Approximation of algebraic functions by rational functions

(on functional analogues of the Diophantine approximants)

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Result

A. I. Aptekarev and M. L. Yattselev,

Pade approximants for functions with branch points – strong asymptotics of Nuttall-Stahl polynomials,

Acta Math., 215:2 (2015), 217 -280, (ArXiv 1109.0332)

Rational approximants of algebraic numbers

Continued fractions

Notations for numbers:

Natural $\mathbb{N}$; Integer $\mathbb{Z}$; Rational $\mathbb{Q}$; Algebraic $\mathcal{A}$; Real $\mathbb{R}$:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathcal{A} \subset \mathbb{R}$$

Eucledian Algorithm : $\alpha \in \mathbb{R}_+ \to$ Continued Fraction

$\{a_l\}$ in $\mathbb{N}$ (incomplete quotients):

$$\alpha = a_0 + \frac{1}{a_1+} \frac{1}{a_2+} \frac{1}{a_3 + \dots} .$$

Convergents – Rational Approximants

$$a_0 + \frac{1}{a_1 + \dots} \frac{1}{a_n} := \frac{p_n}{q_n} \in \mathbb{Q} .$$

Degree of approximation, bounds from above and bellow

- Convergents C.F. are best approximants.

- Bound from above:
  Theorem (Hurwitz-Markov). $\alpha \notin \mathbb{Q} \quad \Rightarrow \quad \exists$ i.m.s.

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{\sqrt{5}}\, q^{-2}, \quad p/q \in \{\text{conver. c. f.}\}\,(\alpha)$$

- Bound from bellow:
  Theorem (Liouville). $\alpha \in \mathcal{A}_k,\ k \geqslant 2 \quad \Rightarrow$

$$\exists\, C(\alpha) \, : \quad \left|\alpha - \frac{p}{q}\right| \geqslant C(\alpha)\, q^{-k}, \quad \forall p, q \in \mathbb{Q}.$$

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Thue–Siegel–Roth Theorem and S.A.I.

- Theorem (Thue-1909, Siegel-21, Dyson-47, Gelfond-48, Roth-55). $\alpha \in \mathcal{A} \quad \Rightarrow$

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- Slowly Approximated Irrationalities $\tilde{\mathcal{A}}$:
$\alpha \in \tilde{\mathcal{A}} \ \Leftrightarrow \ \varepsilon = 0 \ \Leftrightarrow \ \exists C(\alpha) : \ a_n < C, \ \forall n \in \mathcal{A}$

$$\alpha = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 + \ldots} .$$

- We have (Euler–Lagrange) $\mathcal{A}_2 \equiv \tilde{\mathcal{A}}_2$,

$$\alpha \in \mathcal{A}_k, \ \ k > 2 \qquad ???$$

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Example $\alpha := \sqrt[3]{2}$

Numer. computation (A.Bruno 1964, S.Lang-72, A.A.-75)

$$\alpha = 1 + \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{5+}\ \frac{1}{1+\dots}\ .$$

$a_{10} = 14;$

$a_{36} = 543;$

$a_{572} = 7451;$

$a_{620} = 4941;$

Rational approximants of algebraic functions

Rational approximants – continued fraction

- Given a germ

$$f(z)=\sum_{k=0}^{\infty}\frac{f_k}{z^{k+1}}\in\mathbb{C}((z^{-1}))\qquad\leftrightarrow\qquad\alpha\in\mathbb{R}_+.$$

- Eucledian Algorithm :

$$f\ \to \text{Continued Fraction } \{t_l(z)\}_{l=1}^{\infty},:$$

$$f(z)=\frac{1}{t_1(z)+}\,\frac{1}{t_2(z)+}\,\frac{1}{t_3(z)+\dots},\quad t_l\in\mathcal{P}\ \leftrightarrow\ a_l\in\mathbb{N}$$

- Convergents – Rational Approximants

$$\frac{1}{t_1(z)+\dots}\,\frac{1}{t_n(z)}:=\frac{p_n(z)}{q_n(z)}=\pi_n(z)\in\mathcal{R}\ \leftrightarrow\ \mathbb{Q}\,.$$

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Kolchin's conjecture (Functional TSR conjecture)

notations: $Z(f) :=$ ord zero f at ∞;

$$\nu_n(f) = \sup_{r \in \mathcal{R}_n} Z(f - r);$$

$$\|f\|_a := a^{-Z(f)} \text{ for fixed } a > 1.$$

Kolchin's conjecture (1959):
$f \in \mathcal{A}(z)$ or solution of alg. D.E. $\Rightarrow \forall \varepsilon > 0 \quad \exists C(\varepsilon, f)$:

$$(2n \leqslant,) \quad \nu_n(f) < (2 + \varepsilon)n + C(f), \qquad n \in \mathbb{N}\,.$$

S. Uchiyama (1961), Osgood (1984), Chudnovskies (1983, 1984), H.Stahl (1985)

$$\lim_{n\to\infty} \frac{\nu_n(f)}{n} = 2\,.$$

Chudnovskies:

"In the functional case, the solution of Kolchins problem, is equivalent to the normality and the almost normality of Pade approximations. ..."

The Wronskian Formalism for Linear Differential Equations and Pade Approximants, ADVANCES IN MATHEMATICS 53, 28-54 (1984)

Gonchar – Chudnovskies conjecture $\varepsilon = 0$

- Gonchar (1978), Chudnovskies(1984) conjecture:
  $f \in \mathcal{A}(z) \Rightarrow \exists C(f)$:

$$\nu_n(f) \leqslant 2\,n + C(f), \qquad n \in \mathbb{N}\,.$$

- Chudnovskies: "We want to note that our conjecture that $\varepsilon = 0$ in Kolchin's problem is the feature of the rational approximation problem in the functional case only. For numbers it seems highly implausible that one can have $\varepsilon = 0$."

- for algebraic functions (branch points are in a generic position) validity of the G-Ch cojecture follows from Ap-Ya result.

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Rational Pade approximants

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Diagonal PA $\pi_n(z) = p_n/q_n$ to $f = \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}}$,

is $Z\left(f(z) - \pi(z)\right) \quad = \sup_{r\in\mathcal{R}_n} Z(f-r) =: \nu_n(f).$

To determine q_n we have the linear system

$$R_n(z) := q_n(z)f(z) - p_n(z) = O\left(1/z^{n+1}\right) \quad \text{as} \quad z \to \infty$$

Normal index $n \in \mathcal{N} \quad \Leftrightarrow \quad \deg\, q_n = n$

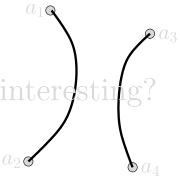
Functions with branch points

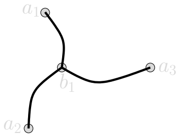
- f be an analytic (and multi-valued)

$$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A), \qquad \sharp A < \infty,$$

A - branch points.

- Behavior of the poles of the diagonal PA ?





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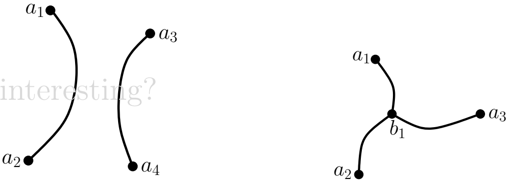

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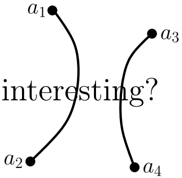

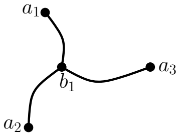

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Nuttall's conjecture and domain of holomorphicity

- J. Nuttall. On convergence of Padé approximants to functions with branch points. Padé and rational Approximation, (E. B. Saff, R. S. Varga, eds.), Academic Press, New York, 1977, pp. 101–109.

- Convergence in capacity:

$$\forall \varepsilon > 0 , \quad \lim_{n\to\infty} \operatorname{cap}(\{z \in K : \quad |f(z) - [n/n]_f(z)| > \varepsilon\}) = 0 ,$$

$$K \Subset \mathbb{C} \setminus \Delta$$

- Compact of minimal capacity Δ:

$$\operatorname{cap}(\Delta) = \min_{\partial D : D \in \mathcal{D}_f} \operatorname{cap}(\partial D) , \qquad \mathcal{D}_f := \{D \ : \ f \in \mathcal{H}(D)\}$$

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Stahl's Theorem

H. Stahl 1985-1986. $f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)$:

$$\operatorname{cap}(A) = 0$$

- the existence of a domain $D^* \in \mathcal{D}_f$;
- weak (n-th root) asymptotics for the denominators q_n

$$\lim_{n\to\infty} \frac{1}{n} \log|q_n(z)| = -V^\lambda(z), \qquad z \in D^*,$$

where $V^\lambda := -\int \log|z-t|\, d\lambda(t)$
equilibrium measure λ – minimizer
$I(\mu) := \int V^\mu(z)\, d\mu(z)$:

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Our goals

the strong (or Szegő type) asymptotics

$$\lim_{n\to\infty} q_n(z) = ? \quad z \in D^*,$$

of the Nuttall-Stahl polynomials q_n, i.e., of the polynomials that are the denominators of the diagonal Padé approximants for

$$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)\,, \qquad A - \text{branch points}, \sharp A < \infty.$$

Motivation: Uniform convergence; Spurious poles; .....

Statement of the result

Class of functions

$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)\,, \qquad A := \{a_j\}, \quad 2 \leq \sharp A < \infty.$

- Restrictions:

  (I) Character of singularities at $\{a_j\}$

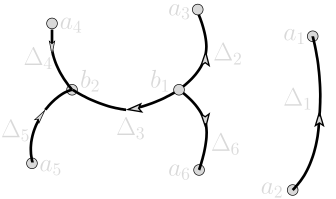
  (II) Disposition of the branch points $\{a_j\}$

- (I) Algebro-logarithmic branching condition:

- (II) Generic position condition:

  $\Delta = \overline{\mathbb{C}} \setminus D^* = E \cup \bigcup \Delta_k,$

  $E = \{a_1, \ldots a_p,\, b_1, \ldots b_q,\}$



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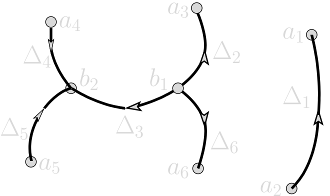

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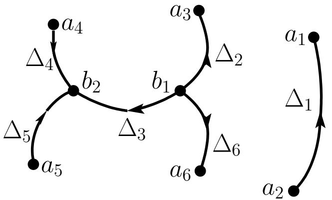

Class of functions. Restrictions:

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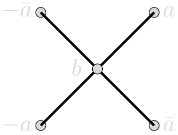
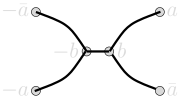
$$\rho(z) := w_k(z)(z-e_{k,1})^{\alpha_{k,1}}(z-e_{k,2}))^{\alpha_{k,2}}, \qquad z \in \Delta_k,\ w_k, w_k^{-1} \in \mathcal{H}(\Delta_k).$$

- GP "constellations"





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- GP “constellations”

Gonchar – Chudnovskies conjecture

Riemann surface $\mathfrak{R}$ – double of the extremal domain of holomorphicity D^*.

COROLLARY of the result for generic case:

$f \in \mathcal{A}(z)$ + GP condition $\Rightarrow$ $\exists C(f)$:

$$\nu_n(f) < 2n + C(f), \quad n \in \mathbb{N}\,, \quad C(f) \leqslant \mathrm{gen}\mathfrak{R}.$$

CONJECTURE for general case:

$f \in \mathcal{A}(z) \Rightarrow \exists C(f) \leqslant \mathrm{gen}\mathfrak{R} + \mathrm{degDiscrim}(f)$:

Thank you for your attention!