Synchronization of Weakly Acyclic Automata

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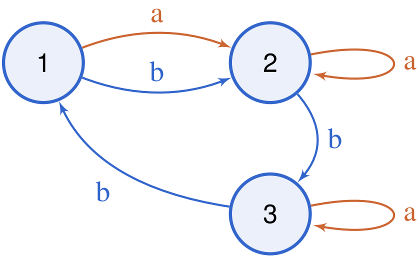
Synchronizing Automata

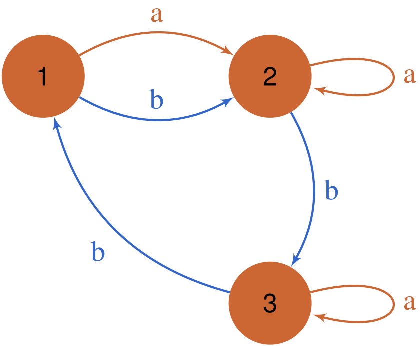
We consider deterministic finite automata without inputs and outputs.

Definition

Automaton $A = (Q, \Sigma, \delta)$ is *synchronizing*, if there exists a word $w \in \Sigma^*$ such that after reading this word A is transited to some particular state regardless of its initial state. Such word is called a *reset word*.



Example



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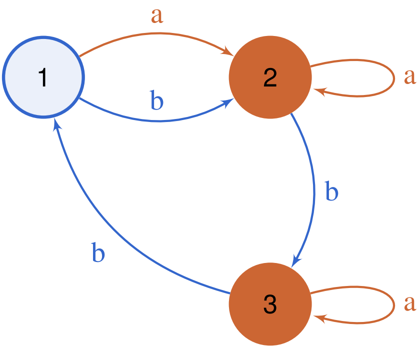

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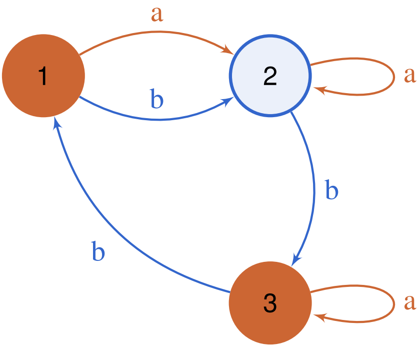

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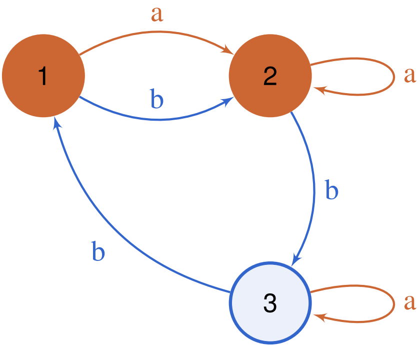

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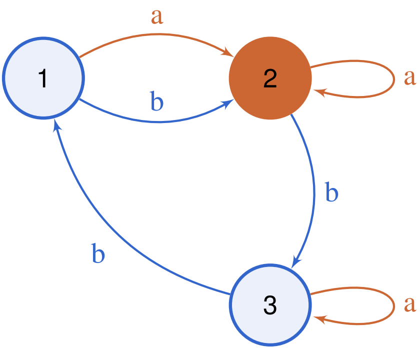

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Applications

1. Orienting parts in manufacturing;
2. Synchronizing codes;
3. Semigroup theory;
4. Symbolic Dynamics.

The Whole Automata

Theorem (Černý)

Checking whether an automaton is synchronizing can be done in polynomial time.

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An automaton is synchronizing $\iff$ each pair of its states can be synchronized.

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Conjecture (Černý)

For each automaton with n states there exists a reset word of length $(n-1)^2$.

Proved for orientable, Eulerian, aperiodic, ...

Theorem (Pin)

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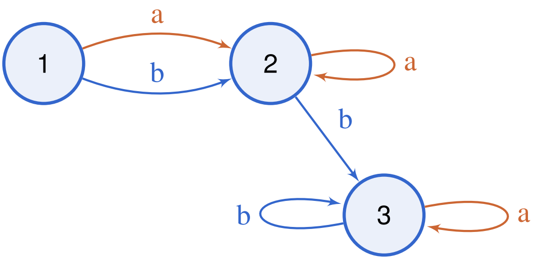
Weakly Acyclic Automata

Definition

A *cycle* in an automaton is a sequence $q_1, \ldots, q_n$ of its states such that there exist letters $x_1, \ldots, x_n \in \Sigma$ with $\delta(q_i, x_i) = q_{i+1}$ for $1 \leq i \leq n-1$ and $\delta(q_n, x_n) = q_1$. A cycle is a *self-loop* if it consists of one state. An automaton is called *weakly acyclic* if all its cycles are self-loops.

Called sometimes *acyclic* or *partially ordered*.



Motivation

Why study synchronization in weakly acyclic automata?

1. It's a natural notion;
2. Many hard problems for general automata are still hard for them – we get tighter results;
3. Already considered implicitly in several complexity reductions.

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The Main Idea of the Talk

Weakly acyclic automata admit short reset words, even for their subsets of states.

Despite of that, most of synchronization problems that are hard for general automata are still extremely hard for weakly acyclic automata.

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Short Words

Definition

Given an automaton A, the *rank* of a word w is the number $|\{\delta(s,w) \mid s \in Q\}|$

Theorem (R)

Let A be a weakly acyclic automaton, and w be a word of rank r with respect to A. Then there exists a word of length $n-r$ and rank r with respect to A.

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Synchronizing Set

Definition

A set $S \subseteq Q$ of states in an automaton A is called *synchronizing* if there exists a word $w \in \Sigma^*$ and a state $q \in Q$ such that the word w maps each state $s \in S$ to the state q.

In particular, an automaton is synchronizing $\iff$ the whole set Q of its states is synchronizing.

Theorem (Vorel)

There exist infinitely many binary strongly connected automata with an exponential lower bound on the length of the shortest reset word for a synchronizing subset of states.

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Synchronizing Set

Theorem (R)

Let S be a synchronizing set of states of size k in a weakly acyclic n-state automaton A. Then the length of a shortest reset word for S is at most $\frac{k(2n-k-1)}{2}$.

It is almost tight ($(k-1)(n-k)+1$):

Also good for monotonic automata and automata of rank 2.

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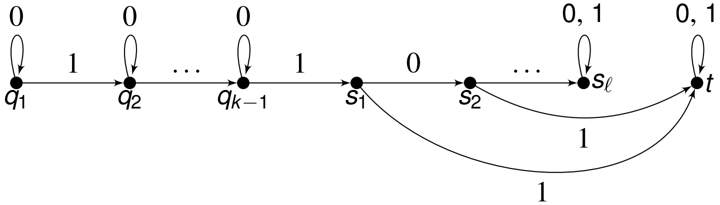

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Short Synchronizing Words Complexity

Theorem (Eppstein)

Finding a shortest reset word for binary weakly acyclic automata is an NP-hard problem.

Theorem (Berlinkov)

For any $\gamma > 0$, the problem of finding a reset word of minimum length in weakly acyclic synchronizing automata with alphabet of size $n^{1+\gamma}$ can not be approximated within a factor of $d \log n$ for any $d < c_{sc}$ unless P = NP.

Question

Improve this bounds or find approximation algorithms.

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Checking whether a given set of states in an automaton is synchronizing is PSPACE-complete...

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Checking whether a given set of states in a binary weakly acyclic automaton is synchronizing is NP-complete.

SET SYNC WORD

Input: An automaton A and a synchronizing subset S of its states;

Output: The shortest reset word for S.

Theorem (R)

The problem SET SYNC WORD cannot be approximated in n-state binary weakly acyclic automata within a factor of $O(n^{\frac{1}{4}-\varepsilon})$ for any $\varepsilon > 0$ unless P = NP.

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The problem of computing the rank of an automaton can be solved in polynomial time.

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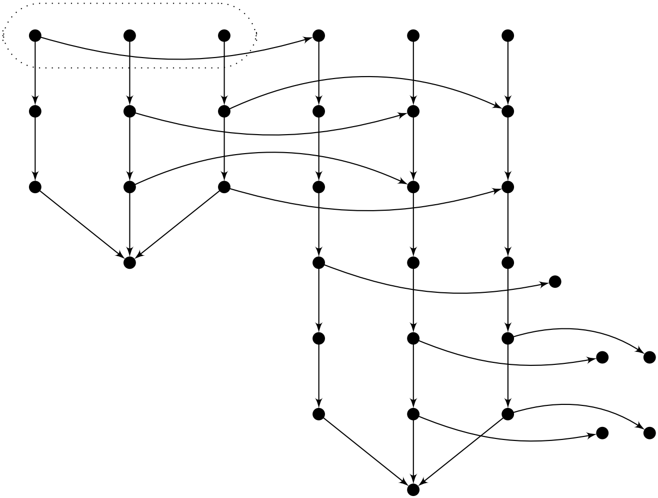
Idea of the Proof



MAX SYNC SET
Input: An automaton A;
Output: A synchronizing set of states of maximum size in A.

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For any $\varepsilon > 0$, unless P = NP, the problem MAX SYNC SET for n-state automata cannot be approximated in polynomial time within a factor ...

Theorem (R)

...$O(n^{1-\varepsilon})$ in the class of weakly acyclic automata over an alphabet of size $O(n)$.

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Find better inapproximability hierarchy for the problem of computing the rank of a subset.

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Find approximation algorithms for considered problems.

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Consider discussed questions for other classes of automata.

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