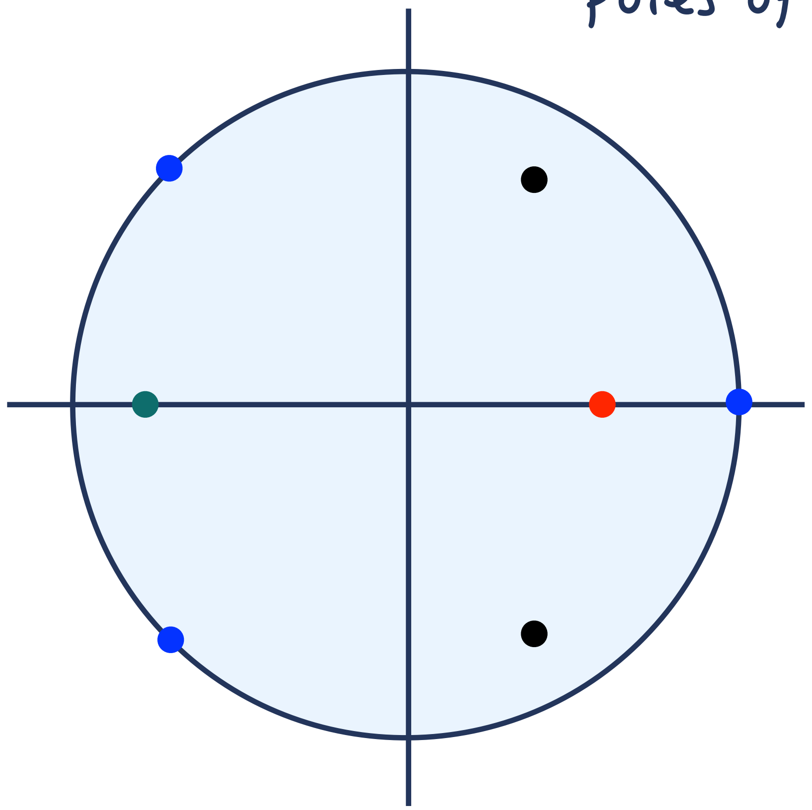
They aren't poles of f!



Logarithmic capacity

- Let K be a compact set in the complex plane and σ a positive measure supported on K. The energy of σ is defined as

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$$\boxed{\operatorname{cap}(K) = \exp\left\{ -\min_{\sigma} I(\sigma) \right\}}$$

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Every function bounded on a domain G and harmonic on $G \setminus K$ admits a harmonic extension to the whole of G if and only if $\mathrm{cap}(K) = 0$

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- If K is connected, then $\boxed{\frac{\mathrm{diam}(K)}{4} \leq \mathrm{cap}(K) \leq \frac{\mathrm{diam}(K)}{2}}$

Convergence in capacity

- The sequence of functions $\{f_n\}$ converges in capacity to the function f on compact subsets of the domain D if

$$\forall \epsilon > 0, \forall K \subset D \lim_{n\to\infty} \mathrm{cap}\{z \in K : |f(z) - f_n(z)| > \epsilon\} = 0.$$

As convergence in measure!

Notation: $f_n \xrightarrow{C} f$ inside D

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f_n meromorphic in D with $\leq$ m poles + f meromorphic in D with exactly m poles $\Longrightarrow$ Uniform convergence of f_n inside $D \setminus \{\text{Poles of } f\}$

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We are done with the direct results!

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How can we calculate $R_m(\alpha)$?

Buslaev Theorem

- Suppose that

(1) The m poles of $\Pi_{n,m}$ have a limit as $n \to \infty$.

(2) The table of interpolations points on the continuum Σ is extremal:

$$\lim_{n\to\infty} \frac{w_n(z)}{[h(z)]^n} = g(z) \neq 0,$$

where h is the conformal mapping of $\mathbb{C} \setminus \Sigma$ onto $\mathbb{C} \setminus \overline{\mathbb{D}}$.

- Then

- The limit points of the poles are singularities of the function f.
- $R_m(\alpha)$ is determined by the limit points with smallest potential.
- Those with larger potential are the only poles of the function f in $D_m(\alpha)$

Exact rate of convergence

Propagation of convergence

- Example: $$\boxed{f(z) = \frac{1+\sqrt[3]{2}\,z}{1-z^3}}$$

- $$\boxed{\limsup_{n\to\infty} \|f - \Pi_{n,2}\|_K^{1/n} = \rho < 1}$$

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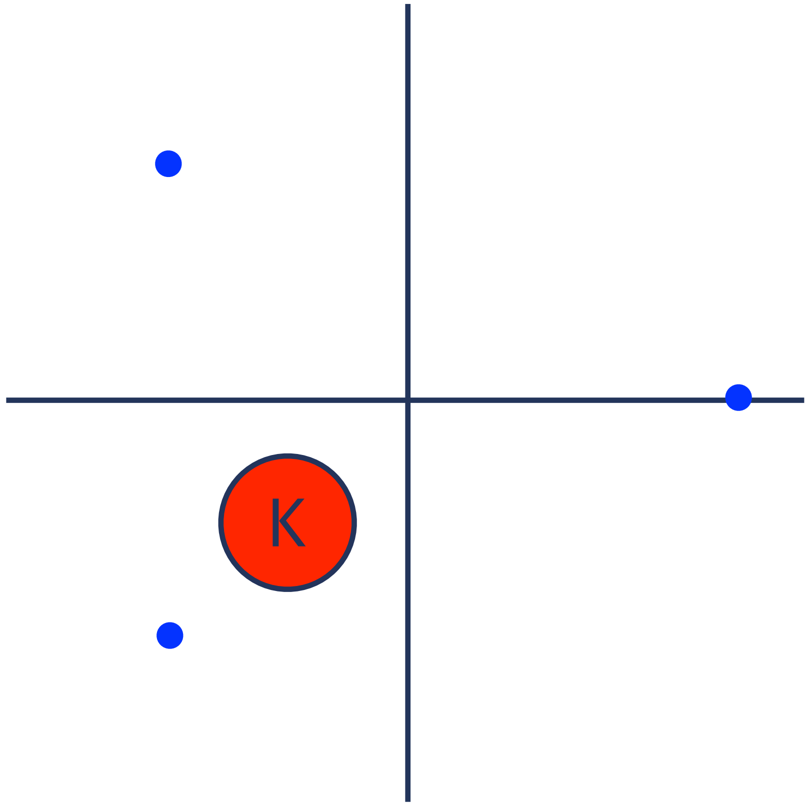

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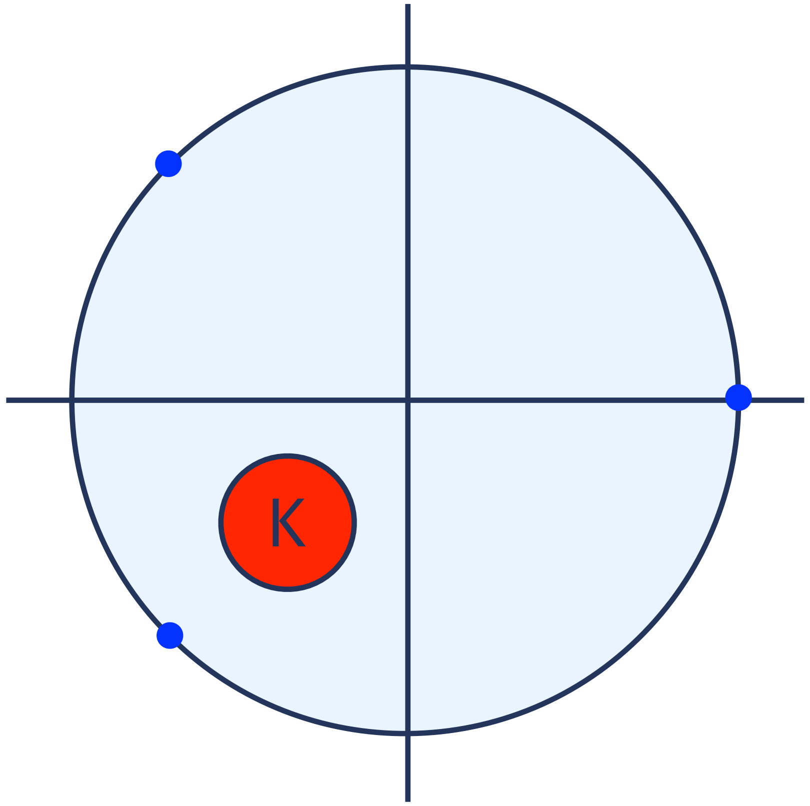

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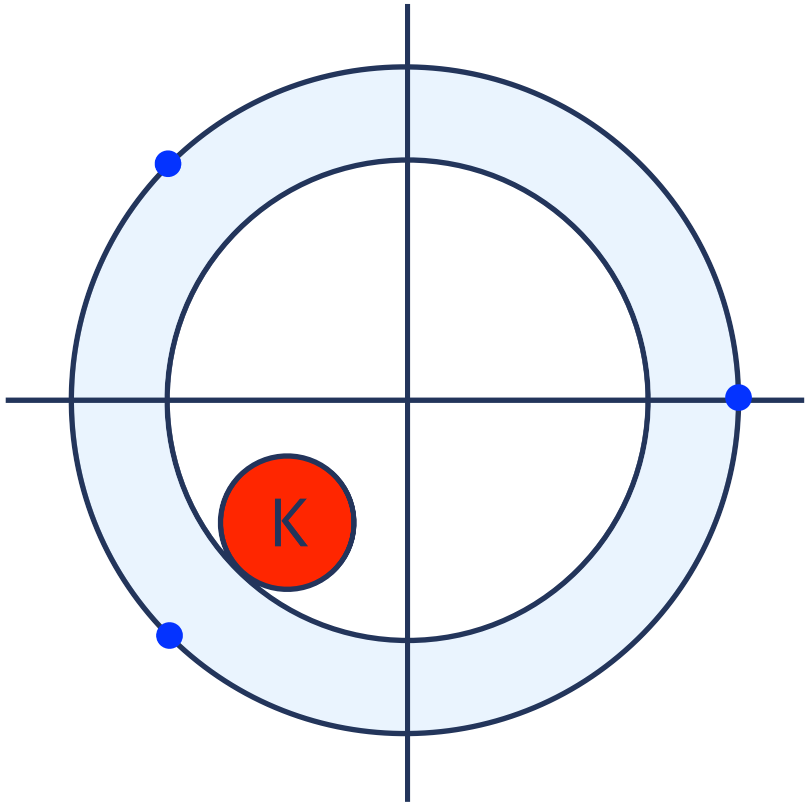

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Best approximants

- Let f be a function defined on a regular compact set $K \subset \mathbb{C}$.

- Let us consider $\boxed{d_{n,m}(K) = \min\{\|f - r\|_K \; : \; r \in R_{n,m}\}}$ where

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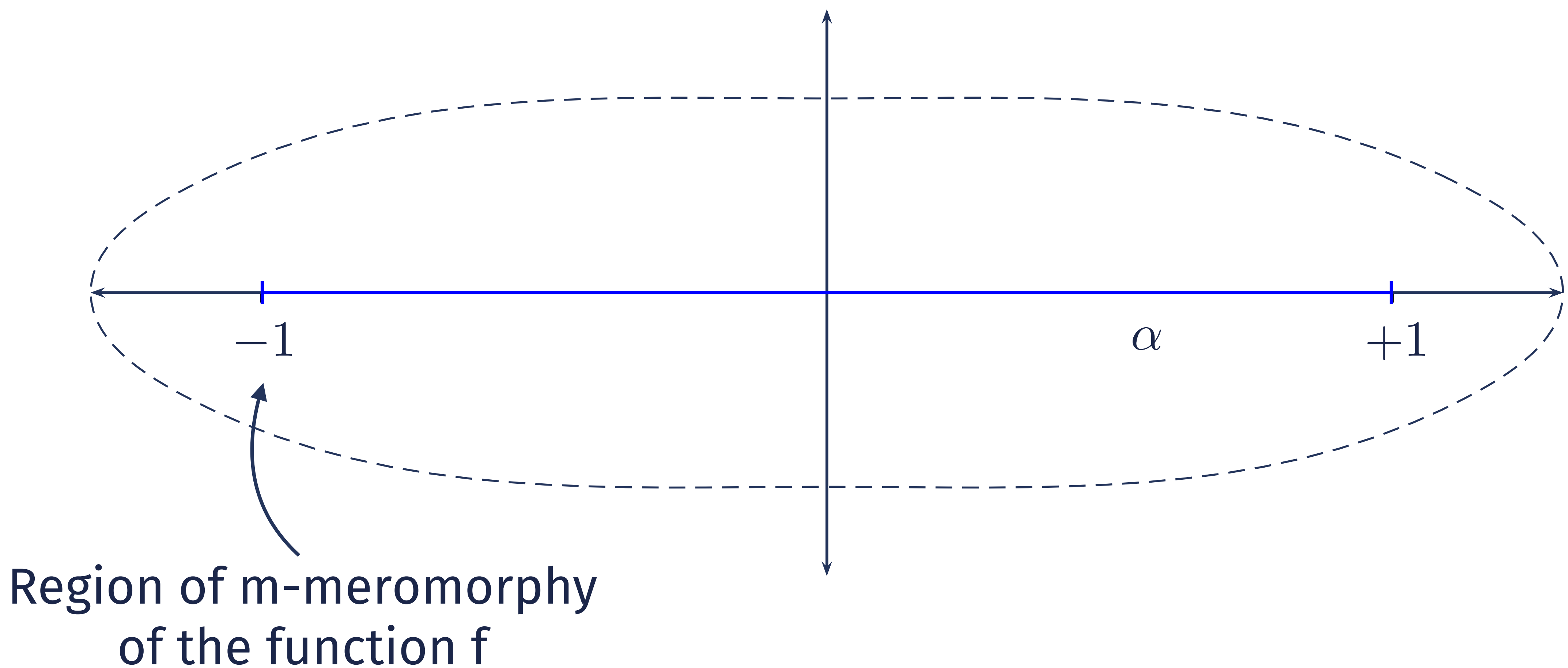
- Suppose that $\limsup_{n\to\infty} \sqrt[n]{d_{n,m}(K)} = \rho < 1$. Then

$$\boxed{\rho = \frac{\mathrm{cap}(K)}{R_m(\mu_K)}}$$

where μ_K is the equilibrium measure of the compact set K.

- Let f be an analytic function on a neighborhood of $[-1, 1]$. Let us interpolate f along a table of points in $[-1, 1]$ with arbitrary distribution α.



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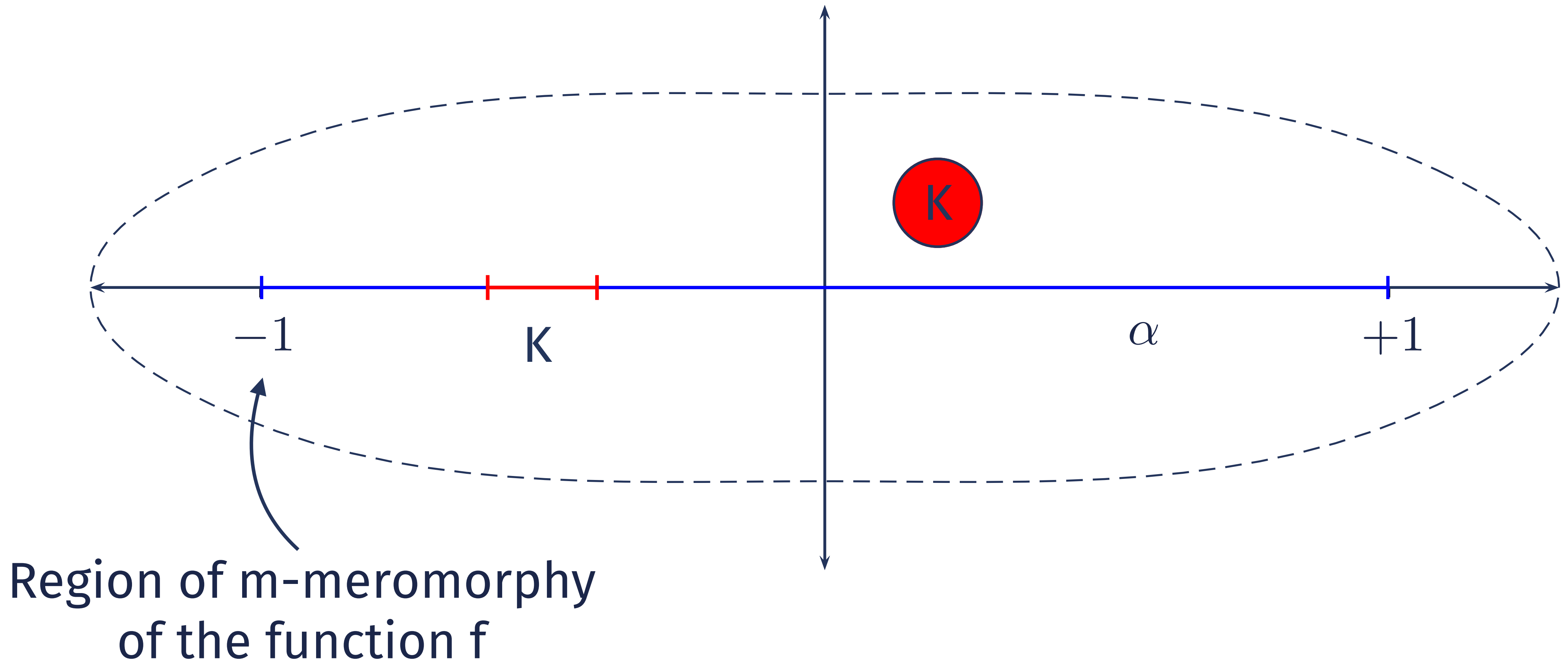

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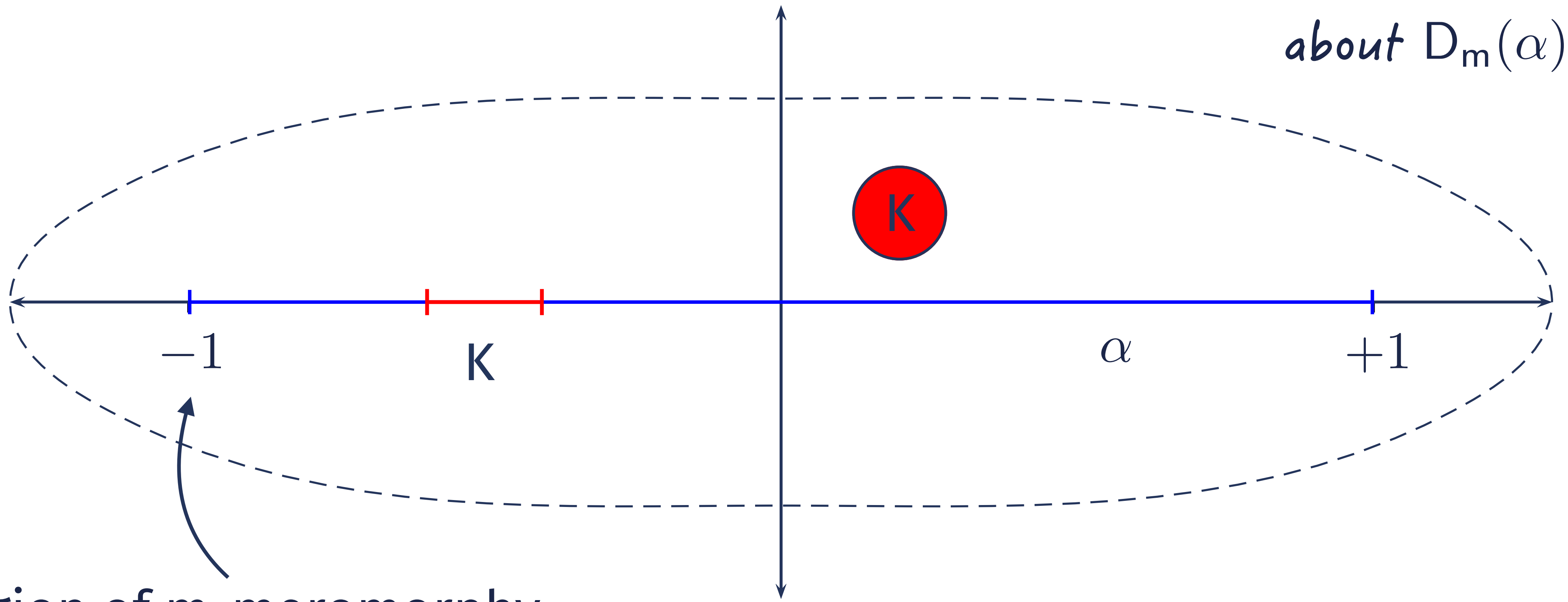
What can be said about $D_m(\alpha)$?



Region of m-meromorphy of the function f

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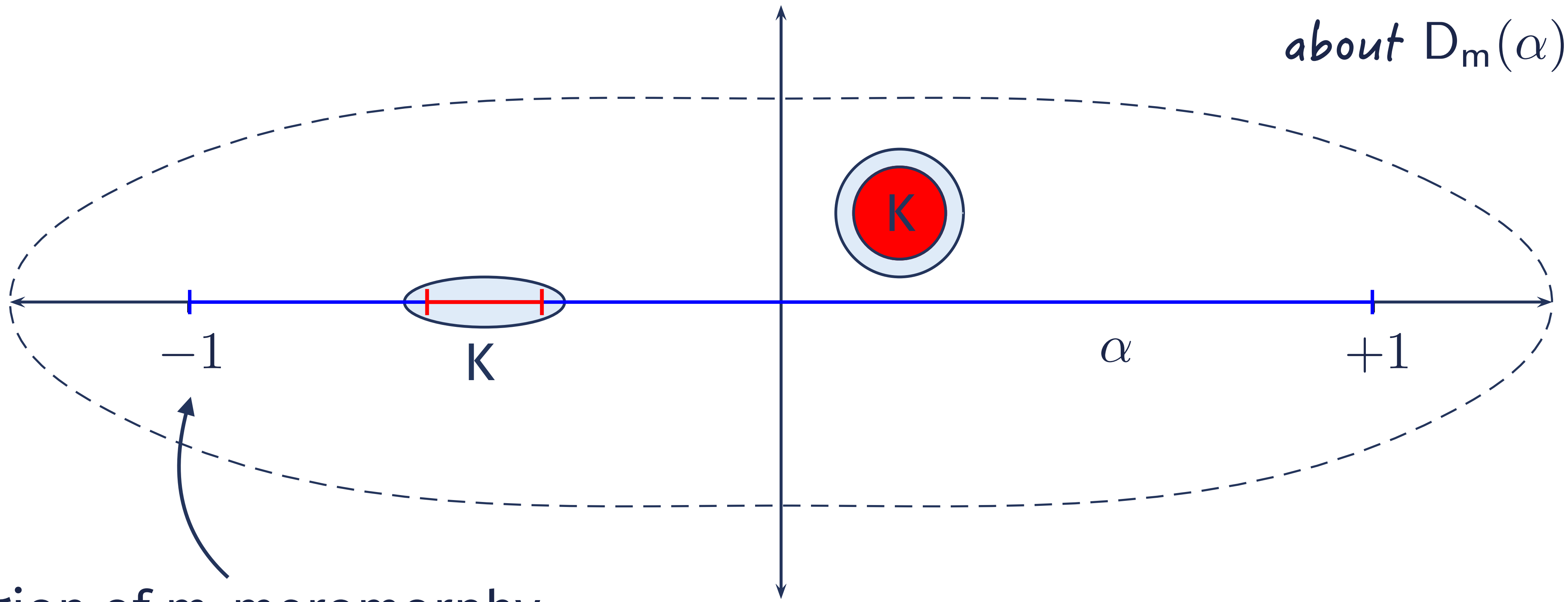

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Theorem

- Suppose that the function f is defined on K and verifies

$$\boxed{\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} = \rho < 1}$$

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- K is a regular compact set on which the minimum value of the potential of α is attained at a point that does not belong to the interior of Σ.

- Then

$$\boxed{\rho = \frac{\alpha(K)}{R_m(\alpha)}} \qquad \alpha(K) = \exp\left\{-\min_{z\in K} P(\alpha; z)\right\}$$

• Let f be an analytic function on a neighborhood of $[-1,1]$. Let us interpolate f along a table of points in $[-1,1]$ with arbitrary distribution α.

• Suppose that $\boxed{\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} = \rho < 1}$ (K is regular)

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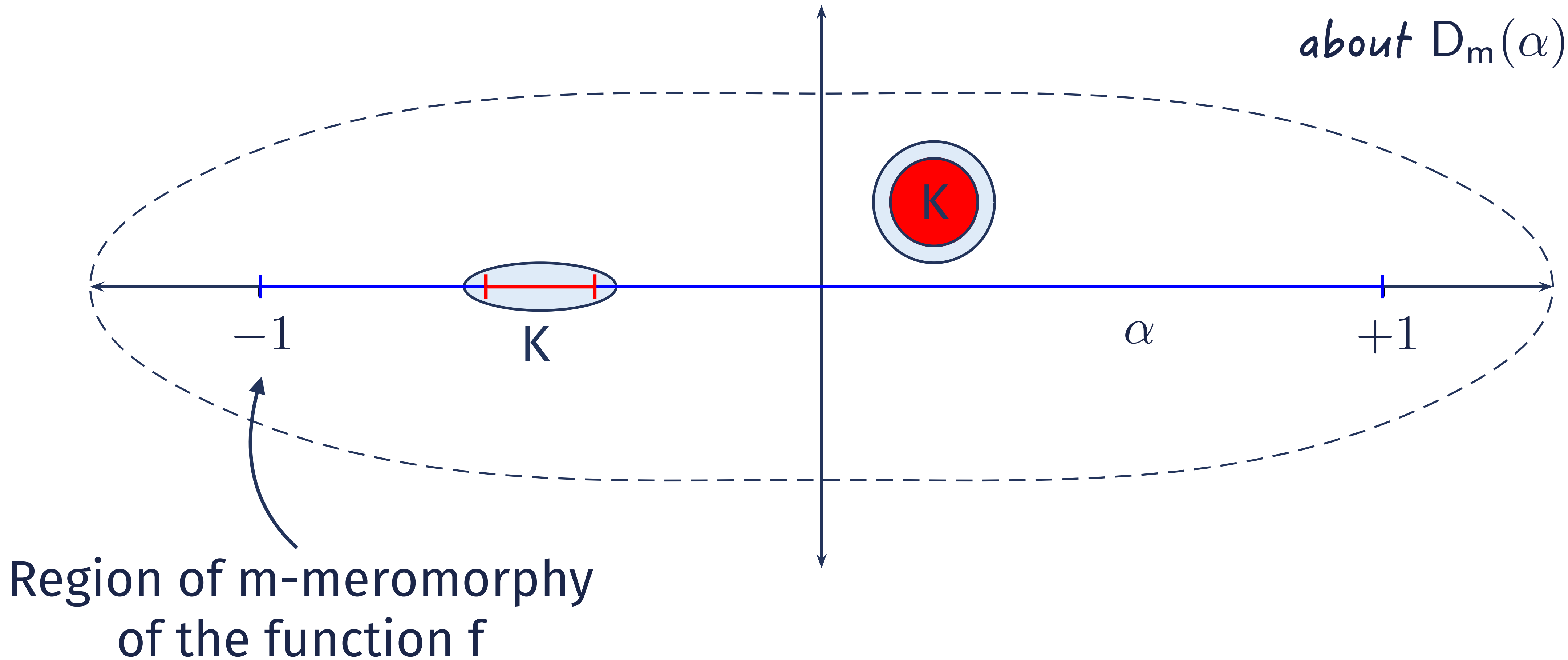

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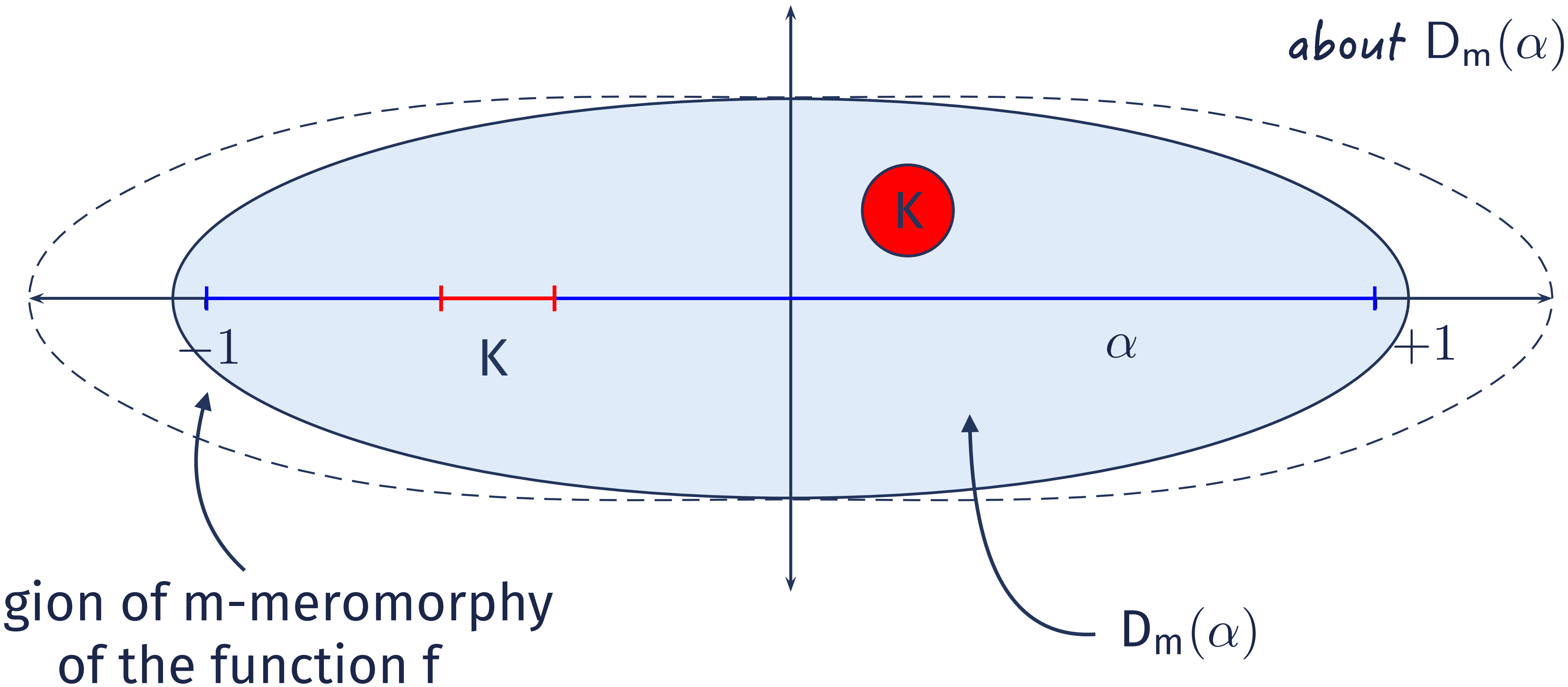

Faster rate of convergence

Overconvergence

- For each continuum $Q \subset D_m(\alpha)$ (not a single point) far away from the poles of the function f and the set of interpolation Σ, we have

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- Suppose that there exists a subsequence $\Lambda \subset \mathbb{N}$ such that for a continuum Q it holds that

$$\lim_{n\in\Lambda} \|Q_{n,m}(f - \Pi_{n,m})\|_Q^{1/n} < \frac{\alpha(Q)}{R_m(\alpha)}$$

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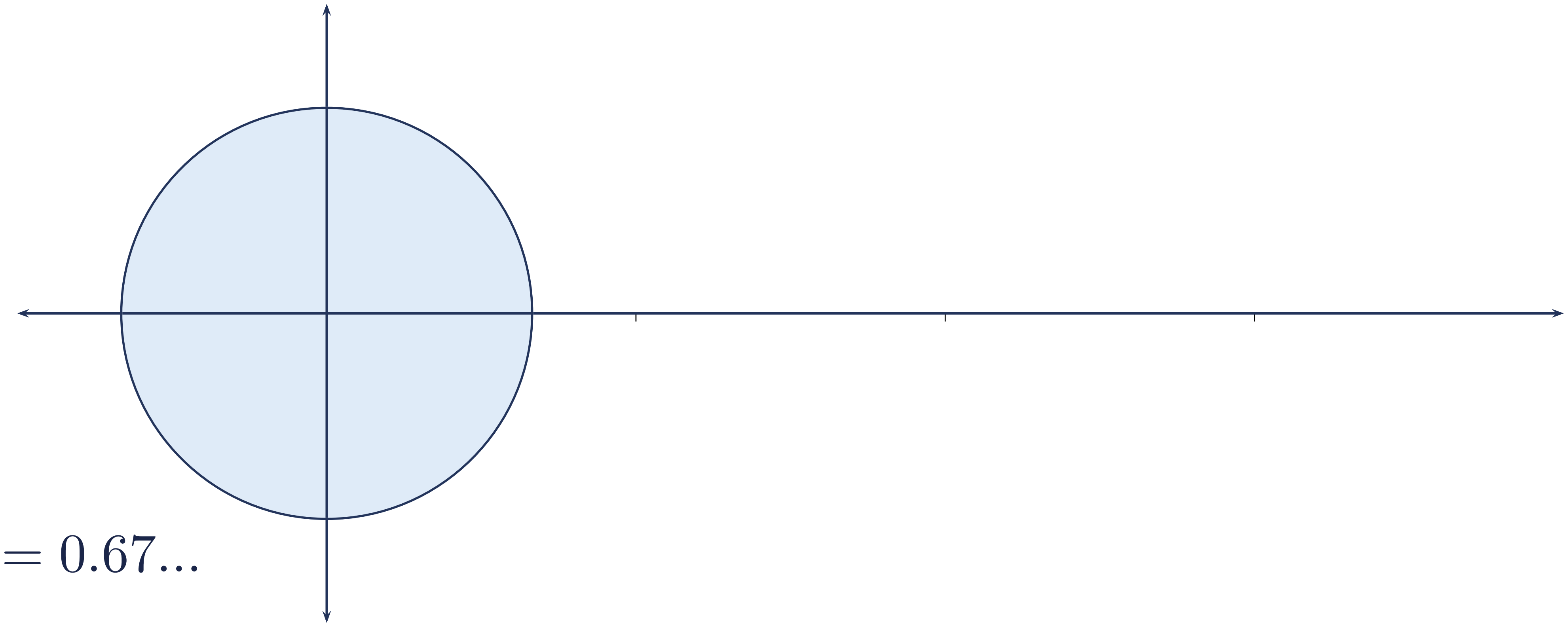
$$\lim_{n\in\Lambda} \|Q_{n,m}(f - \Pi_{n,m})\|_Q^{1/n} < \frac{\alpha(Q)}{R_m(\alpha)}$$

Faster subsequences want to converge beyond the domain of m-meromorphy

Porter (1906)

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{z(z-3)}{\sqrt{6}} \right)^{3^k}$$



- Convergence and analyticity domain of f: $|z(z-3)| < \sqrt{6}$

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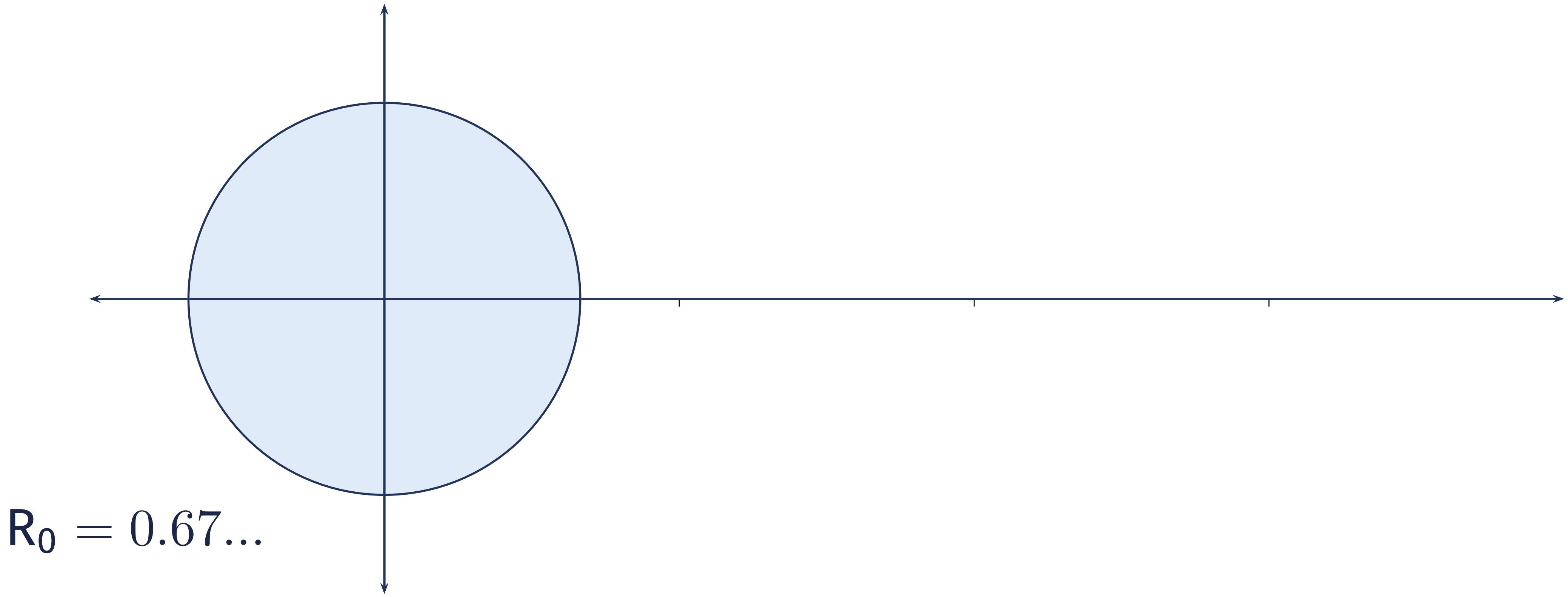
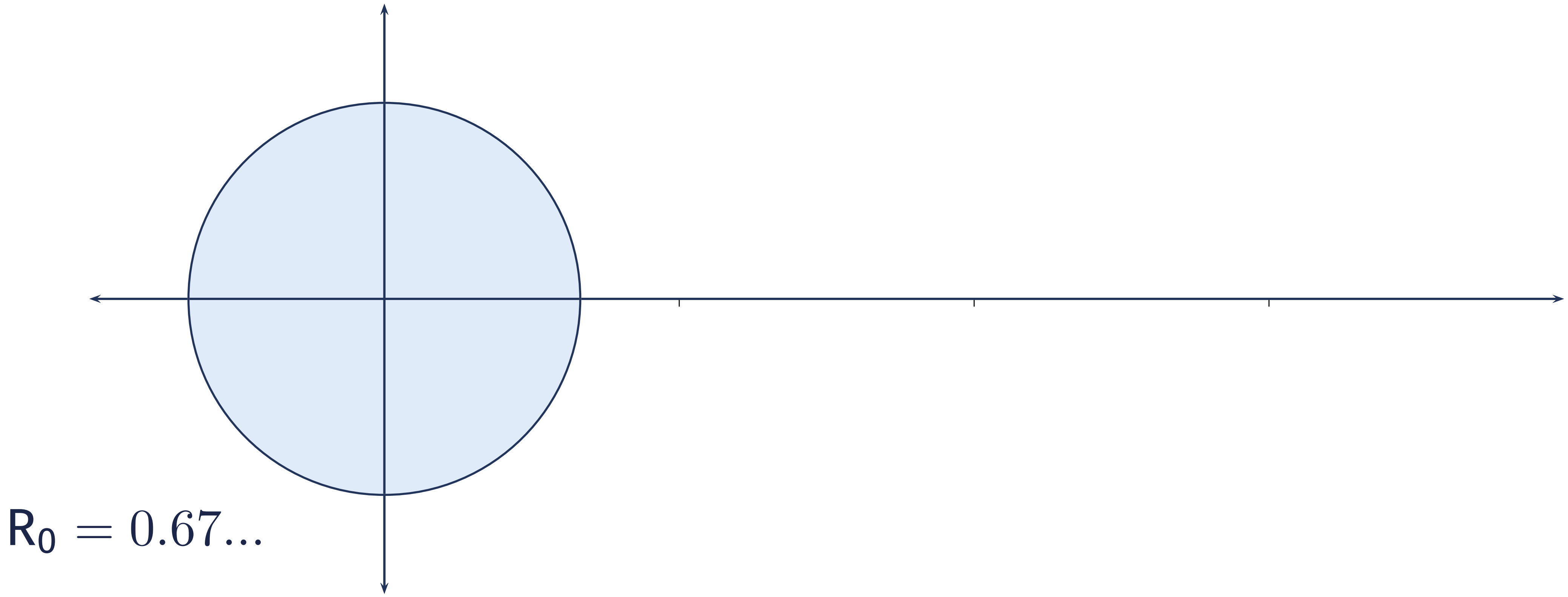
The partial sums of the series are a subsequence of the Taylor polynomials



- Convergence and analyticity domain of f: $|z(z-3)| < \sqrt{6}$

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{z(z-3)}{\sqrt{6}} \right)^{3^k} = 1 + \left(\frac{z(z-3)}{\sqrt{6}} \right)^{3} + \left(\frac{z(z-3)}{\sqrt{6}} \right)^{9} + \left(\frac{z(z-3)}{\sqrt{6}} \right)^{27} + \cdots$$



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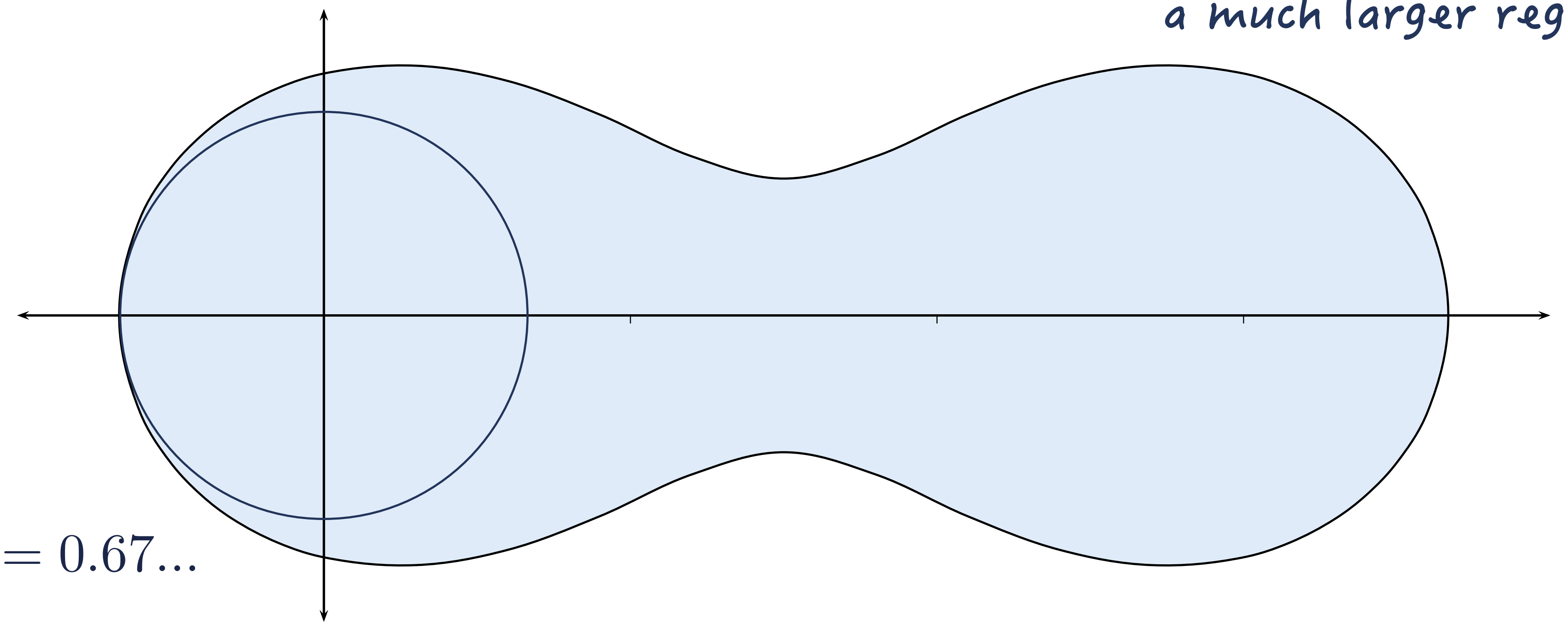
The subsequence converges in a much larger region!



$R_0 = 0.67\ldots$

- Convergence and analyticity domain of f: $\boxed{|z(z-3)| < \sqrt{6}}$

Results

Increasingly large gaps ⇒ Overconvergence around any regular point of the circle of convergence

Results

Increasingly large gaps ⇒ Overconvergence around any regular point of the circle of convergence

⇕

Faster subsequences

Results

Increasingly large gaps ⟹ Overconvergence around any regular point of the circle of convergence

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Faster subsequences ⟹ Overconvergence in capacity around any regular point or pole of $\partial D_m(\alpha) \setminus \Sigma$

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- For Taylor polynomials:

Increasingly large gaps ⇒ There are no poles on the circle of convergence

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- For Taylor polynomials:

Increasingly large gaps ⇒ There are no poles on the circle of convergence

Very large gaps ⇒ The domains of analytic and meromorphic continuation of f coincide

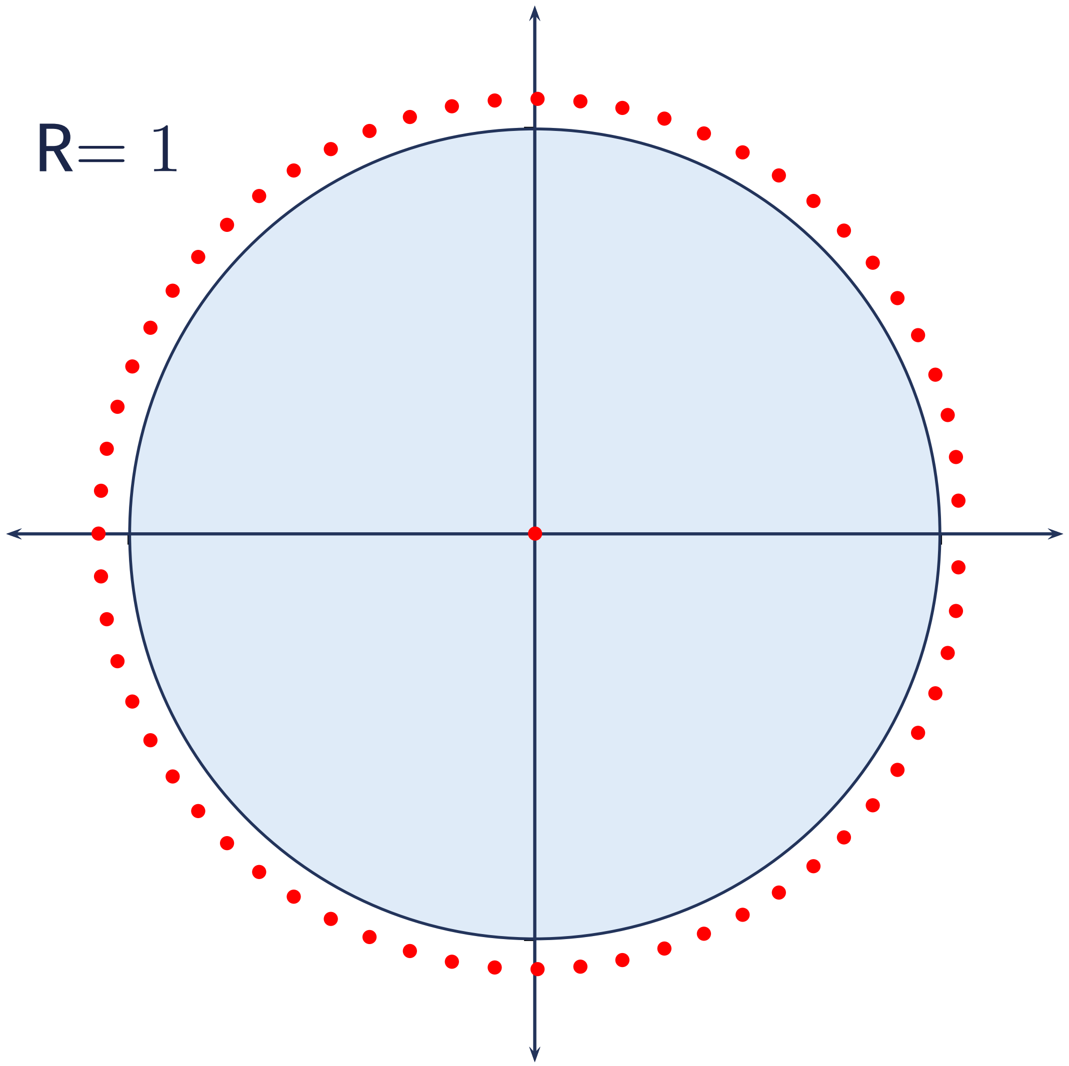
With any amount of poles!

Distribution of zeros

The Jentzsch-Szegő Theorem

$$-\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

Zeros of $\sum_{k=1}^{n} \frac{z^k}{k}$ for $n = 64$



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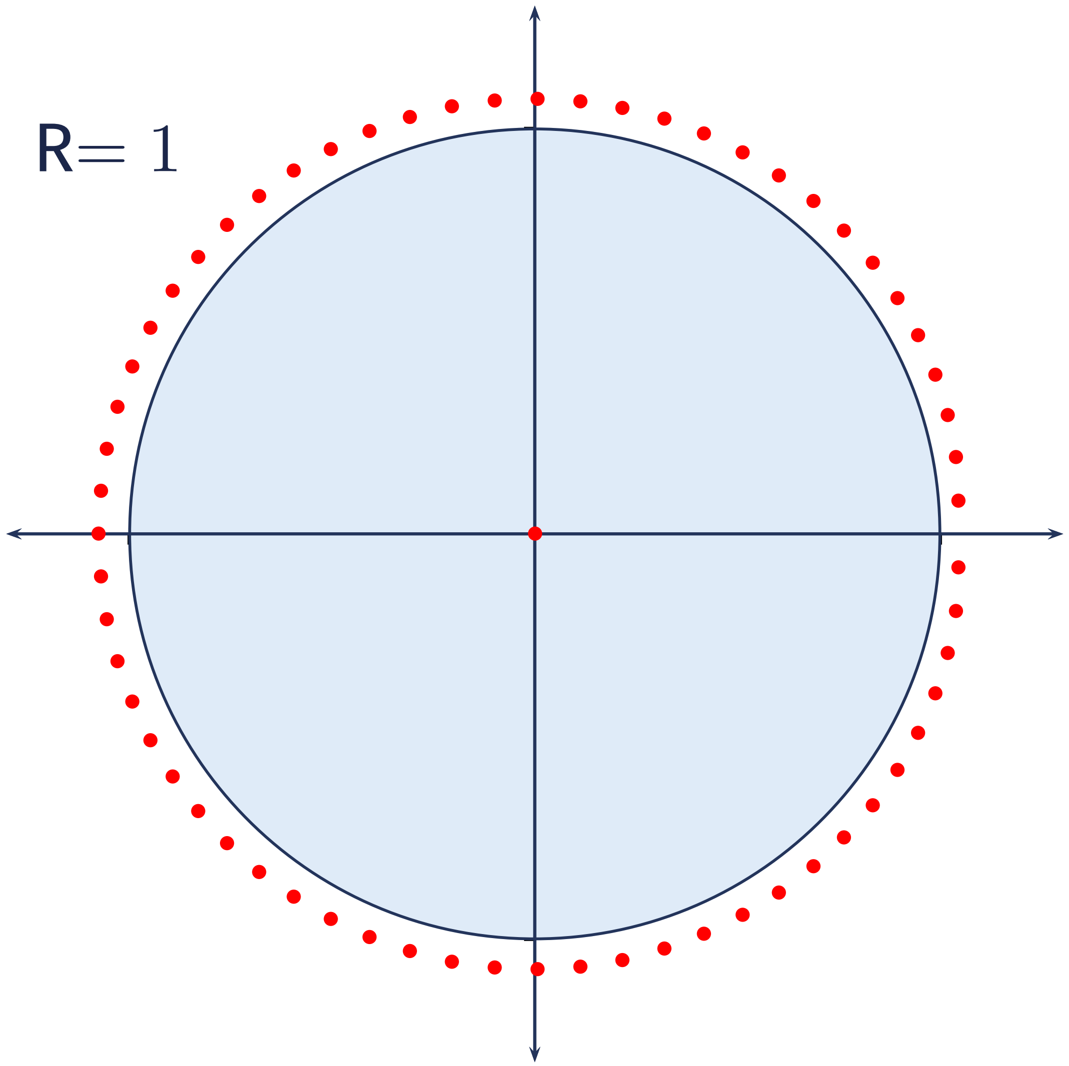


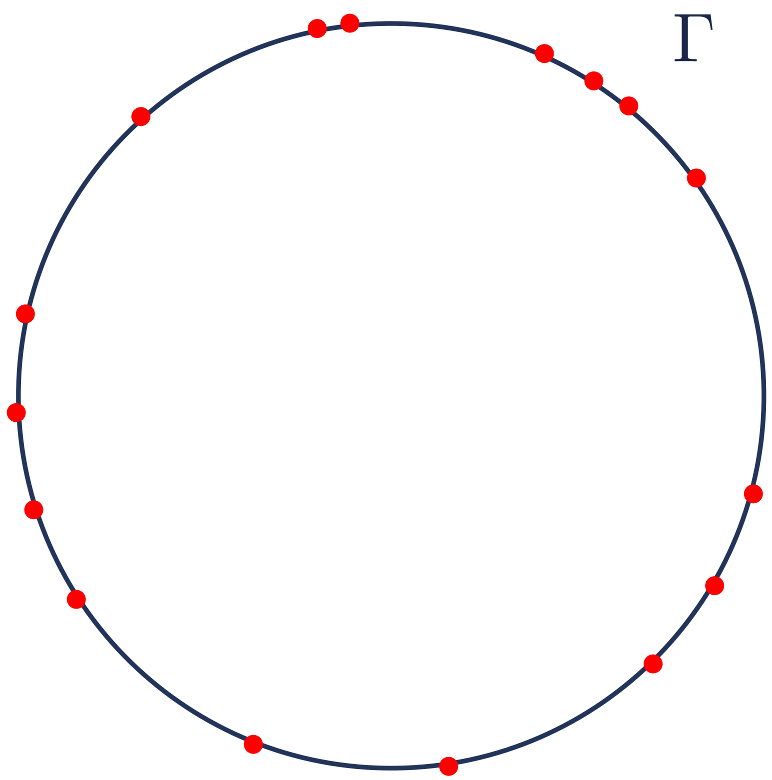
Zeros of $\sum_{k=1}^{n} \frac{z^k}{k}$ for $n = 64$

- Given any expansion with $0 < R_m < +\infty$:

There exists a subsequence of the m-th row of the Padé approximants whose zero limit distribution is the equilibrium measure of ∂D_m

Extensions

There exist many generalizations for extremal approximants



$$Q_n(z) = \prod_{i=1}^{n} (z - a_i), \quad a_i \in \Gamma$$

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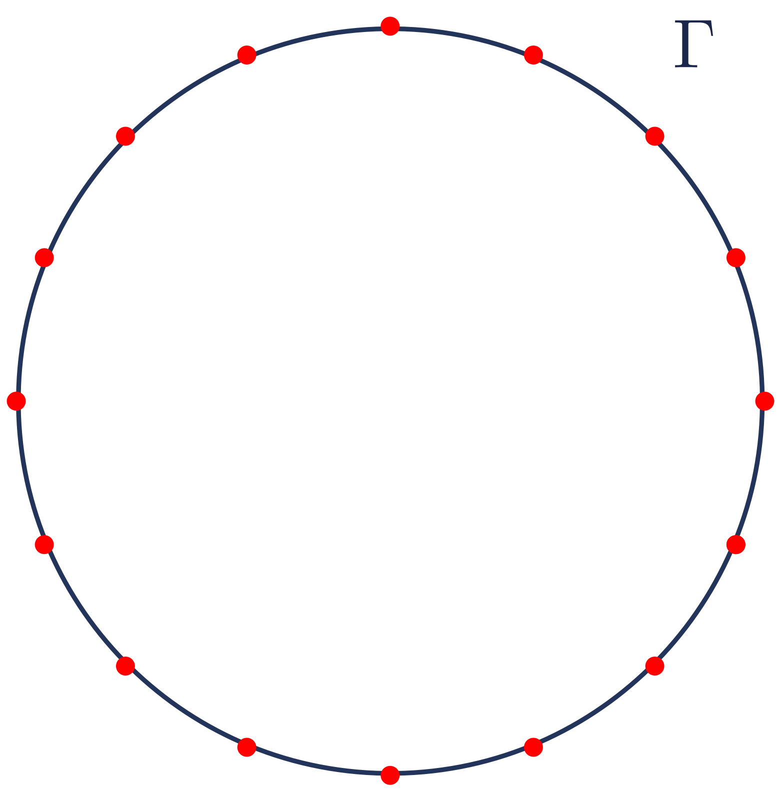


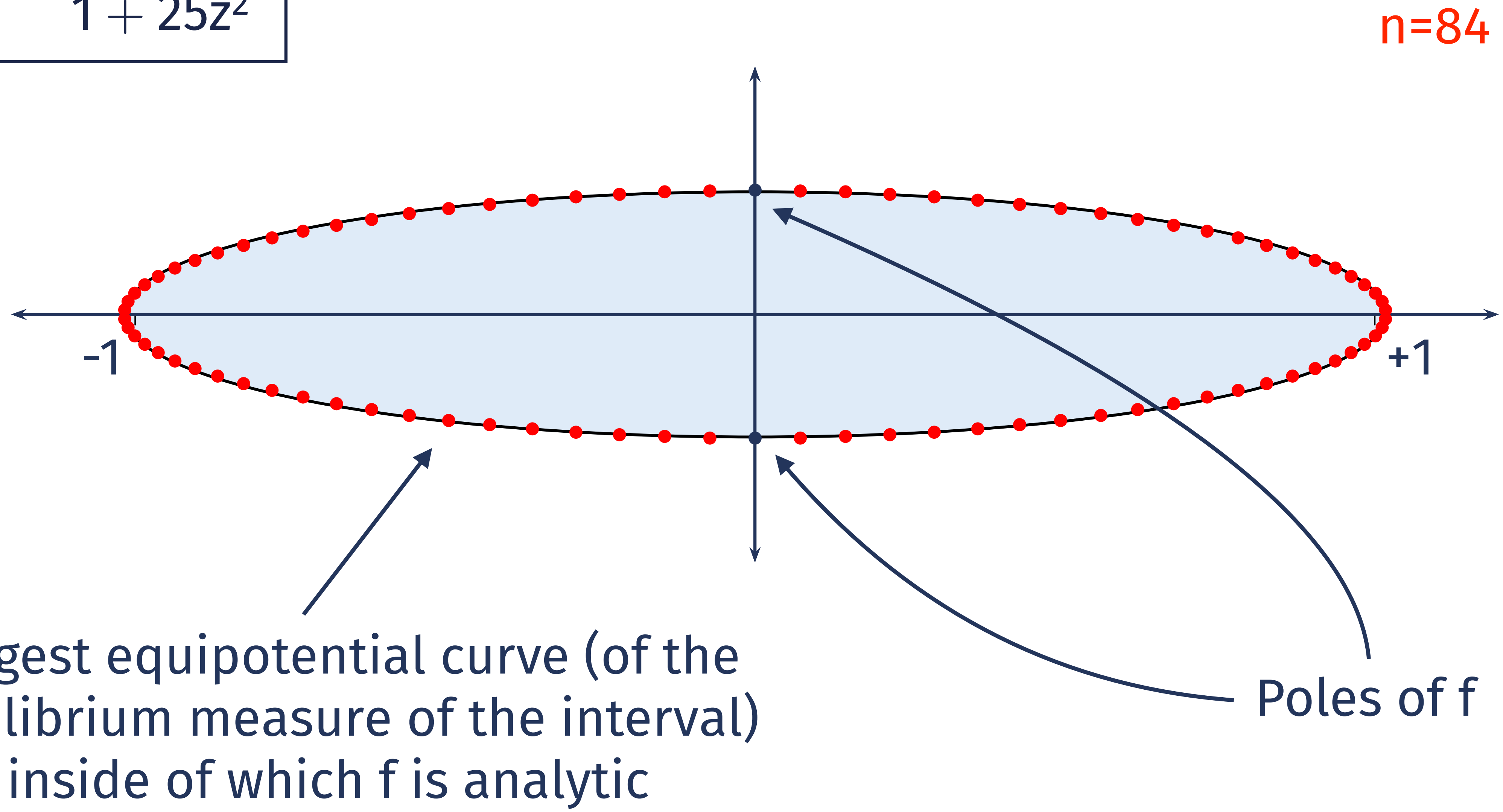
$\Longleftrightarrow$ $\|Q_n\|_\Gamma$ is minimum

$$Q_n(z) = \prod_{i=1}^{n}(z - a_i), \quad a_i \in \Gamma$$

Examples

• Zeros of the orthogonal expansion in the interval $[-1, 1]$ of the function f by means of Chebyshev polynomials.

$$f(z) = \frac{1}{1 + 25z^2}$$

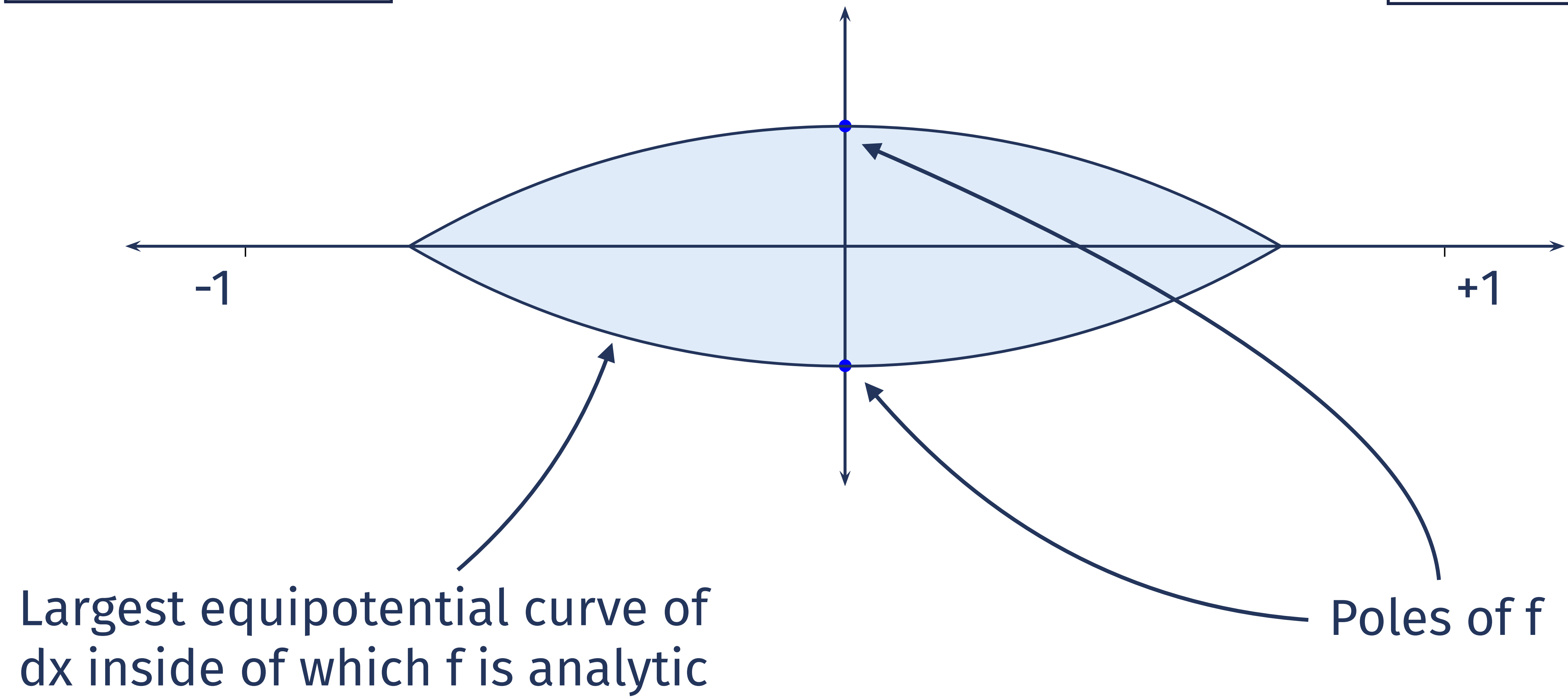


Examples

• Let P_n be the polynomial that interpolates the function f at the $n+1$ equidistant nodes of $[-1,1]$

$$f(z) = \frac{1}{1+25z^2}$$

$$dw_n \xrightarrow{*} \frac{dx}{2}$$



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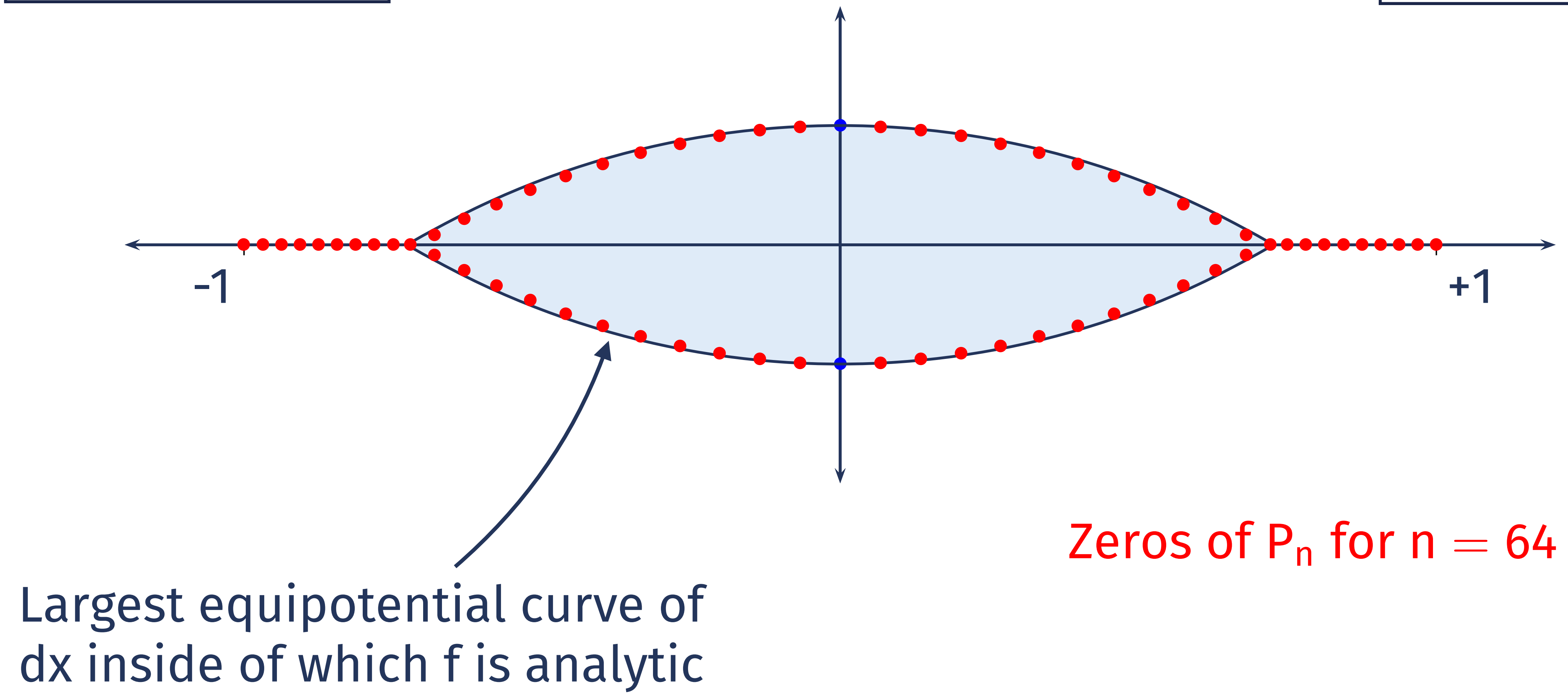

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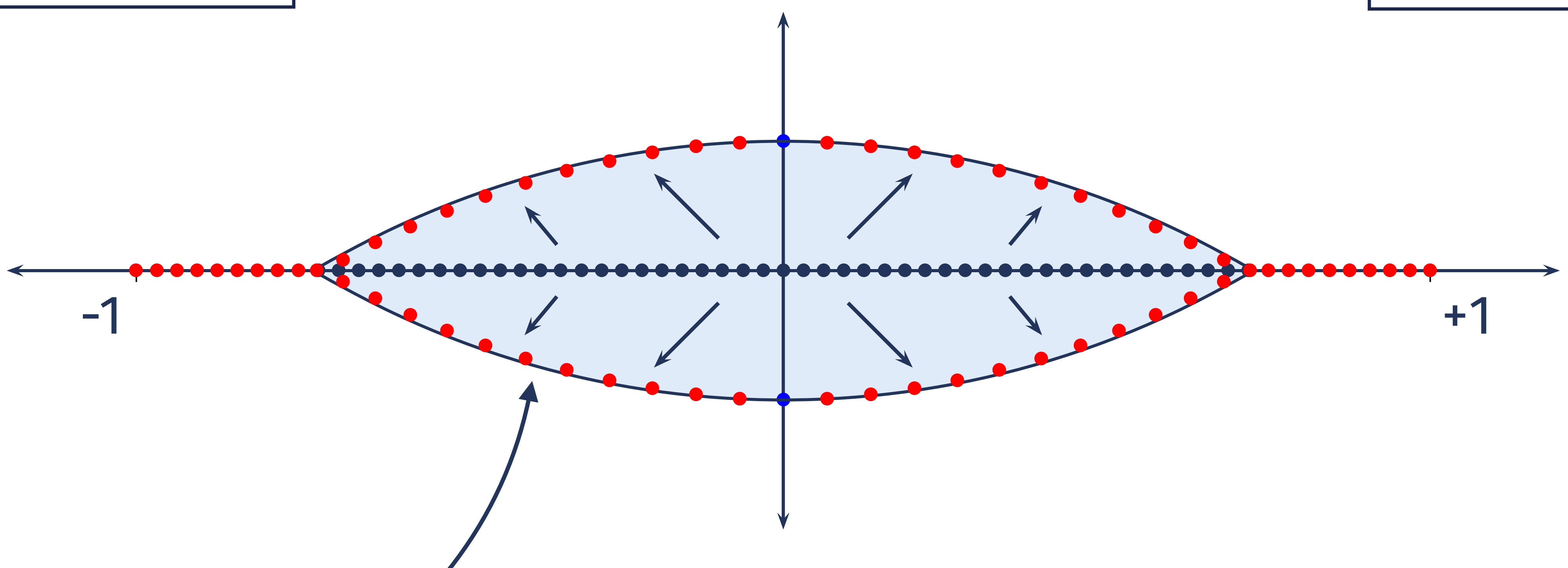
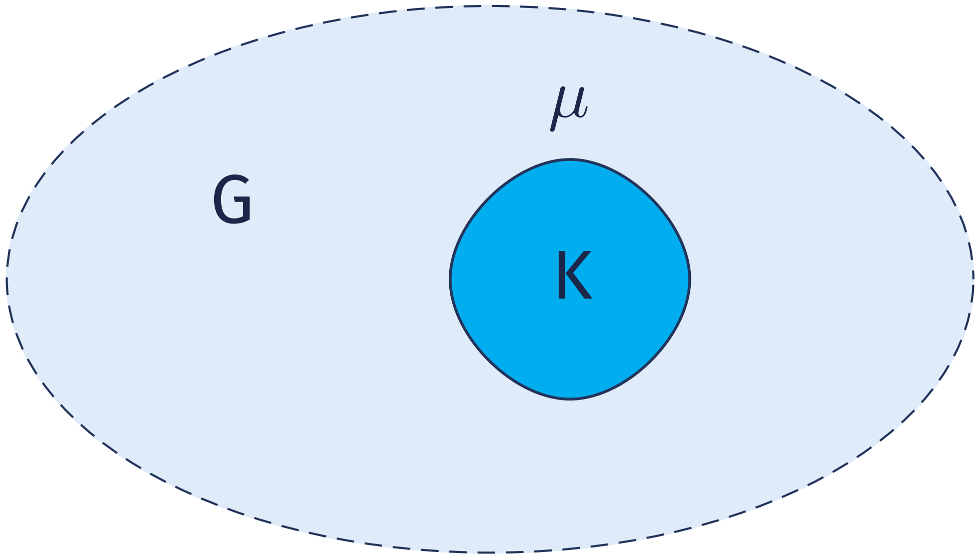


The interpolation points are swept towards the boundary!

Zeros of P_n for $n = 64$

Balayage measures

- Let μ be a probability measure supported on a compact set K which is contained in a bounded domain G.



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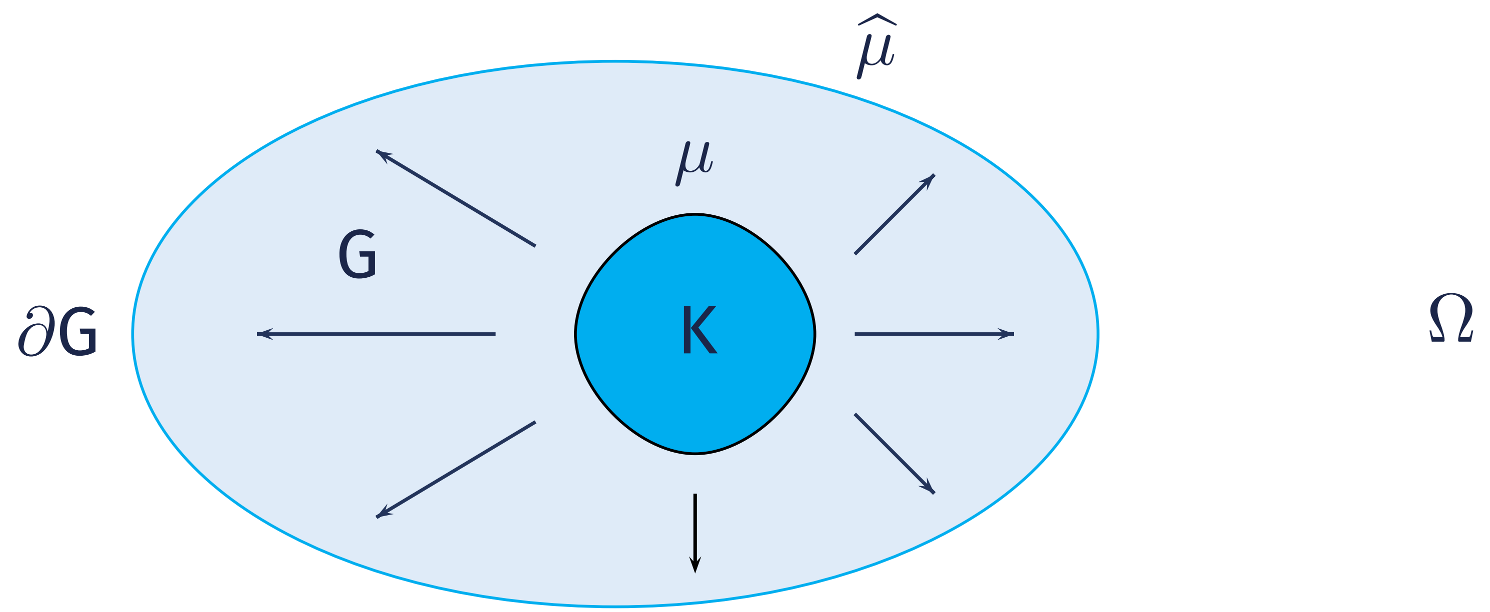

• The balayage measure of μ onto ∂G is the unique probability measure $\widehat{\mu}$ supported on ∂G such that

$$P(\widehat{\mu}; z) = P(\mu; z), \quad z \in \overline{\Omega} = \mathbb{C} \setminus G$$

Balayage measures

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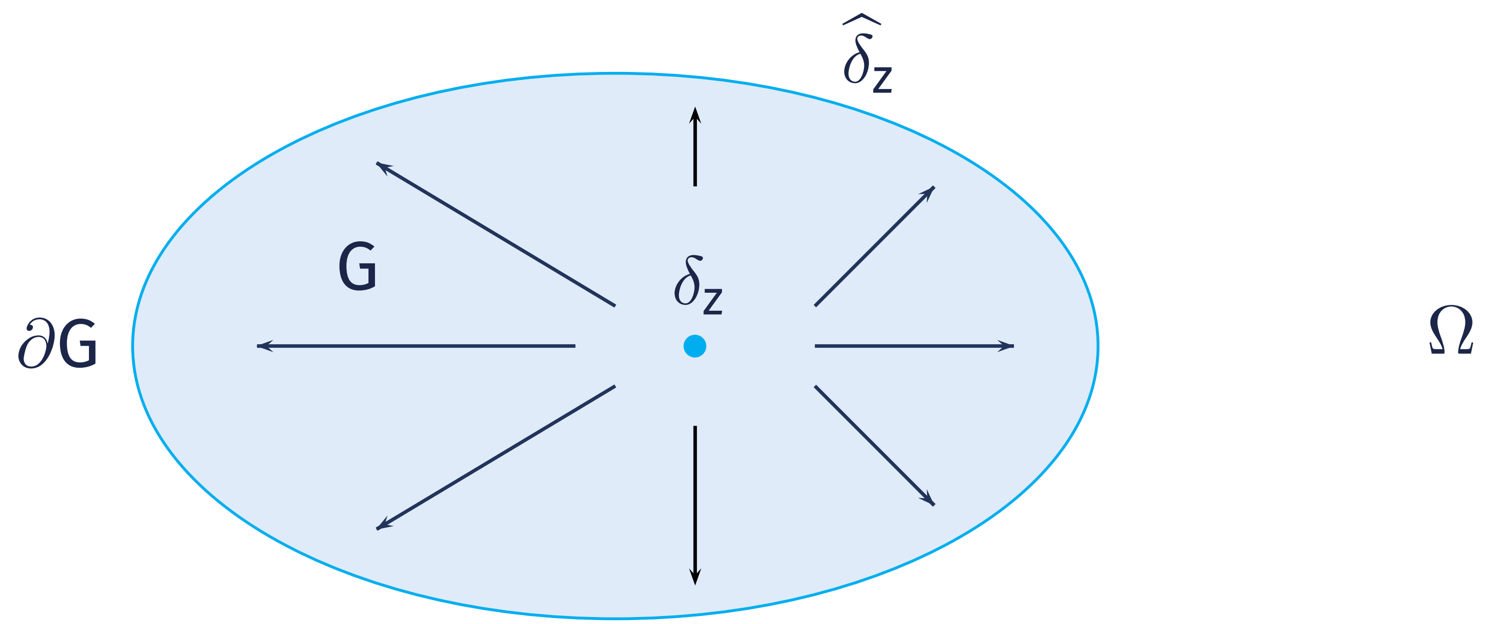

- The balayage measure of δ_z onto ∂G is the harmonic measure corresponding to z and G.

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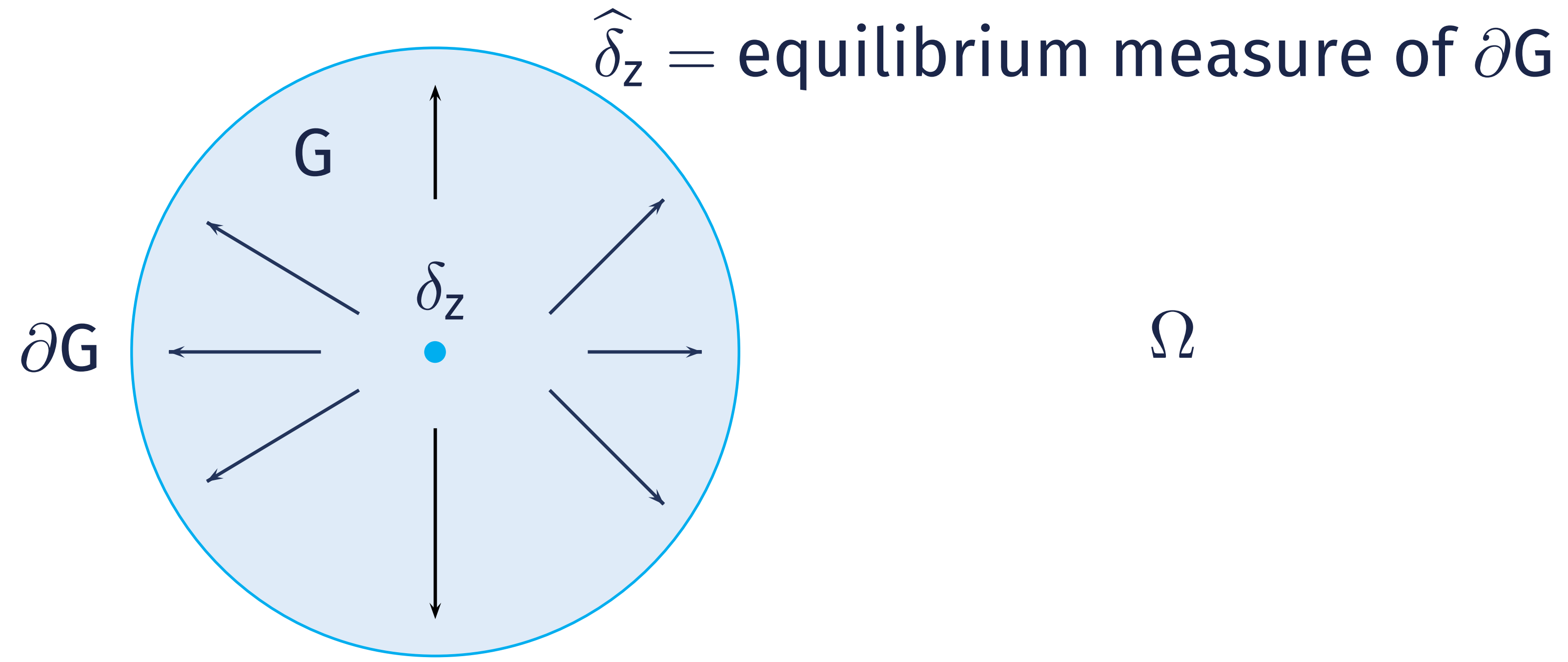

• The balayage measure of δ_z onto ∂G is the harmonic measure corresponding to z and G.

• The balayage measure of μ onto an equipotential curve of μ is the equilibrium measure of the curve.

Exact harmonic majorant

• The harmonic function h is an exact harmonic majorant of a sequence of subharmonic functions $\{u_n\}_{n\in\Lambda}$ on a domain D if

$$\boxed{\lim_{n\in\Lambda}\left\{\max_{z\in Q} u_n(z)\right\} = \max_{z\in Q} h(z)} \quad \text{for any continuum } Q \subset D$$

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$u_n = \dfrac{1}{n}\log|f_n|$ with f_n analytic in D

+

$\{u_n\}$ has an exact harmonic majorant on D

$\Longrightarrow$

There are $o(n)$ zeros of $\{f_n\}$ in compact subsets of D

Exact harmonic majorant

Exact rate of convergence of the interpolants on $D_m(\alpha)$

Existence of an exact harmonic majorant of $\frac{1}{n}\log|P_{n,m}|$ on the complement of $\Sigma \cup \overline{D_m(\alpha)}$ for a subsequence $\Lambda \subset \mathbb{N}$

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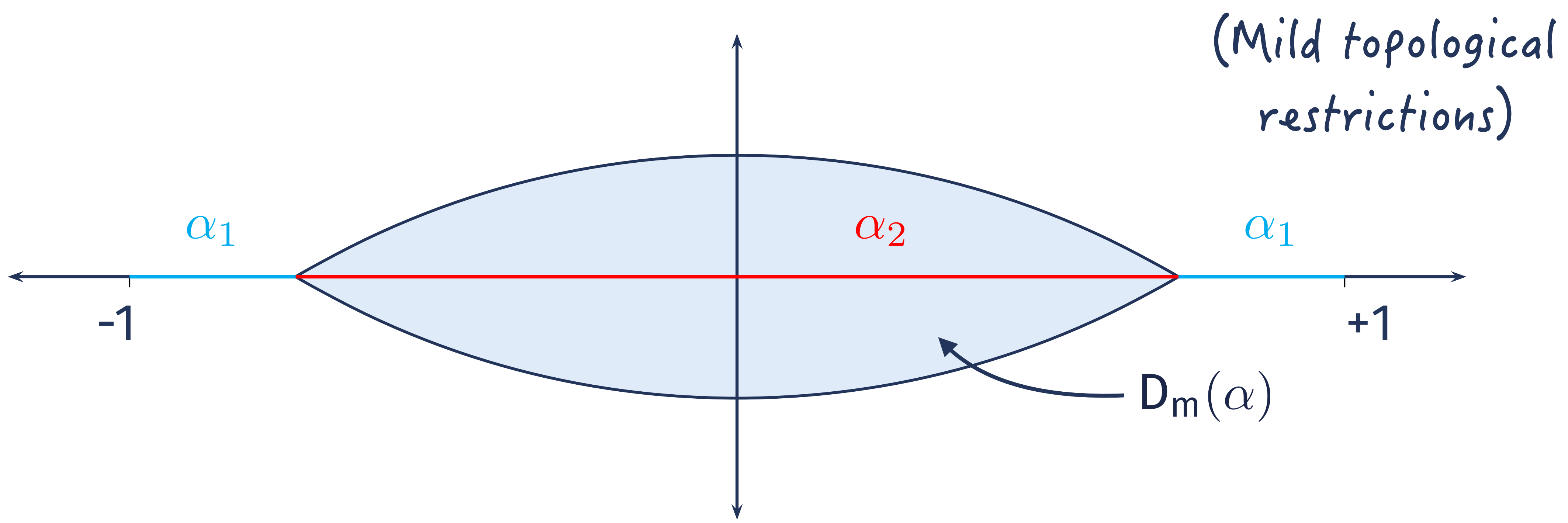
There are $o(n)$ zeros of $P_{n,m}$ in compact subsets of the complement of $\Sigma \cup \overline{D_m(\alpha)}$ for $n \in \Lambda$

The zero limit distribution of $P_{n,m}$ is supported on $\left(\Sigma \cup \overline{D_m(\alpha)}\right) \setminus D_m(\alpha)$

Theorem

- Let $\alpha = \alpha_1 + \alpha_2$ be the limit distribution of the interpolation points.



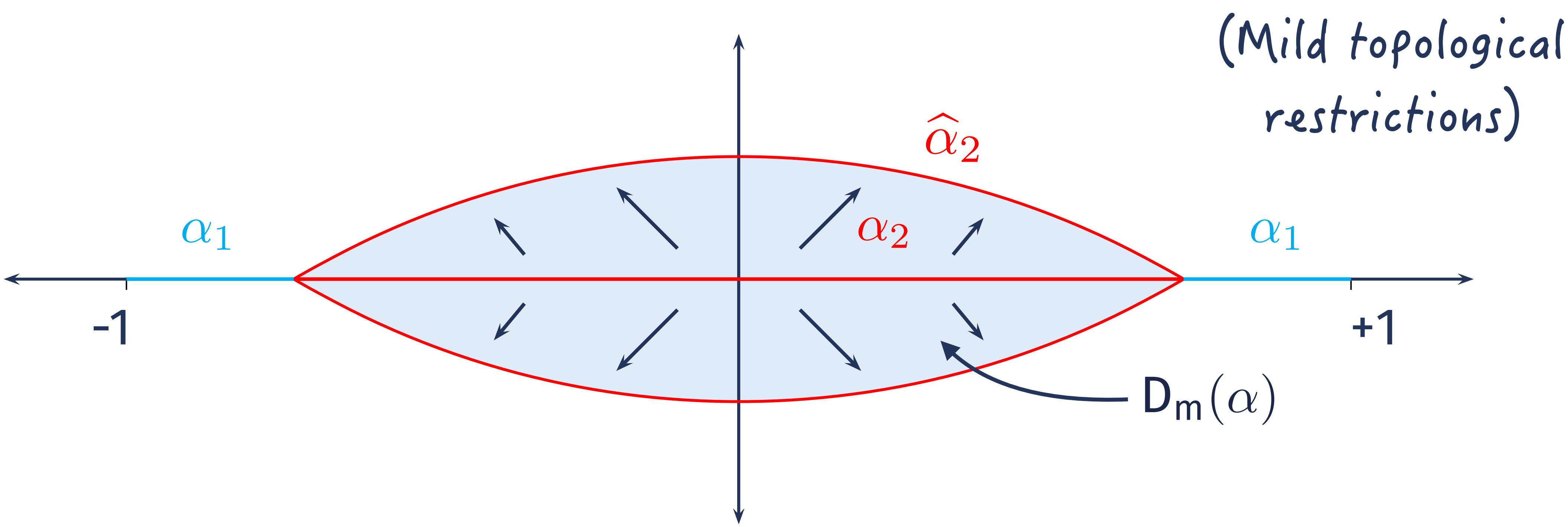
Theorem

- Let $\alpha = \alpha_1 + \alpha_2$ be the limit distribution of the interpolation points.

Then there exists $\Lambda \subset \mathbb{N}$ such that

$$\mathrm{d}P_{n,m} \xrightarrow{*} \alpha_1 + \widehat{\alpha}_2, \quad n \in \Lambda,$$

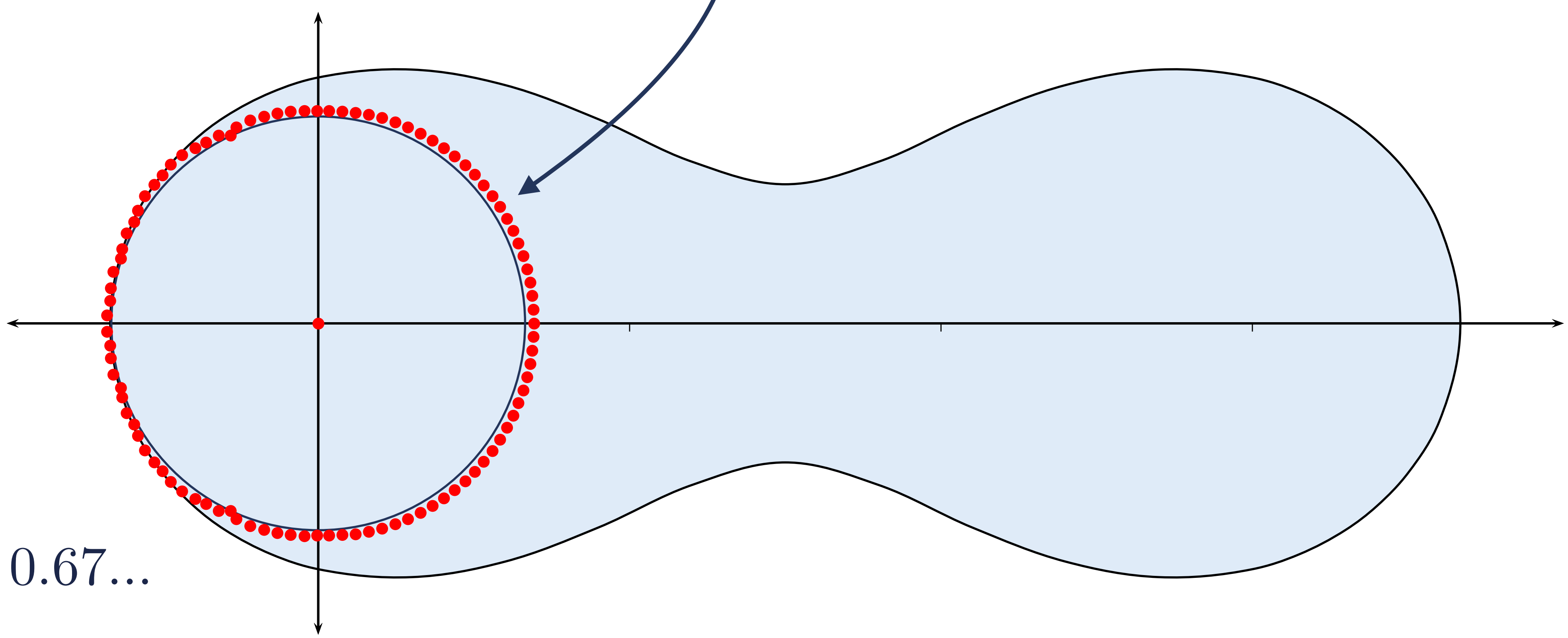
where $\widehat{\alpha}_2$ is the balayage of α_2 onto $\partial D_m(\alpha)$.



An open problem

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{z(z-3)}{\sqrt{6}} \right)^{3^k}$$

Zeros of the Taylor polynomial of the function f of degree 100



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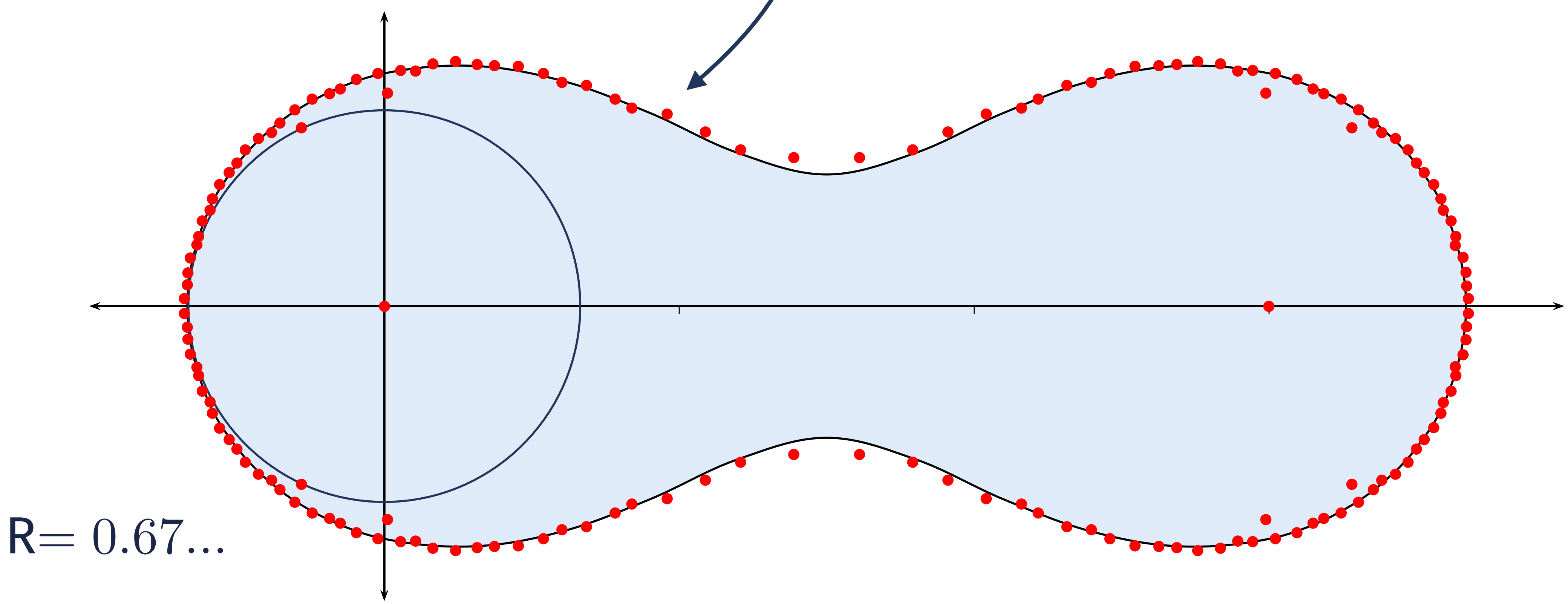
Zeros of the Taylor polynomial of the function f of degree 162=2x3^4



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The partial sums of the series are also interpolating polynomials at z=0 and z=3

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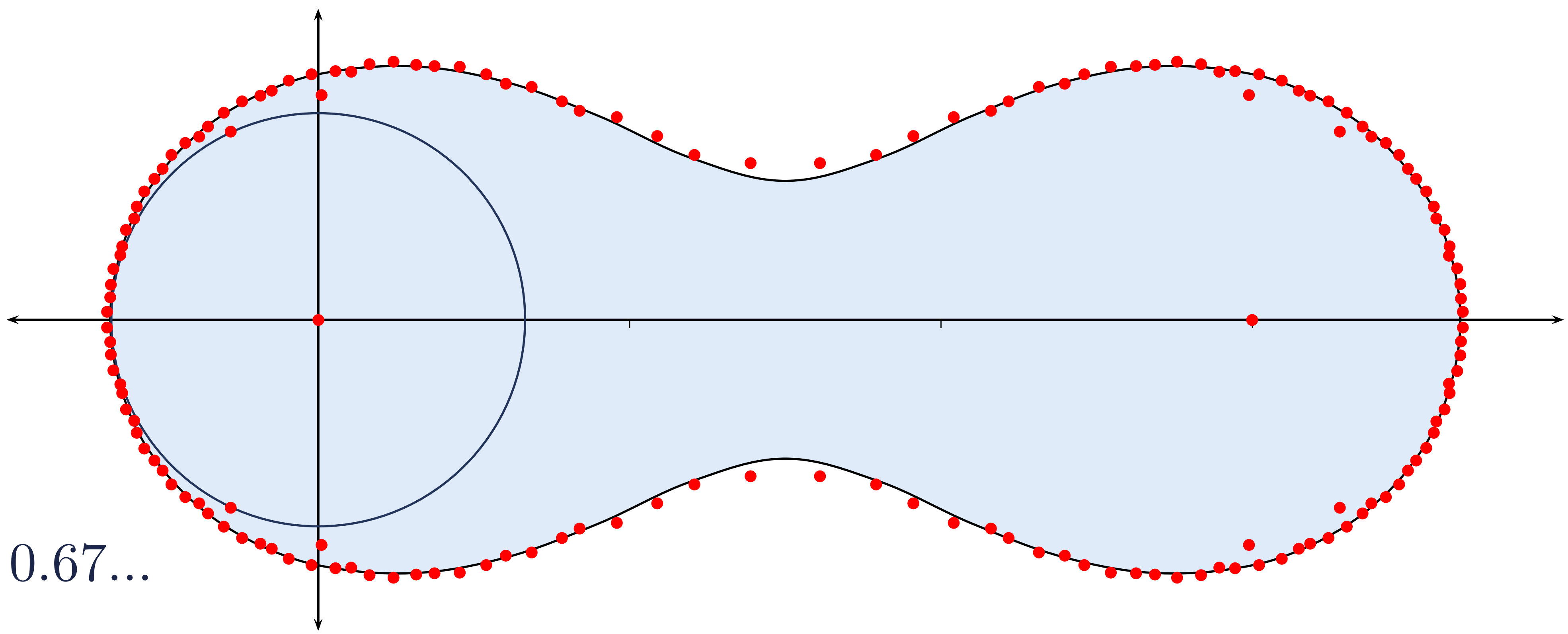

R $= 0.67\ldots$

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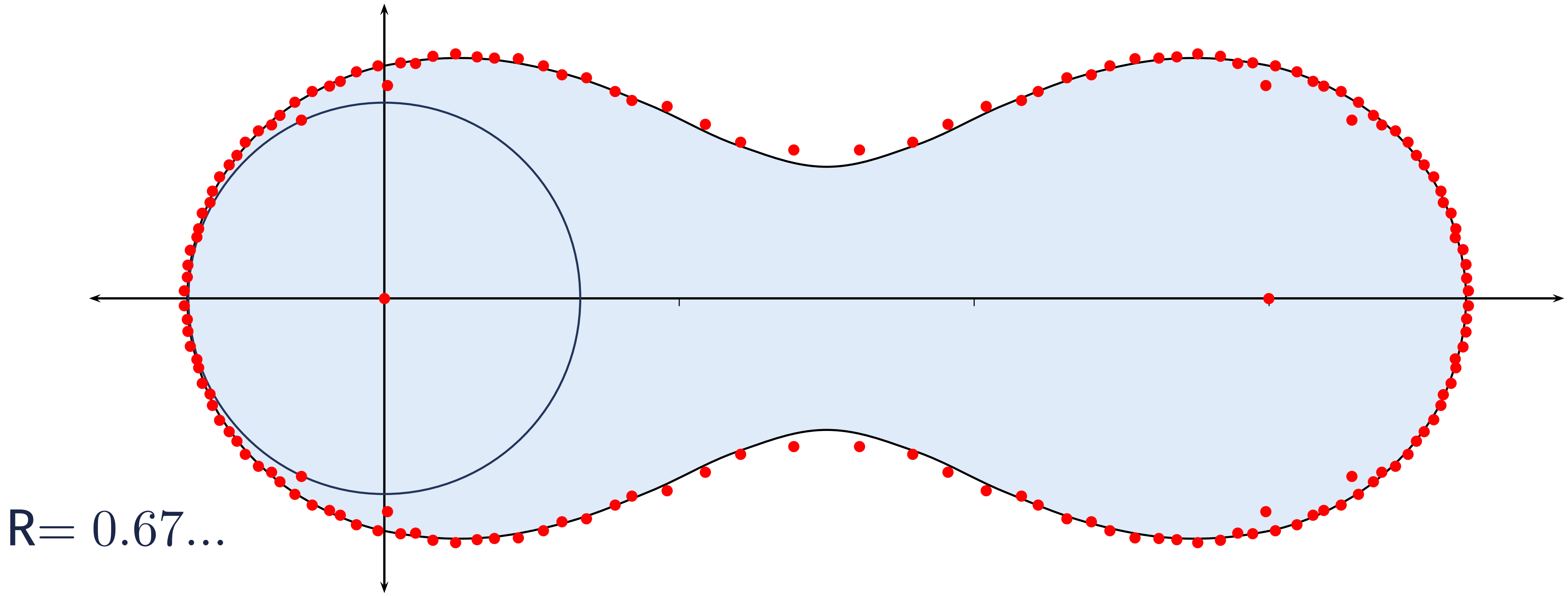


Is this true in general?

References

- **M. Bello Hernández, B. de la Calle Ysern,** *Meromorphic continuation of functions and arbitrary distribution of interpolation points,* J. Math. Anal. Appl. 284 (2013) 155-70.

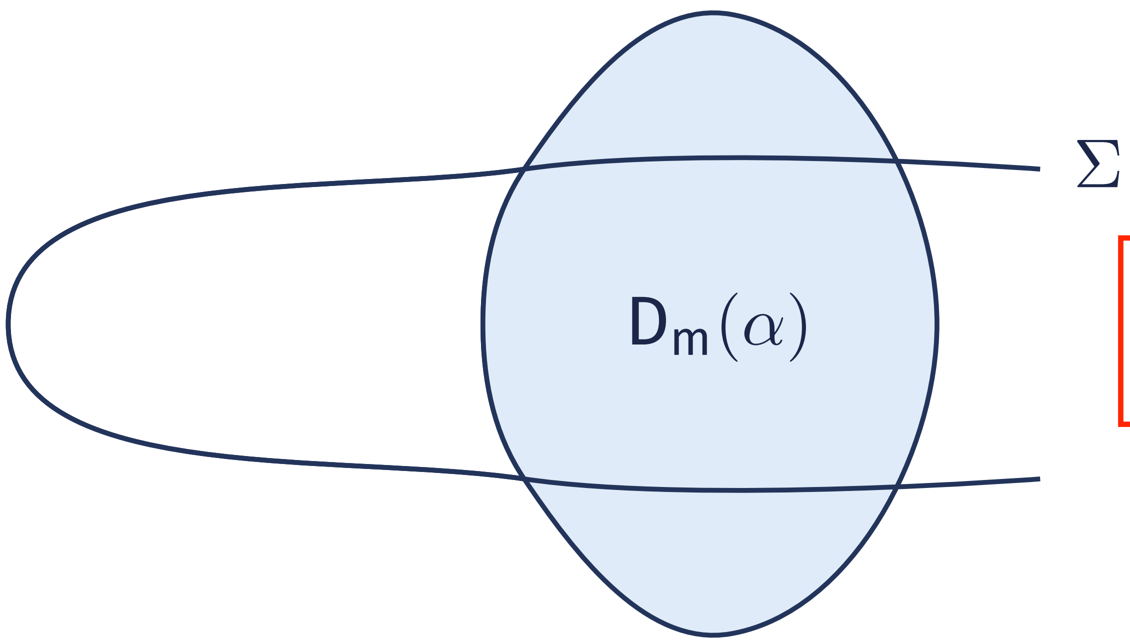
- **B. de la Calle Ysern,** *The Jentzsch-Szegö theorem and balayage measures,* Constr. Approx. 40 (2014) 307-327.

- **B. de la Calle Ysern, J. Mínguez Ceniceros,** *Rate of convergence of row sequences of multipoint Padé approximants,* J. Comput. Appl. Math. 284 (2015) 155-70.

- **B. de la Calle Ysern, J. Mínguez Ceniceros**, *Zero distribution of incomplete Padé and Hermite–Padé approximations,* J. Approx. Theory 201 (2016) 13-29.

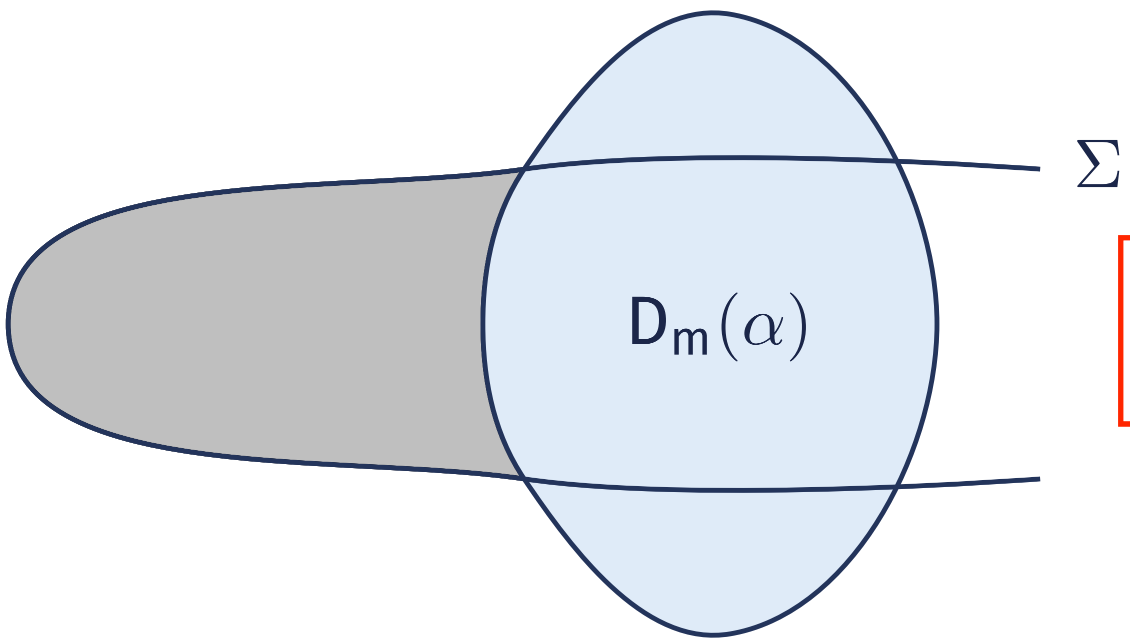
Thank you so much for your attention!

Topological restrictions



$\Omega = \overline{\mathbb{C}} \setminus (\overline{D_m(\alpha)} \cup \Sigma)$ is a connected set

Topological restrictions



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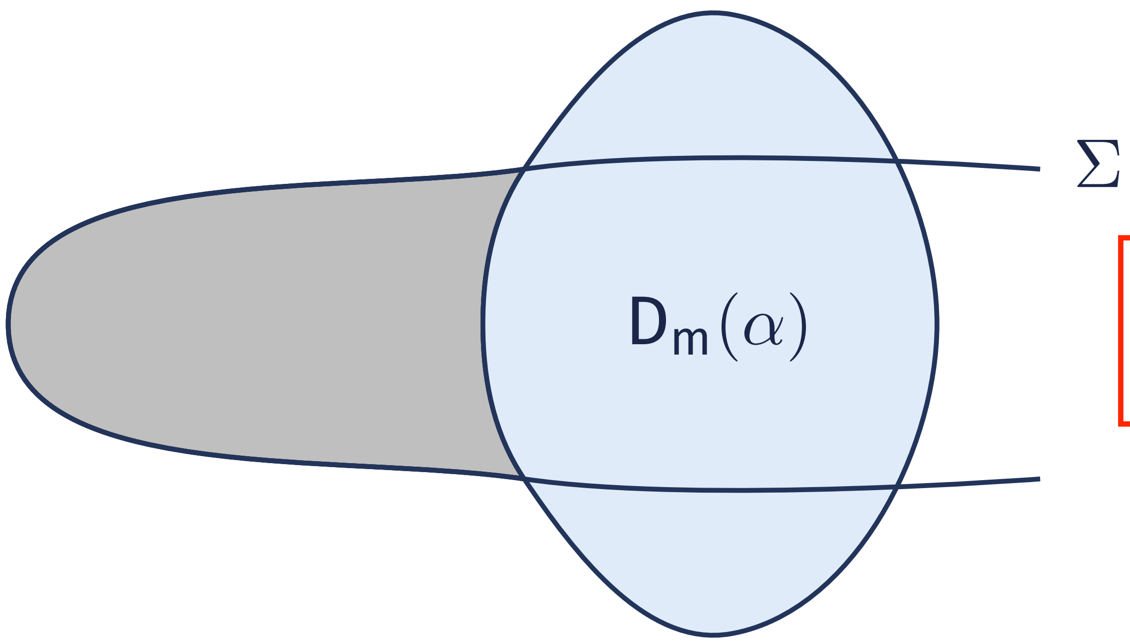

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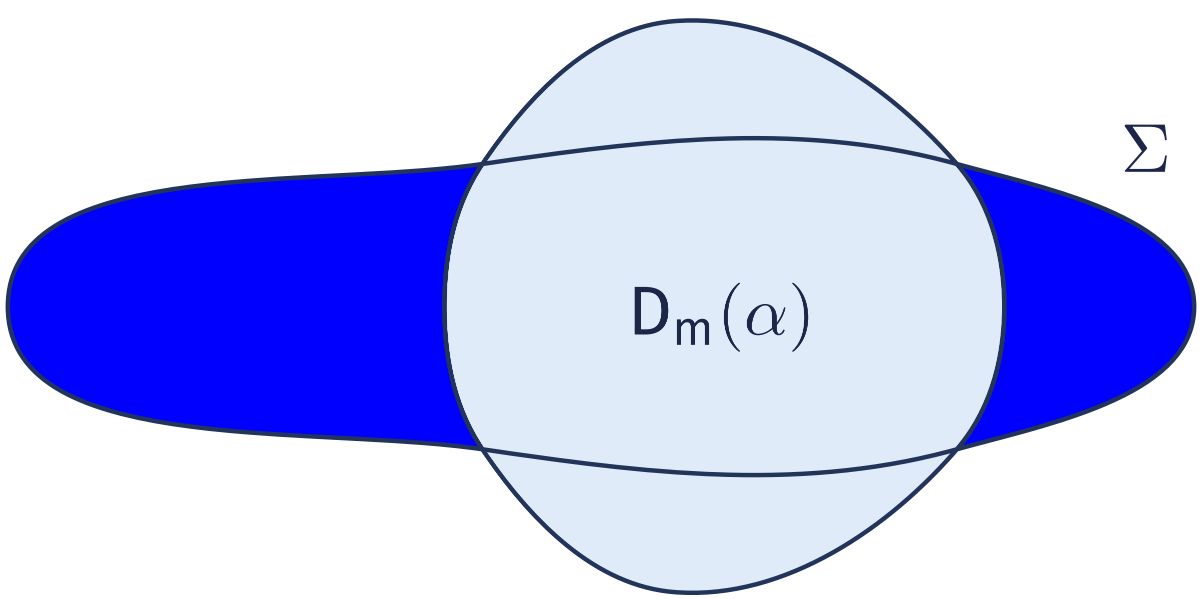
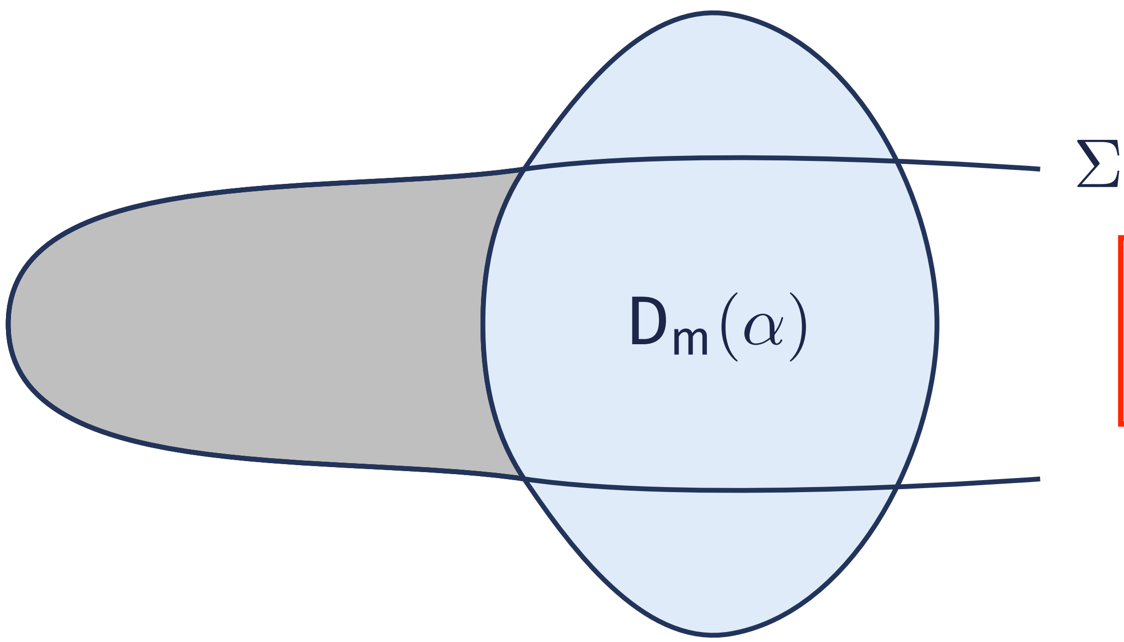


$\partial\Omega = (\overline{D_m(\alpha)} \cup \Sigma) \setminus D_m(\alpha)$

Topological restrictions



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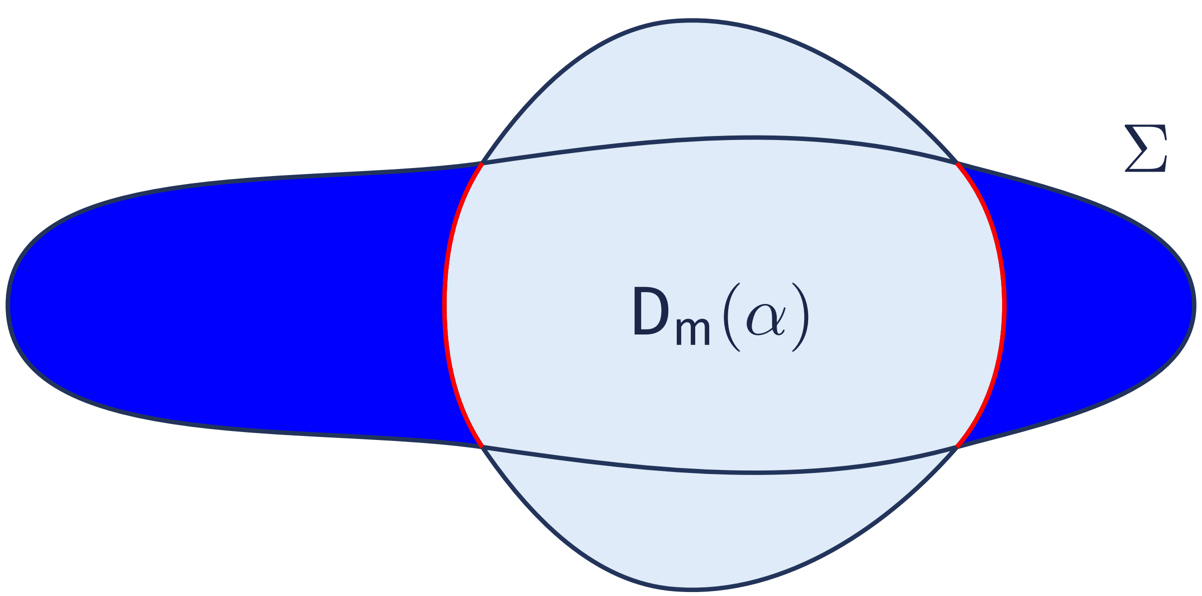

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