Convergence of Dimension Elevation Algorithms: Only a Typical CAGD Issue ?

Rachid Ait-Haddou

Osaka (Japan)

(Joint work with Marie-Laurence Mazure)

Joseph Fourier University

Grenoble (France)

Outline

- Dimension Elevation Algorithms
  - Degree Elevation of Bézier Curves
  - Dimension Elevation in Chebyshev Spaces

Outline

- Dimension Elevation Algorithms
  - Degree Elevation of Bézier Curves
  - Dimension Elevation in Chebyshev Spaces
- Dimension Elevation in Müntz Spaces
  - Characterization of Convergence
  - Dimension Elevation Versus Density

Outline

- Dimension Elevation Algorithms
  - Degree Elevation of Bézier Curves
  - Dimension Elevation in Chebyshev Spaces
- Dimension Elevation in Müntz Spaces
  - Characterization of Convergence
  - Dimension Elevation Versus Density
- Dimension Elevation Versus Bernstein Operators

Outline

- Dimension Elevation Algorithms
  - Degree Elevation of Bézier Curves
  - Dimension Elevation in Chebyshev Spaces
- Dimension Elevation in Müntz Spaces
  - Characterization of Convergence
  - Dimension Elevation Versus Density
- Dimension Elevation Versus Bernstein Operators
- Dimension Elevation For Rational Spaces
  - Characterization of Convergence
  - Connection to Pólya's Theorem on Positive Polynomials

Bézier Curves

Bernstein Basis (Bernstein, 1912)

$$B_k^n(t) = \binom{n}{k} \left(\frac{t-a}{b-a}\right)^k \left(\frac{b-t}{b-a}\right)^{n-k} \qquad k = 0, 1, \ldots, n.$$

- Basis of the linear space $\mathbb{P}_n$ of polynomials of degree n.
- $0 \leq B_k^n(t) \leq 1 \quad$ for $t \in [a, b]$.
- $\sum_{k=0}^n B_k^n(t) = 1 \quad$ for any $t \in \mathbb{R}$.
- $\frac{d}{dt} B_k^n(t) = n(b-a) \left(B_{k-1}^{n-1}(t) - B_k^{n-1}(t)\right).$

Bézier curves (Bézier-de Casteljau 1960)

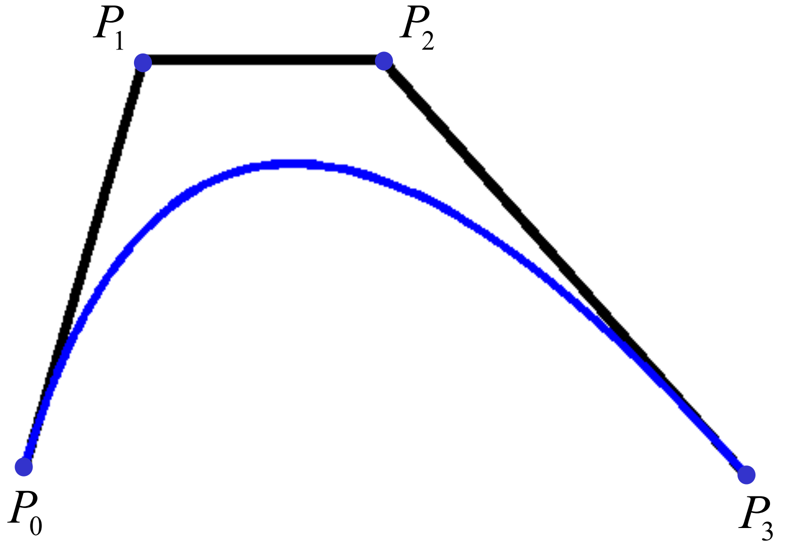
$$\Gamma : P(t) = \sum_{k=0}^{n} B_k^n(t) P_k; \quad t \in [a,b]; \quad P_k \in \mathbb{R}^d.$$

- $P(a) = P_0$ and $P(b) = P_n$.
- The curve Γ lies in the convex hull of the points $P_0, P_1, \dots, P_n$.
- $P'(a) = n(b-a)(P_1 - P_0)$ and $P'(b) = n(b-a)(P_n - P_{n-1})$.

⇨ The curve Γ is tangent to the end-segments of the polygon $(P_0, P_1, \dots, P_n)$.

Bernstein Representation of Polynomials

$$\Gamma : P(t) = \sum_{k=0}^{n} B_k^n(t) P_k; \quad t \in [a,b]; \quad P_k \in \mathbb{R}^d$$



(de Casteljau algorithm, blossom, degree elevation, splines,...)

Degree Elevation Algorithm

Degree Elevation of Bézier Curves

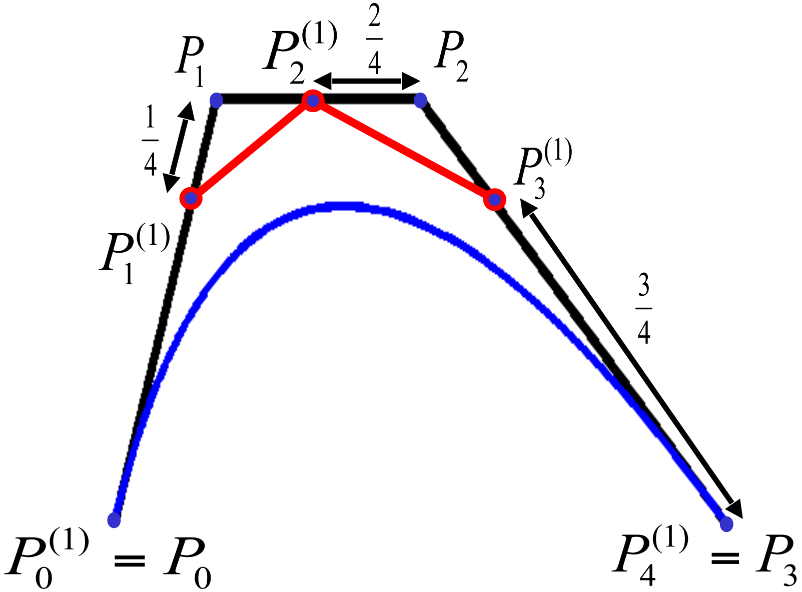
- $\mathbb{P}_n \subset \mathbb{P}_{n+1}$

$$P(t) = \sum_{k=0}^{n} B_k^n(t) P_k = \sum_{k=0}^{n+1} B_k^{n+1}(t) P_k^{(1)}.$$

- New control points

$$P_0^{(1)} = P_0, \quad P_{n+1}^{(1)} = P_n.$$

$$P_k^{(1)} = \frac{k}{n+1} P_{k-1} + \left(1 - \frac{k}{n+1}\right) P_k.$$



- $\mathbb{P}_n \subset \mathbb{P}_{n+1} \subset \mathbb{P}_{n+2} \subset \dots$

⇨ Iterating degree elevation leads to a sequence of control polygons.

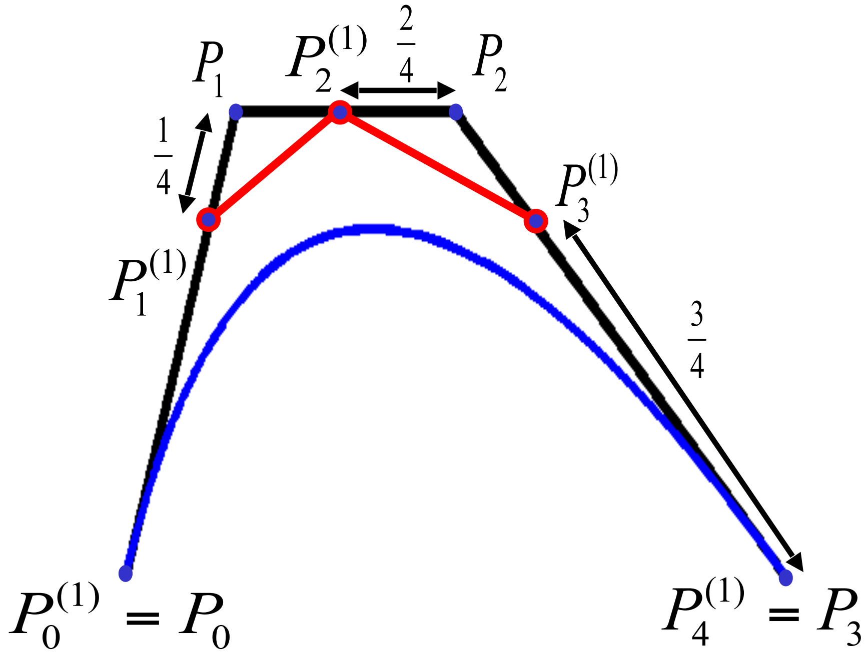
Degree Elevation

Degree Elevation of Bézier Curves

$$P(t) = \sum_{k=0}^{3} B_k^3(t) P_k = \sum_{k=0}^{4} B_k^4(t) P_k^{(1)}.$$

$$[t^0 \quad t^1 \quad t^2 \quad t^3 \quad | \quad t^4]$$

$$[0 \quad 1 \quad 2 \quad 3 \quad | \quad 4\ ]$$

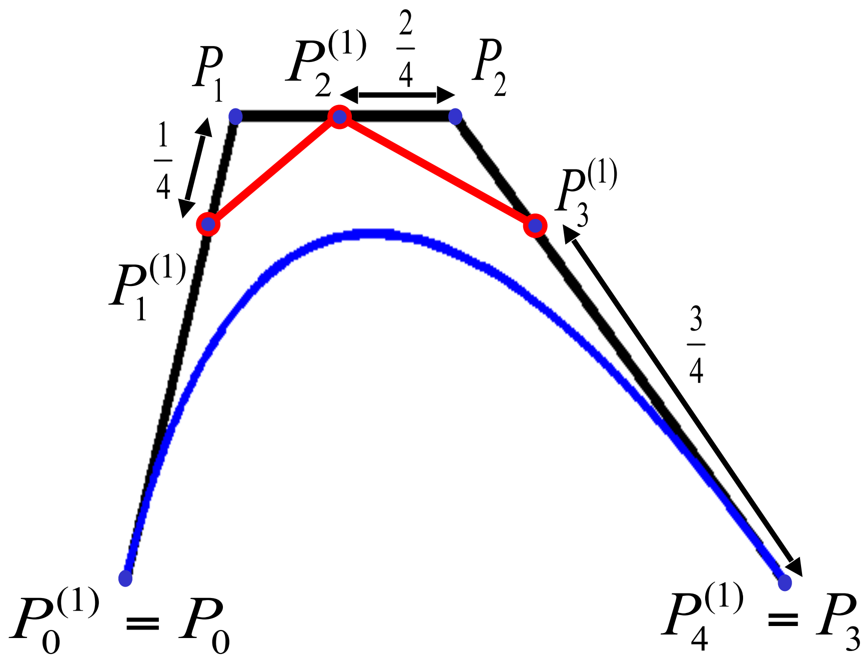


Degree Elevation of Bézier Curves

$$P(t) = \sum_{k=0}^{3} B_k^3(t) P_k = \sum_{k=0}^{4} B_k^4(t) P_k^{(1)}.$$

$$[t^0 \quad t^1 \quad t^2 \quad t^3 \quad | \quad t^4]$$

$$[0 \quad 1 \quad 2 \quad 3 \quad | \quad 4\,]$$



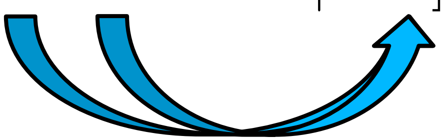
Degree Elevation

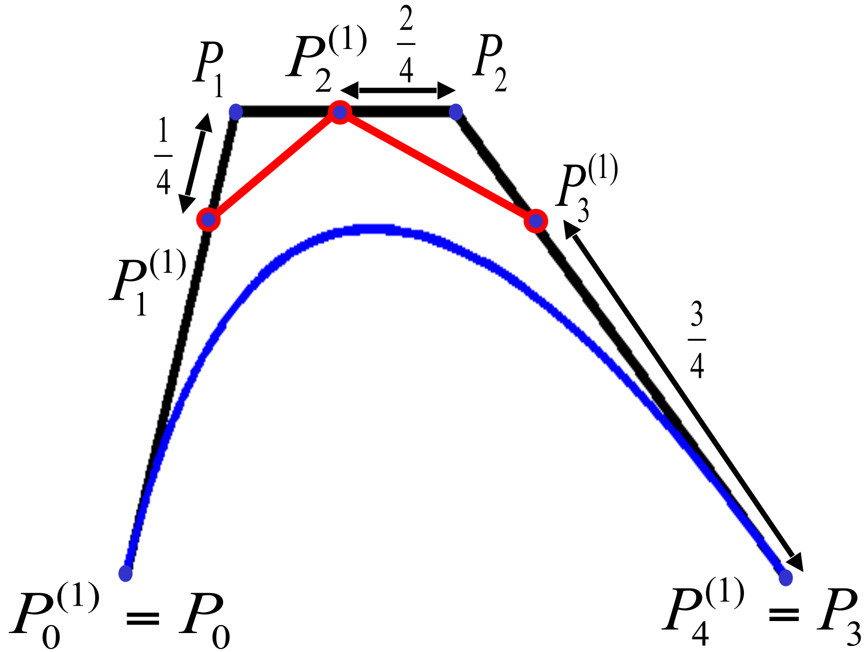
Degree Elevation of Bézier Curves

$$P(t) = \sum_{k=0}^{3} B_k^3(t) P_k = \sum_{k=0}^{4} B_k^4(t) P_k^{(1)}.$$

$$[t^0 \quad t^1 \quad t^2 \quad t^3 \quad | \quad t^4]$$

$$[0 \quad 1 \quad 2 \quad 3 \quad | \quad 4\,]$$



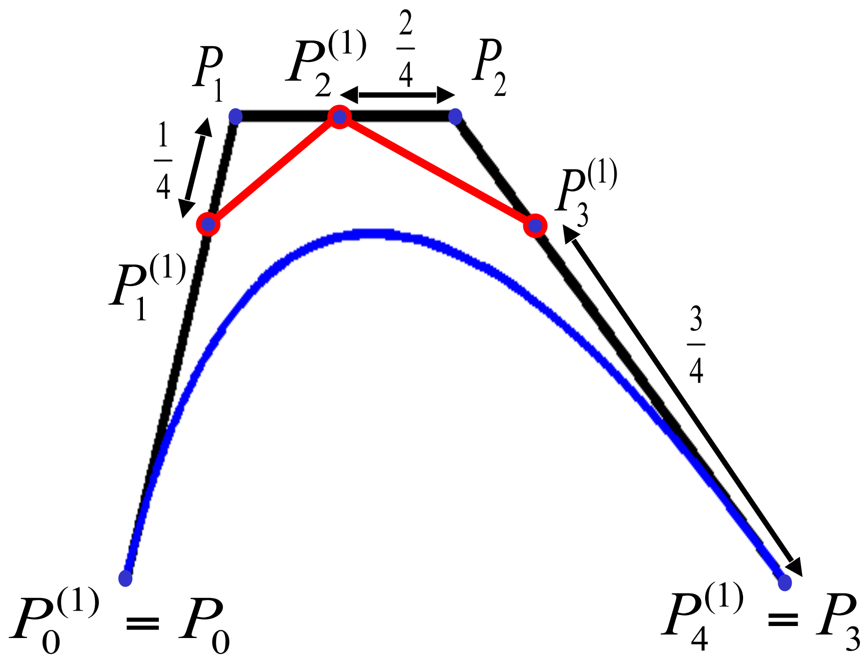


Degree Elevation of Bézier Curves

$$P(t) = \sum_{k=0}^{3} B_k^3(t) P_k = \sum_{k=0}^{4} B_k^4(t) P_k^{(1)}.$$

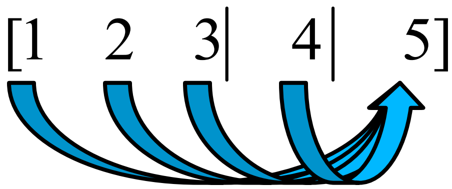
$$[t^0 \quad t^1 \quad t^2 \quad t^3 \quad | \quad t^4]$$

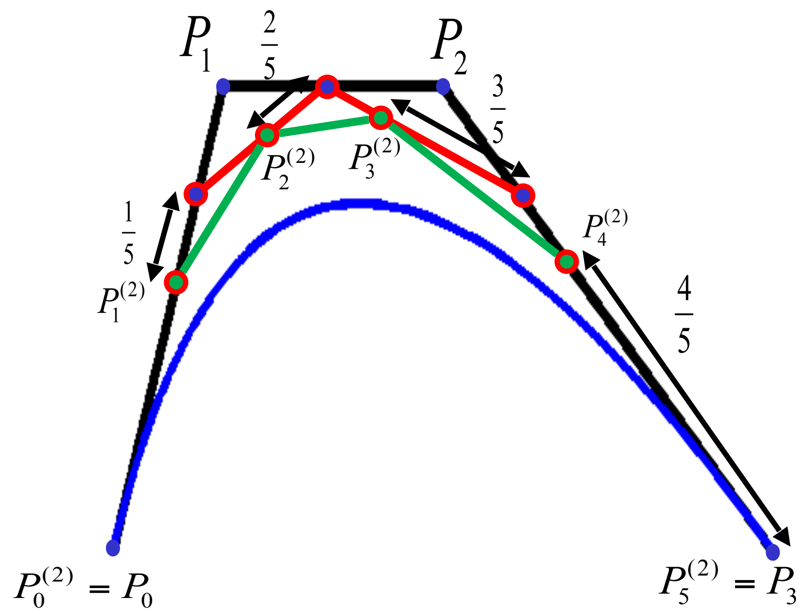
$$[0 \quad 1 \quad 2 \quad 3 \quad | \quad 4\ ]$$



Degree Elevation of Bézier Curves

$$P(t) = \sum_{k=0}^{3} B_k^3(t) P_k = \sum_{k=0}^{5} B_k^5(t) P_k^{(2)}.$$

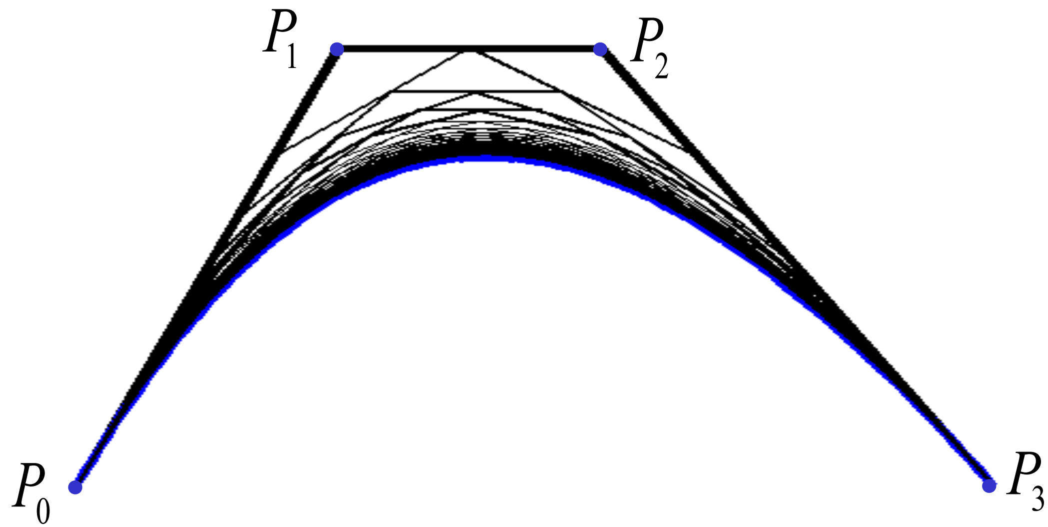




Degree Elevation

Convergence of Degree Elevation

$$P(t) = \sum_{k=0}^{n} B_k^n(t) P_k = \sum_{k=0}^{m} B_k^m(t) P_k^{(m)}.$$



Theorem (Farin, 1969)
The sequence of control polygons generated by the degree elevation algorithm converges uniformly to the underlying Bézier curve.

Generalizing Degree Elevation Algorithm

Nested Sequence of Linear Function Spaces

$$1 \in \mathbb{P}_1 \subset \mathbb{P}_2 \subset \ldots \subset \mathbb{P}_n \subset \ldots \subset C^\infty([a,b])$$

$\mathbb{P}_n$: the linear space of polynomials of degree n over $[a,b]$.

Bernstein Basis in Each Space

$$B_k^n(t) = \binom{n}{k} \left(\frac{t-a}{b-a}\right)^k \left(\frac{b-t}{b-a}\right)^{n-k} \qquad k = 0, 1, \ldots, n.$$

- $B_k^n(t) > 0 \quad$ for $t \in]a,b[$.
- $\sum_{k=0}^n B_k^n(t) = 1$ for any $t \in [a,b]$.
- B_k^n vanishes exactly k times at a and $(n-k)$ times at b.

Extended Chebyshev Spaces

Definition

An $(n+1)$-dimensional space $\mathbb{E}_n \subset C^\infty([a,b]$ is said to be an Extended Chebyshev space (in short, EC-space) on $[a,b]$ if any non-zero element $F \in \mathbb{E}_n$ vanishes at most n times on $[a,b]$ counting multiplicities.

Definition

$(B_0^n, B_1^n, \ldots, B_n^n)$ is said to be a Bernstein basis of $\mathbb{E}_n$ over (a,b) if

- $B_k^n(t) > 0$ for $t \in]a,b[$.
- $\sum_{k=0}^n B_k^n(t) = 1$ for any $t \in [a,b]$.
- B_k^n vanishes exactly k times at a and $(n-k)$ times at b.

Existence of Bernstein Bases

Theorem (Mazure, 2009)

Assume that $1 \in \mathbb{E}_n$. $\mathbb{E}_n$ possesses a Bernstein basis over (a, b) if and only if $D\mathbb{E}_n$ is an (n-dim) EC-space over $[a, b]$, where $D\mathbb{E}_n = \{F'/F \in \mathbb{E}_n\}$.

Definition

$\mathbb{E}_n$ is said to be good for design if it contains constants and $D\mathbb{E}_n$ is an (n-dim) EC-space over $[a, b]$.

⟹ Development of all CAGD algorithms in $\mathbb{E}_n$.

(Bézier curves, de Casteljau algorithm, blossom, dimension elevation, splines,...)

Spaces Good for Design

Examples

- $(1, t, t^2, \ldots, t^{n-2}, \cosh t, \sinh t)$ span a space good for design on any interval $[a, b] \subset \mathbb{R}$.

- $(1, t, t^2, \ldots, t^{n-2}, \cos t, \sin t)$ span a space good for design on any interval $[a, b] \subset [\alpha, \alpha + \pi[$. (Cycloidal spaces)

- Given real numbers $0 = r_0 < r_1 < \ldots r_n$, $(t^{r_0}, t^{r_1}, \ldots, t^{r_n})$ span a space good for design on any $[a, b] \subset]0, \infty[$. (Müntz spaces)

- Given any real numbers $a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus [a, b]$, $(1, \frac{1}{t-a_1}, \frac{1}{t-a_2}, \ldots, \frac{1}{t-a_n})$ span a space good for design over $[a, b]$.

Dimension Elevation

Setting

- $[a,b]$ a fixed interval. $\mathbb{E}_n \subset \mathbb{E}_{n+1}$ both good for design over $[a,b]$.
- $(B_0^n, B_1^n, \ldots, B_n^n)$ Bernstein basis of $\mathbb{E}_n$ over (a,b).
- $(B_0^{n+1}, B_1^{n+1}, \ldots, B_{n+1}^{n+1})$ Bernstein basis of $\mathbb{E}_{n+1}$ over (a,b).

$$P(t) = \sum_{k=0}^{n} B_k^n(t) P_k = \sum_{k=0}^{n+1} B_k^{n+1}(t) P_k^{(1)}, \quad P_k, P_k^{(1)} \in \mathbb{R}^d.$$

Theorem

There exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in]0,1[$, independent of P, such that

$$P_0^{(1)} = P_0,\ P_{n+1}^{(1)} = P_n$$

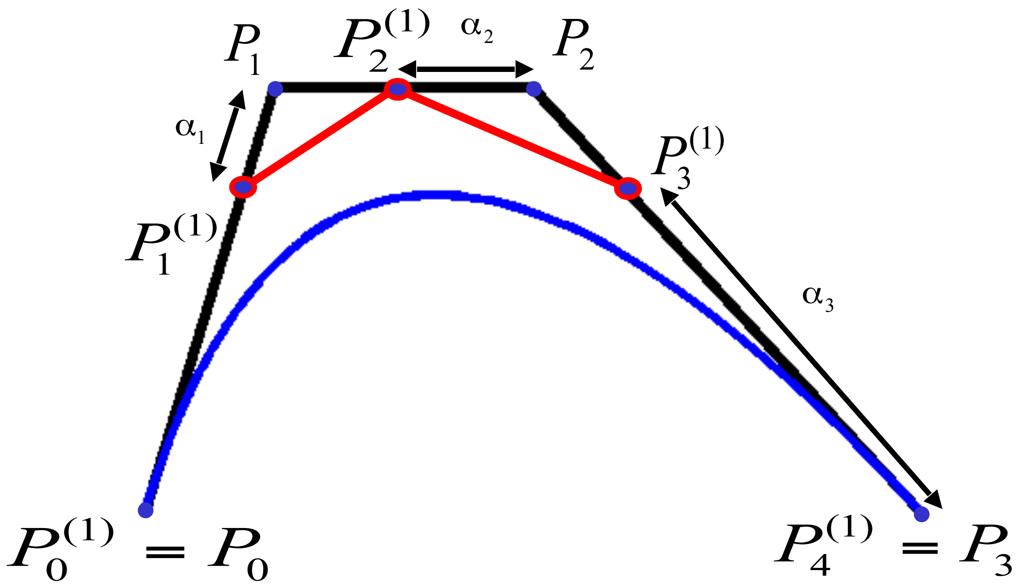
$$P_k^{(1)} = \alpha_k P_{k-1} + (1-\alpha_k) P_k,\ k = 1, 2, \ldots, n$$

Dimension Elevation

Geometrical Interpretation

- $[a, b]$ a fixed interval. $\mathbb{E}_n \subset \mathbb{E}_{n+1}$ both good for design over $[a, b]$.

$$P(t) = \sum_{k=0}^{n} B_k^n(t) P_k = \sum_{k=0}^{n+1} B_k^{n+1}(t) P_k^{(1)}, \quad P_k, P_k^{(1)} \in \mathbb{R}^d.$$

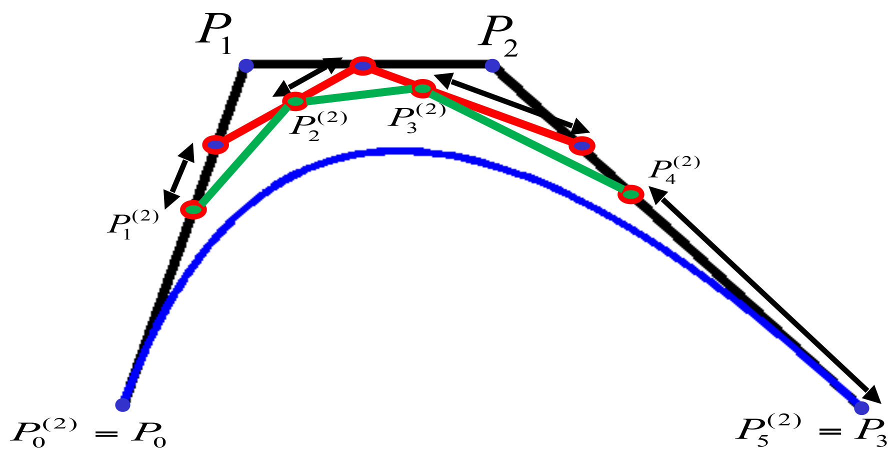


Dimension Elevation Algorithm

Nested Sequence of Spaces Good for Design

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

$$P(t) = \sum_{k=0}^{n} B_k^n(t) P_k = \sum_{k=0}^{m} B_k^m(t) P_k^{(m)}$$



Main Question

When does the sequence of control polygons, generated by the dimension elevation algorithm, converges to the underlying Bézier curve ?

Nested Sequence of Müntz Spaces

- $\Lambda_\infty = (r_1, r_2, \ldots, r_n, \ldots)$ a strictly increasing sequence of positive real numbers.

- $\mathbb{E}_n(\Lambda_\infty) := Span\{1, t^{r_1}, t^{r_2}, \ldots, t^{r_n}\}$ is a space good for design on any interval $[a, b] \subset]0, \infty[$.

$$1 \in \mathbb{E}_1(\Lambda_\infty) \subset \mathbb{E}_2(\Lambda_\infty) \ldots \subset \mathbb{E}_n(\Lambda_\infty) \subset \ldots \subset C^\infty([a, b]).$$

Main Question: Characterize the sequences Λ_∞ for which the dimension elevation algorithm converges to the underlying curve ?

Schur Functions

- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a real partition if

$$\lambda_1 > \lambda_2 - 1 > \lambda_3 - 2 > \ldots > \lambda_n - (n-1) > -n.$$

- The Schur function S_λ indexed by a real partition λ is defined by

$$S_\lambda(u_1, \ldots, u_n) = \frac{\det(u_i^{\lambda_j + n - j})_{1 \le i,j \le n}}{\prod_{1 \le i < j \le n}(u_i - u_j)}, \quad \tilde{S}_\lambda(u_1, \ldots, u_n) := \frac{S_\lambda(u_1, \ldots, u_n)}{S_\lambda(1, 1, \ldots, 1)}$$

- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive integers, then we recover the classical notion of integer partitions and Schur functions

$$S_\lambda(u_1, u_2, \ldots, u_n) \in \mathbb{Z}[u_1, u_2, \ldots, u_n].$$

Schur Functions and Bernstein Bases (R.A, 2013)

- $\mathbb{E}_n(\Lambda_\infty) := Span\{1, t^{r_1}, t^{r_2}, \ldots, t^{r_n}\}$.

Theorem. The Bernstein basis of the space $\mathbb{E}_n(\Lambda_\infty)$ over (a, b) is given by

$$B^n_{k,\Lambda_\infty}(t) = \frac{\tilde{S}_{\lambda^{(0)}}(a^{[n-k]}, b^{[k]})\tilde{S}_\lambda(a^{[n-k]}, b^{[k]}, ab/t)}{\tilde{S}_\lambda(a^{[n+1-k]}, b^{[k]})\tilde{S}_\lambda(a^{[n-k]}, b^{[k+1]})} t^{\lambda_1} B^n_k(t),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda^{(0)} = (\lambda_2, \ldots, \lambda_n)$ with

$$\lambda_k = r_n - r_{k-1} - (n - k + 1), \quad k = 1, 2, \ldots, n.$$

- The ratio b/a is a shape parameter.

⇨ It is sufficient to study dimension elevation over intervals of the form $[a, 1], a > 0$.

Dimension Elevation in Müntz spaces

Dimension Elevation Algorithm

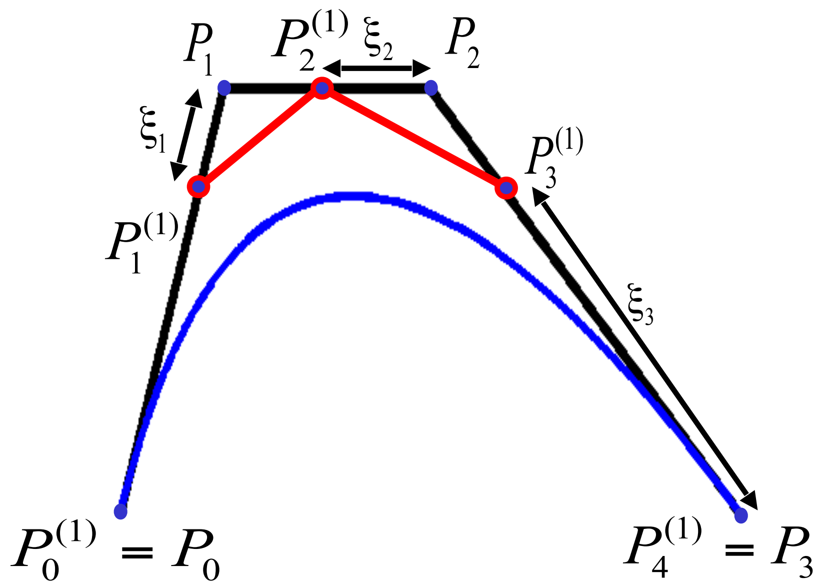
- $\Lambda_n = (r_1, r_2, \ldots, r_n), \quad \Lambda_{n+1} = (r_1, r_2, \ldots, r_n, r_{n+1})$

$$P(t) = \sum_{k=0}^{n} B^n_{k,\Lambda_\infty}(t) P_k = \sum_{k=0}^{n+1} B^{n+1}_{k,\Lambda_\infty}(t) P^{(1)}_k.$$

- New control points

$$P^{(1)}_0 = P_0, \quad P^{(1)}_{n+1} = P_n$$

$$P^{(1)}_k = \xi_k P_{k-1} + (1-\xi_k) P_k.$$



➡ ξ_k have very complicated expression

➡ A direct proof of a convergence theorem is unlikely

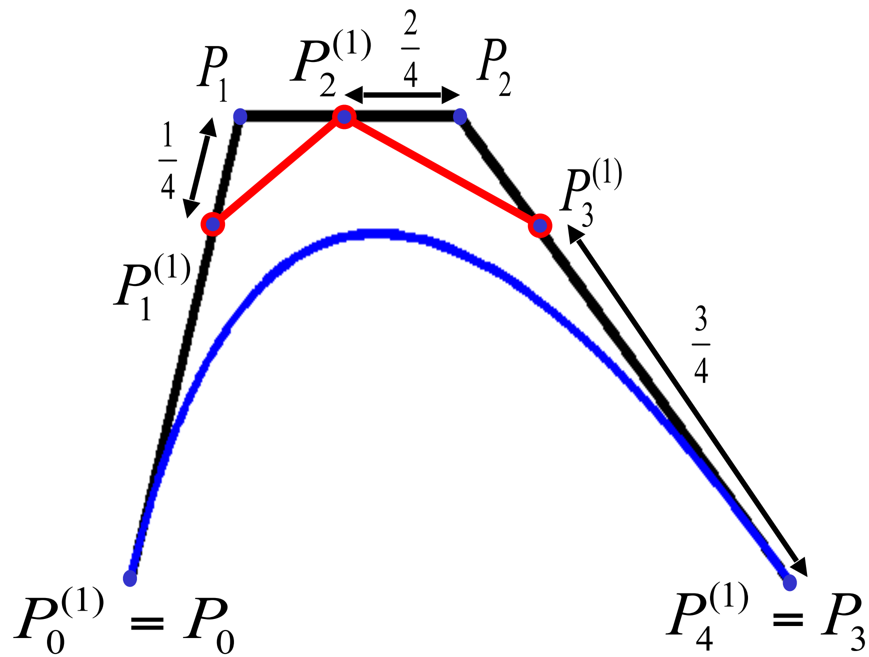
Degree Elevation

Degree Elevation of Bézier Curves

$$P(t) = \sum_{k=0}^{3} B_k^3(t) P_k = \sum_{k=0}^{4} B_k^4(t) P_k^{(1)}.$$

$$[t^0 \quad t^1 \quad t^2 \quad t^3 \quad | \quad t^4]$$

$$[0 \quad 1 \quad 2 \quad 3 \quad | \quad 4\;]$$



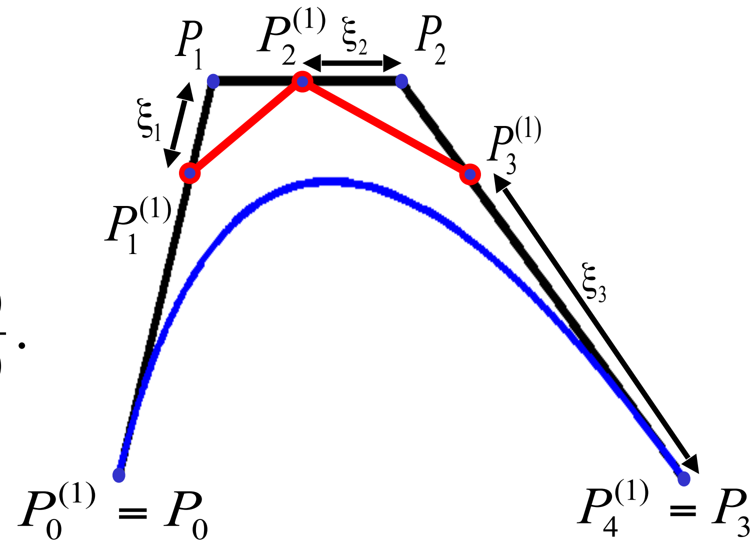
Dimension Elevation in Müntz spaces

Example

- $\Lambda_3 = (1, 3, 4), \quad \Lambda_{n+1} = (1, 3, 4, 6)$
- $[0 \quad 1 \quad 3 \quad 4 \quad | \quad 6]$

$$\xi_1(a, b) = a^{-1} \frac{\tilde{S}_{(1)}(a,a,b)\tilde{S}_{(2,2,1,1)}(a,a,a,a,b)}{\tilde{S}_{(1,1)}(a,a,a,b)\tilde{S}_{(2,1,1)}(a,a,a,b)}.$$

- $\lim_{a \to 0} \xi_1(a, 1) = \frac{1}{6}.$



Working with $H^n_{k,\Lambda_\infty}(t) := \lim_{a \to 0} B^n_{k,\Lambda_\infty}(t)$.
(Gelfond-Bernstein basis)

❑ Dimension Elevation for Gelfond-Bézier Curves

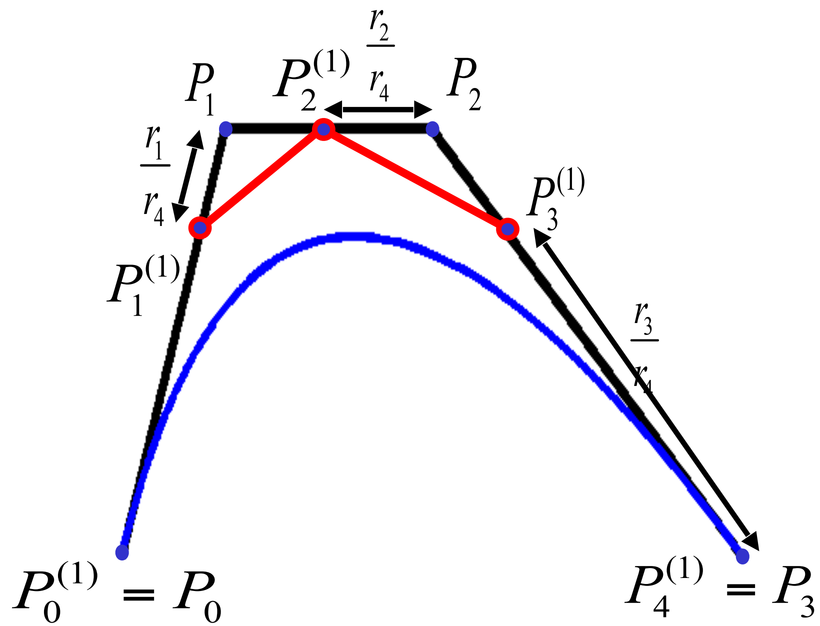
- $\Lambda_n = (r_1, r_2, \ldots, r_n) \qquad \Lambda_{n+1} = (r_1, r_2, \ldots, r_n, r_{n+1})$

$$P(t) = \sum_{k=0}^{n} H_{k,\Lambda_\infty}^{n}(t) P_k = \sum_{k=0}^{n+1} H_{k,\Lambda_\infty}^{n+1}(t) P_k^{(1)}$$

- New control points

$$P_0^{(1)} = P_0 \qquad P_{n+1}^{(1)} = P_n$$

$$P_k^{(1)} = \frac{r_k}{r_{n+1}} P_{k-1} + \left(1 - \frac{r_k}{r_{n+1}}\right) P_k$$

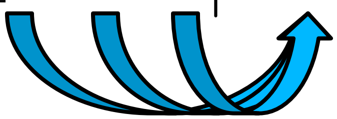


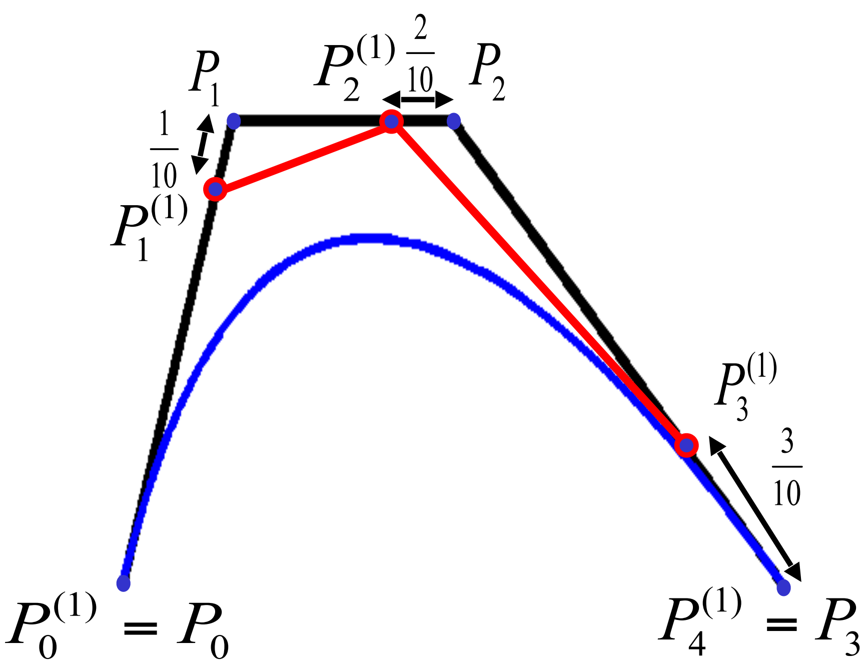
Corner Cutting Schemes

❑ Dimension Elevation for Gelfond-Bézier Curves

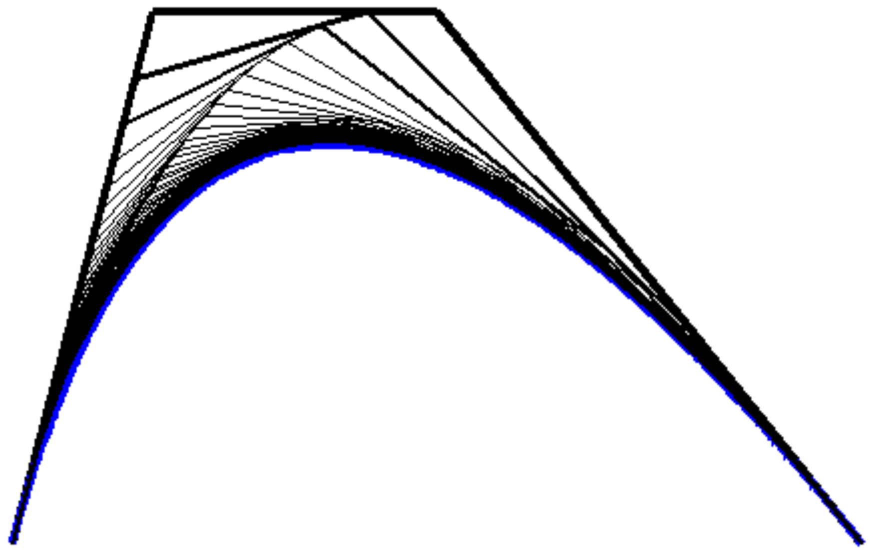
$$P(t) = \sum_{k=0}^{3} B^3_{k,\Lambda_\infty}(t) P_k$$

$[1 \quad 2 \quad 3| \quad 10| \quad 15]$





Example 1

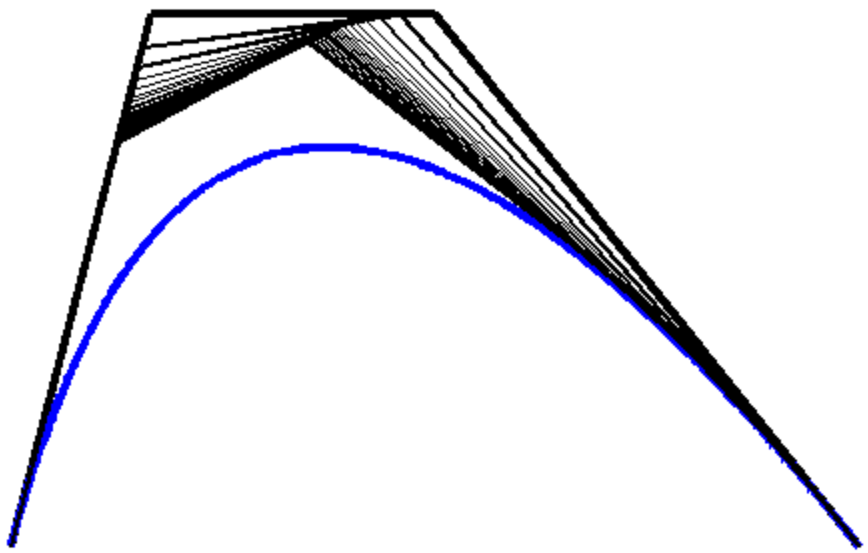


100 iterations

$$\Lambda_3 = [1,2,3] \quad \Rightarrow \quad \Lambda_\infty = [1,2,3\,|\,8,10,\ldots,2i,\ldots]$$

Corner Cutting Schemes

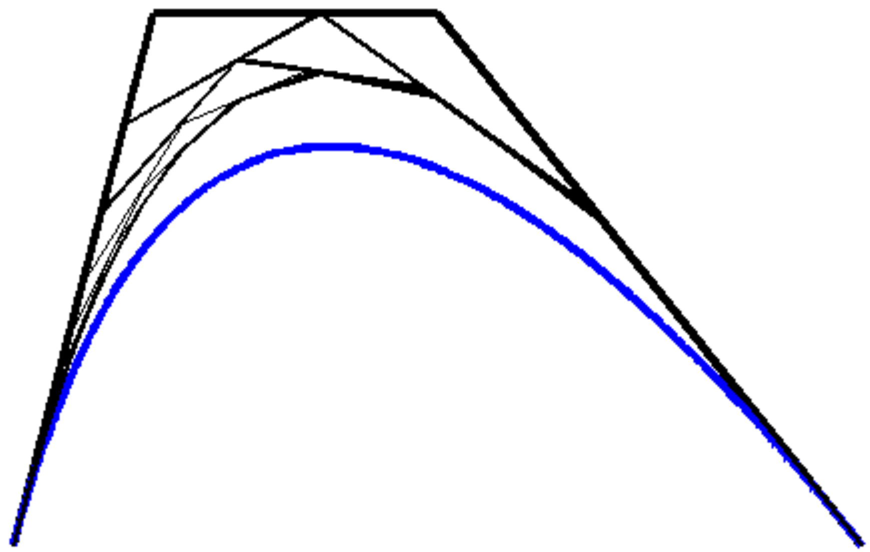
Example 2



100 iterations

$$\Lambda_3 = [1,2,3] \quad \Rightarrow \quad \Lambda_\infty = [1,2,3 \,|\, 16, 25, \ldots, i^2, \ldots]$$

Example 3



100 iterations

$$\Lambda_3 = [1,2,3] \quad \Rightarrow \quad \Lambda_\infty = [1,2,3 \,|\, 4-\frac{1}{4}, 4-\frac{1}{5}, \ldots, 4-\frac{1}{i}, \ldots]$$

Corner Cutting Schemes

Examples

Convergent $\Lambda_3 = [1,2,3] \Rightarrow \Lambda_\infty = [1,2,3 | 8,10,\ldots,2i,\ldots]$ $\sum_i \frac{1}{2i} = +\infty$

Non convergent $\Lambda_3 = [1,2,3] \Rightarrow \Lambda_\infty = [1,2,3 | 16,25,\ldots,i^2,\ldots]$ $\sum_i \frac{1}{i^2} < +\infty$

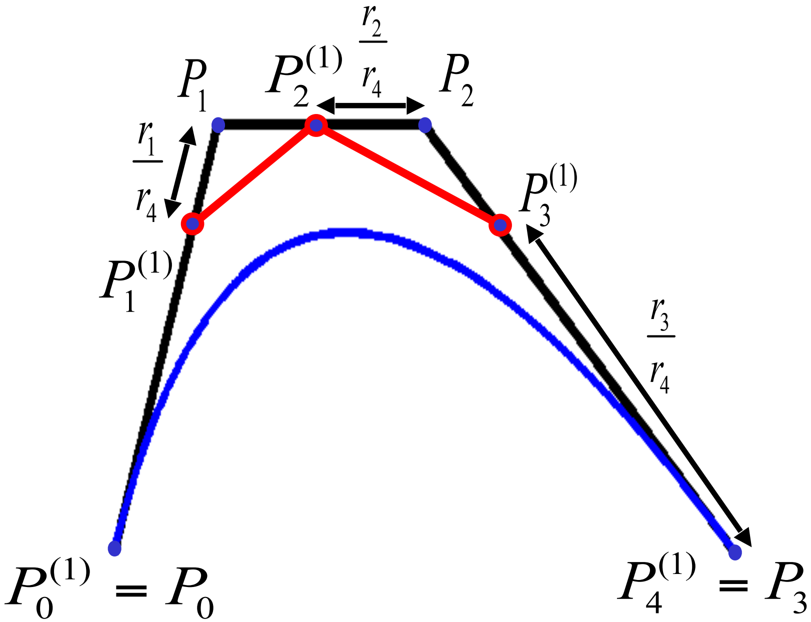
Non convergent $\Lambda_3 = [1,2,3] \Rightarrow \Lambda_\infty = [1,2,3 | 4-\frac{1}{4}, 4-\frac{1}{5},\ldots,4-\frac{1}{i},\ldots]$

$$\sum_i \frac{1}{4-\frac{1}{i}} = +\infty \Rightarrow \lim_{n\to\infty} r_n = +\infty \text{ and } \sum_i \frac{1}{r_i} = +\infty$$

Dimension Elevation of Gelfond-Bézier Curves

Convergence Theorem

- $\Lambda_\infty = (r_1, r_2, \ldots, r_n, r_{n+1}, r_{n+2}, \ldots)$



$$P_0^{(1)} = P_0 \qquad P_{n+1}^{(1)} = P_n$$

$$P_k^{(1)} = \frac{r_k}{r_{n+1}} P_{k-1} + \left(1 - \frac{r_k}{r_{n+1}}\right) P_k$$

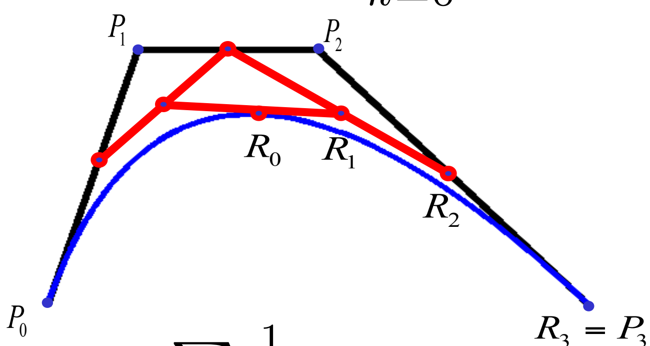
Theorem (R.A, 2013)
The sequence of control polygons obtained by dimension elevation converges uniformly to the underlying Gelfond-Bézier curve
if and only if

$$\lim_{n\to\infty} r_n = +\infty \quad \text{and} \quad \sum_i \frac{1}{r_i} = +\infty$$

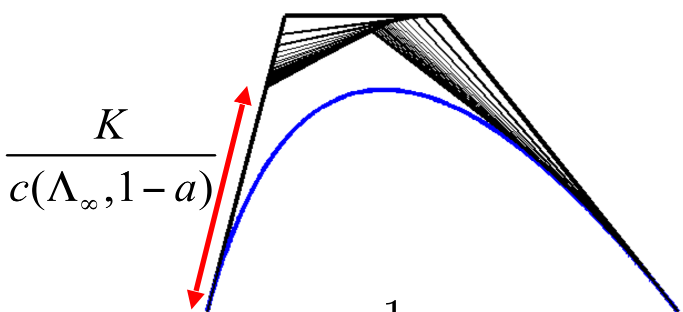
Dimension Elevation of Müntz-Bézier Curves

How to Go Back to the Interval [a,1] ?

$$P(t) = \sum_{k=0}^{n} H^n_{k,\Lambda_\infty}(t) P_k = \sum_{k=0}^{n} B^n_{k,\Lambda_\infty}(t) R_k$$



$$\sum_k \frac{1}{r_k} = \infty$$



$$\sum_k \frac{1}{r_k} < \infty$$

- The de Casteljau algorithm and Chebyshev blossoming still make sense over $[0,1]$
- Clarkson-Erdös-Schwartz inequality $||P'||_{[0,1-\epsilon]} < c(\Lambda_\infty, \epsilon)||P||_{[1-\epsilon,1]}$

A New Emergence of the Müntz Condition

Theorem (R.A, 2013)
For a sequence $\Lambda_\infty = (r_1, r_2, \ldots, r_n, \ldots)$ of strictly increasing positive real numbers such that $\lim_{n\to\infty} r_n = \infty$, the convergence of dimension elevation to the underlying curves is equivalent to

$$\sum_n \frac{1}{r_n} = +\infty$$

Theorem(Müntz, 1912; Schwartz, 1944)

The space $Span(1, t^{r_1}, t^{r_2}, \ldots, t^{r_n}, \ldots)$ is a dense subset of $C([a,b])$ with $[a,b] \subset [0,\infty]$ if and only if

$$\sum_n \frac{1}{r_n} = +\infty$$

Dimension Elevation vs Density

Corollary

For a sequence $\Lambda_\infty = (r_1, r_2, \dots, r_n, \dots)$ of strictly increasing positive real numbers such that $\lim_{n\to\infty} r_n = \infty$, the convergence of dimension elevation to the underlying curves is equivalent to the density of the space $\cup_{n\geq 1}\mathbb{E}_n$ in $C([a,b])$ endowed with the uniform norm.

Question: Is this equivalence an isolated fact or a general one ?

Polynomial Bernstein Operators

Explicit Expression

$$1 \in \mathbb{P}_1 \subset \mathbb{P}_2 \subset \ldots \subset \mathbb{P}_n \subset \ldots \subset C^\infty([a,b])$$

- $\mathbb{B}_n : C([a,b]) \longrightarrow \mathbb{P}_n$

$$\mathbb{B}_n F = \sum_{k=0}^{n} F\left(\frac{(n-k)a+kb}{n}\right) B_k^n.$$

- $\mathbb{B}_n$ linear positive operator.
- $\mathbb{B}_n$ reproduces every elements of $\mathbb{P}_1$.

$$t = \sum_{k=0}^{n} \frac{(n-k)a+kb}{n} B_k^n(t).$$

Construction

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

- Let $U \in \mathbb{E}_1$ be a strictly increasing function over $[a,b]$.

$$U(t) = \sum_{k=0}^{n} u_{k,n} B_k^n(t).$$

The Bernstein operator $\mathbb{B}_n : C([a,b]) \longrightarrow \mathbb{E}_n$ reproducing U is defined by:

$$\mathbb{B}_n F = \sum_{k=0}^{n} F(\xi_{k,n}) B_k^n,$$

where $\xi_{k,n} = U^{-1}(u_{k,n})$, $k = 0, 1, \ldots, n$.

Chebyshev-Bernstein Operators

Main Question

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

When does the Bernstein operators $\mathbb{B}_n$ converge to the identity ? i.e.;

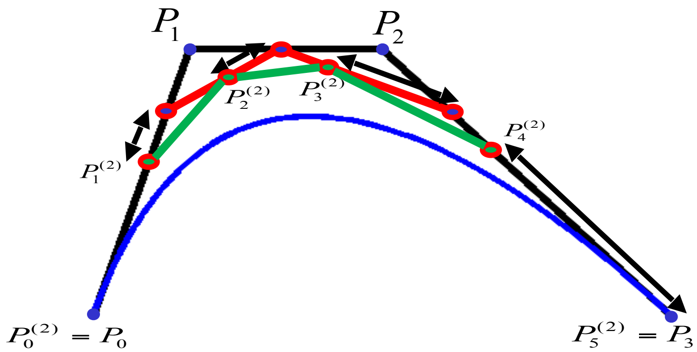
$$\lim_{n\to\infty} ||\mathbb{B}_n F - F||_\infty = 0 \quad \text{for any} \quad F \in C([a,b]).$$

⇨ $\cup_{n=0}^{\infty}\mathbb{E}_n$ is a dense subset of $C([a,b])$ endowed with the uniform norm.

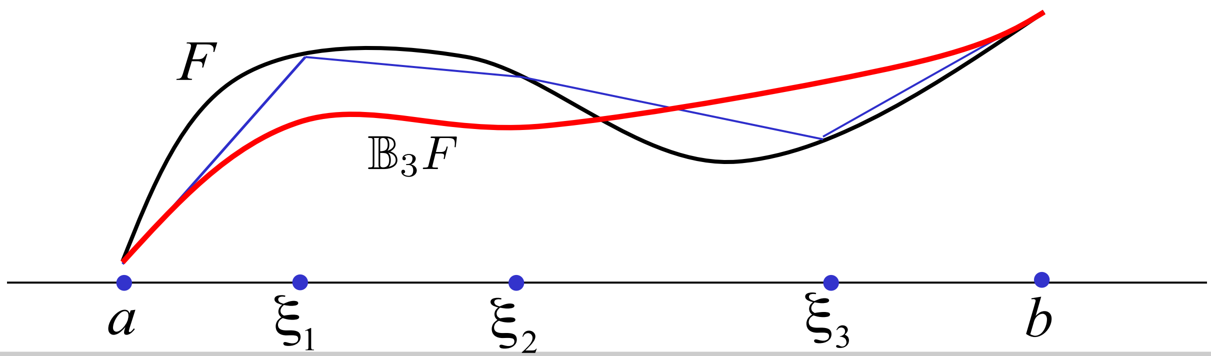
Dimension Elevation vs Bernstein Operators

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

❑ Dimension Elevation (CAGD)



❑ Bernstein Operators (Approximation)



Main Theorem

Dimension Elevation vs Bernstein Operators

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

Theorem(M.-L. Mazure, R. A, 2014)

There is equivalence between:

- Convergence of dimension elevation to the underlying curves
- Convergence of the Bernstein operators to the identity

Corollary

If dimension elevation converges to the underlying curves then $\cup_{n\geq 1}\mathbb{E}_n$ is dense in $C([a,b])$ endowed with the uniform norm

Converse Question

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^{\infty}([a,b])$$

Does the density of $\cup_{n \geq 1} \mathbb{E}_n$ implies the convergence of dimension elevation to the underlying curves ?

Converse Question

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

Does the density of $\cup_{n\geq 1}\mathbb{E}_n$ implies the convergence of dimension elevation to the underlying curves ?

Answer: NO

Dimension Elevation for Rational Spaces

Nested Sequence of Rational Spaces

- $\mathcal{A}_\infty = (a_1, a_2, \ldots, a_n, \ldots)$ a sequence of real numbers in $\mathbb{R} \setminus [a, b]$.

- $\mathbb{E}_n(\mathcal{A}_\infty) := Span\{1, \frac{1}{t-a_1}, \frac{1}{t-a_2}, \ldots, \frac{1}{t-a_n}\}$ is a space good for design on the interval $[a, b]$.

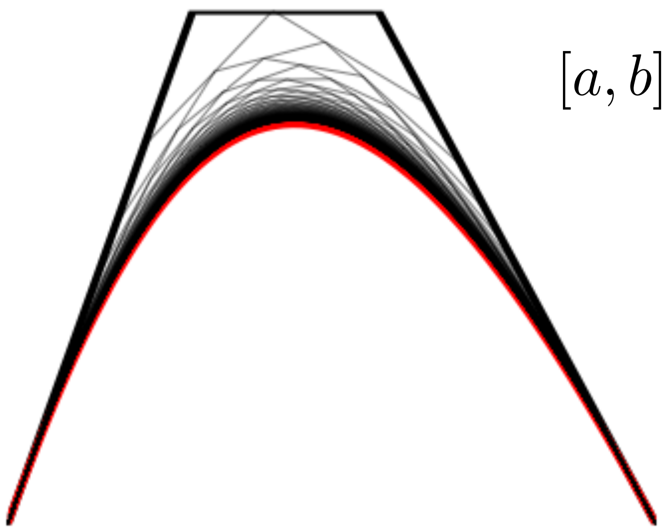
$$1 \in \mathbb{E}_1(\mathcal{A}_\infty) \subset \mathbb{E}_2(\mathcal{A}_\infty) \ldots \subset \mathbb{E}_n(\mathcal{A}_\infty) \subset \ldots \subset C^\infty([a, b]).$$

Akhieser's theorem (1956)

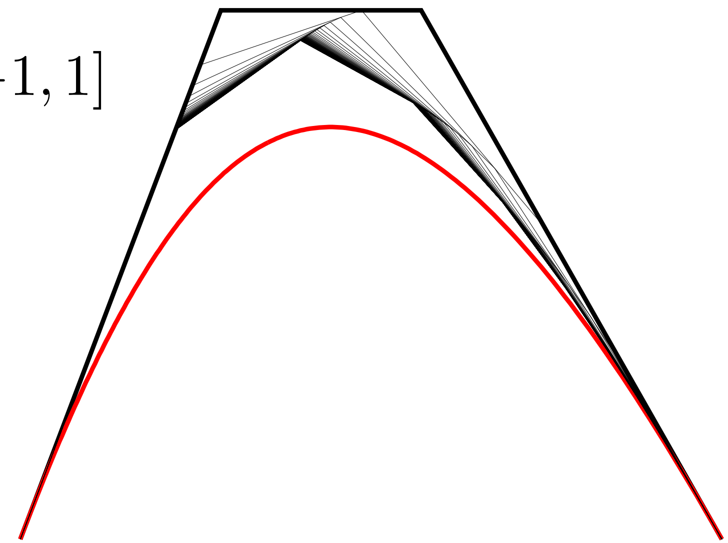
$\cup_{n \geq 1} \mathbb{E}_n(\mathcal{A}_\infty)$ is dense in $C([a, b])$ if and only if

$$\sum_{n=1}^{\infty} \sqrt{(a_n - a)(a_n - b)} = +\infty.$$

Counter-Example



$$[a,b]=[-1,1]$$



$$[2 \quad 3 \quad -7 \quad |8 \quad 10 \ldots 2n \ldots] \qquad [2 \quad 3 \quad -7 \quad |\tfrac{17}{15} \quad \tfrac{26}{24} \ldots \tfrac{n^2+1}{n^2-1} \ldots]$$

$$\sum_{n=1}^{\infty}\sqrt{(a_n-a)(a_n-b)}=+\infty.$$

Pólya's Theorem and Dimension Elevation

Pólya Theorem (1928)

Let P be a real univariate polynomial which is positive on the interval $[0, \infty[$. Then there exists an integer m such that the coefficients of the polynomial $(t+1)^m P(t)$ are positive.

Definition

An infinite sequence $\mathcal{B}_\infty = (b_1, b_2, \ldots, b_n, \ldots)$ of positive numbers is said to be Pólya positive if, for any real polynomial P which is positive on the interval $[0, \infty[$, there exists an integer m such that all the coefficients of the polynomial $(t+b_1)(t+b_2)\ldots(t+b_m)P(t)$ are positive.

⇨ The sequence $(1, 1, \ldots, 1, \ldots)$ is Pólya positive

Main Result for Rational Spaces

Theorem (R.A, M.-L. Mazure, 2016)

A infinite sequence of poles $\mathcal{A}_\infty = (a_1, a_2, \ldots, a_n, \ldots)$ ensures the convergence of dimension elevation to the underlying curves if and only if the infinite sequence

$$\mathcal{B}_\infty = \left(\frac{a-a_1}{b-a_1}, \frac{a-a_2}{b-a_2}, \ldots, \frac{a-a_n}{b-a_n}, \ldots\right)$$

is Pólya positive.

Theorem (Baker and Handelman, 1992)

An infinite sequence $\mathcal{B}_\infty = (b_1, b_2, \ldots, b_n, \ldots)$ of positive numbers is Pólya positive if and only if

$$\sum_{n=1}^{\infty} \min(b_n, \frac{1}{b_n}) = +\infty$$

Conclusion

$$1 \in \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_n \subset \mathbb{E}_{n+1} \subset \ldots \subset C^\infty([a,b])$$

Convergence of dimension elevation to the underlying curves

Müntz spaces

Density of $\cup_{n\geq 1}\mathbb{E}_n$ in $C([a,b])$

Convergence of Bernstein operators to the identity

Rational spaces

$$\lim_{n\to\infty} \max_{k=0,1,\ldots,n-1} |u_{k,n} - u_{k+1,n}| = 0$$

- $u_{k,n}, k = 0, 1, \ldots, n$, the control points of any element of $\mathbb{E}_1$

Thank you for your attention