The Generalized Fermat Equation

$$x^2 + y^3 = z^{11}$$

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Rational points and Algebraic Geometry
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$$\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 \,.$$

Theorem.

- If $\chi > 0$, there are infinitely many solutions.
- If $\chi \leq 0$, there are only finitely many solutions.

Known Solutions

Apart from trivial solutions (with $xyz = 0$),
there are only the following ten solutions known when $\chi \leq 0$:

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$
$$3^5 + 11^4 = 122^2, \quad 17^7 + 76271^3 = 21063928^2, \quad 1414^3 + 2213459^2 = 65^7,$$
$$9262^3 + 15312283^2 = 113^7, \quad 43^8 + 96222^3 = 30042907^2, \quad 33^8 + 1549034^2 = 15613^3$$

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Remark.
The ABC Conjecture (with any $\varepsilon < 1/5$) would imply
that there are only finitely many solutions in total for $\chi \leq 0$.

The Next Case

Heuristically, one expects more solutions when $\chi < 0$ is closer to zero:

$\{p, q, r\}$	$\{2, 3, 7\}$	$\{2, 3, 8\}$	$\{2, 4, 5\}$	$\{2, 3, 9\}$	$\{2, 3, 10\}$	$\{2, 3, 11\}$
$-\chi$	$1/42$	$1/24$	$1/20$	$1/18$	$1/15$	$5/66$
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The five cases that have $\chi < 0$ closest to zero have been completely solved. ($\{2, 3, 8\}$, $\{2, 4, 5\}$, $\{2, 3, 9\}$: N. Bruin; $\{2, 3, 7\}$: B. Poonen, E. Schaefer, MS; $\{2, 3, 10\}$: D. Zureick-Brown and S. Siksek independently)

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The only nontrivial solutions should be $(x, y, z) = (\pm 3, -2, 1)$.

Goal: Solve $x^2 + y^3 = z^{11}$!

Frey Curves

We follow the general approach taken in the proof of FLT.
To a putative solution (a, b, c) of $x^2 + y^3 = z^{11}$
we associate the Frey elliptic curve

$$E_{(a,b,c)}\colon y^2 = x^3 + 3bx - 2a\,.$$

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The 11-torsion Galois module $E_{(a,b,c)}[11]$ is always irreducible.
By the usual level lowering results and modularity (plus some extra work),
we find that (up to quadratic twist) $E_{(a,b,c)}[11] \simeq E[11]$ for some

$$E \in \{27a1, 54a1, 96a1, 288a1, 864a1, 864b1, 864c1\}\,.$$

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$$E \in \{27a1, 54a1, 96a1, 288a1, 864a1, 864b1, 864c1\} \,.$$

Known solutions: $(\pm 1, 0, 1) \leftrightarrow 27a1$, $\pm(0, 1, 1) \leftrightarrow 288a1$, $(\pm 3, -2, 1) \leftrightarrow 864b1$.
The trivial solutions $(\pm 1, -1, 0)$ result in a degenerate Frey curve.

The CM Cases

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Elliptic curves E' such that $E'[11] \simeq 27a1[11]$ or $288a1[11]$ correspond to rational points on the quadratic twists

$$X^{(d)}_{\text{nonsplit}}(11) \longrightarrow X^{+}_{\text{nonsplit}}(11)$$

with $d = -3$ or -1 of the double cover $X_{\text{nonsplit}}(11) \longrightarrow X^{+}_{\text{nonsplit}}(11)$.

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$X^{(d)}_{\text{nonsplit}}(11)$ has genus 4 and can be defined by the equations

$$\begin{aligned} y^2 &= 4x^3 - 4x^2 - 28x + 41 \\ t^2 &= -d(4x^3 + 7x^2 - 6x + 19) \end{aligned}$$

The CM Cases (2)

$$X_{\text{nonsplit}}^{(d)}(11): \quad y^2 = 4x^3 - 4x^2 - 28x + 41, \quad t^2 = -d(4x^3 + 7x^2 - 6x + 19)$$

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Let $K = \mathbb{Q}(\alpha)$ with α a root of $4x^3 - 4x^2 - 28x + 41$.
A rational point on $X^{(d)}_{\text{nonsplit}}(11)$ will give a K-rational point
with rational x-coordinate on

$$u^2 = -d(x-\alpha)(4x^3+7x^2-6x+19) \quad \text{or} \quad u^2 = -d(4-\alpha)(x-\alpha)(4x^3+7x^2-6x+19)\,.$$

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These elliptic curves over K have rank $\leq 2 < [K : \mathbb{Q}]$,
so Elliptic Curve Chabauty applies and can be used to show that the only solutions coming from $27a1$ and $288a1$ are the trivial ones.

The Remaining Curves

We still have to deal with $E = 54a1, 96a1, 864a1, 864b1, 864c1$.

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of the modular curve $X(11)$,
depending on whether the isomorphism acts on the Weil pairing
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It remains to find the rational points on the five twists $X_E(11)$
that correspond to primitive (= coprime integer) solutions of $x^2 + y^3 = z^{11}$.

From $X(11)$ to $X_0(11)$

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Instead, we use the intermediate curve $C := X_0(11)$,

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Then a rational point on $X_E(11)$ maps to a K_E-rational point on C,
whose image under the j-invariant map is in $\mathbb{Q}$.

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Problem:
We need to find generators of a finite-index subgroup of $C(K_E)$,
but are unable to do so.

Selmer Group Chabauty

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The Selmer group sits in the following diagram:

$$\frac{C(K_E)}{2C(K_E)} \hookrightarrow S \xrightarrow{\sigma} \frac{C(K_E \otimes \mathbb{Q}_2)}{2C(K_E \otimes \mathbb{Q}_2)} \hookrightarrow \frac{(L_E \otimes \mathbb{Q}_2)^\times}{(L_E \otimes \mathbb{Q}_2)^{\times 2}}$$

We check that σ is injective for each of our curves E.

Partitioning the j-Line

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We first find the potential images in $\mathbb{Q}_2$ under the j-invariant map of the points we are interested in.
For each curve E, we obtain a finite collection of sets $\{u + vt^n : t \in \mathbb{Z}_2\}$:

$$54a1\colon 1 \text{ set}, \quad 96a1\colon 3 \text{ sets}, \quad 864a1\colon 2 \text{ sets}, \quad 864b1\colon 3 \text{ sets}, \quad 864c1\colon 3 \text{ sets}.$$

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We lift these sets in all possible ways to $C(K_E \otimes \mathbb{Q}_2)$
and check which of them map into $\sigma(S)$ under $\pi\colon C(K_E \otimes \mathbb{Q}_2) \to \frac{C(K_E \otimes \mathbb{Q}_2)}{2C(K_E \otimes \mathbb{Q}_2)}$.
This leaves

$$54a1\colon 1 \text{ set}, \quad 96a1\colon 2 \text{ sets}, \quad 864a1\colon 0 \text{ sets}, \quad 864b1\colon 1 \text{ set}, \quad 864c1\colon 1 \text{ set}.$$

This already rules out $864a1$.

Dealing With the Remaining Sets

For each of the remaining sets D there is a point $P \in C(K_E)$ such that P and all points mapping into D have the same image in $\frac{C(K_E \otimes \mathbb{Q}_2)}{2C(K_E \otimes \mathbb{Q}_2)}$.

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Lemma.
Assume that for all $P \neq Q \in C(K_E \otimes \mathbb{Q}_2)$ with $j(Q) \in D$ there are $n \geq 0$ and $Q' \in C(K_E \otimes \mathbb{Q}_2)$ such that $Q = P + 2^n Q'$ and $\pi(Q') \notin \sigma(S)$.

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Then if $j(P) \in D$, P is the only point $Q \in C(K_E)$ with $j(Q) \in D$, and if $j(P) \notin D$, then there is no such point.

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Proof. Let $Q \in C(K_E)$ with $j(Q) \in D$ and $Q \neq P$.
Then $Q = P + 2^n Q'$ with $Q' \in C(K_E \otimes \mathbb{Q}_2)$ and $\pi(Q') \notin \sigma(S)$.
Using that σ is injective and $C(K_E)[2] = 0$, we obtain $Q' \in C(K_E)$, which implies $\pi(Q') \in \sigma(S)$, a contradiction. $\square$

Finishing the Argument

The point Q' in the Lemma is unique (we have to take n maximal).
The map $Q \mapsto \pi(Q')$ is locally constant on any lift of D in an explicit way.
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It turns out that the assumption holds in all cases.
This leaves us with three points P such that $j(P) \in D$,
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(This point comes from the 'tautological point' on $X_{864b1}(11)$.)

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We finally obtain:

Theorem.
Assume GRH. The only coprime integer solutions of $x^2 + y^3 = z^{11}$ are

$$(\pm 1, 0, 1)\,, \quad \pm(0, 1, 1)\,, \quad (\pm 1, -1, 0)\,, \quad (\pm 3, -2, 1)\,.$$

Thank You!