Automorphisms of low complexity subshifts 2

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Any subshift has a non empty minimal subshift.

Example:

- Periodic sequences.
- Sturmian subshift, substitutive,...
- Toeplitz subshift.

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To understand $\mathrm{Aut}(X, \sigma)$: first understand the minimal case !

Automorphism of minimal systems

Lemma

Let (X, T) be a minimal dynamical system. The action of $Aut(X, T)$ on X

$$\begin{array}{rcl} Aut(X,T)\times X & \rightarrow & X \\ (\phi, x) & \mapsto & \phi(x), \end{array}$$

is free (the stabilizer of any point is trivial).

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Proof. For any automorphism ϕ, the set

$$\{x; \phi(x) = x\}$$

is closed and T invariant.

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Examples of minimal subshift (X, σ), with $\mathrm{Aut}(X, \sigma)$ isomorphic to

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Examples of minimal subshift (X, σ), with $\mathrm{Aut}(X, \sigma)$ isomorphic to

- $\mathbb{Q}$, with 1 identified with σ (BLR, 88)
- $\langle \sigma \rangle \oplus G$ for an arbitrarily finite group G (Host-Parreau - Lemańczyk-Mentzen, 89)
- $\langle \sigma \rangle \oplus G$ for an arbitrarily f. g. abelian group G (eventually G trivial)

Non superlinear complexity

Theorem (Donoso-Durand-Maass & P., Cyr & Kra (15))

Let (X,σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X,\sigma)/\langle\sigma\rangle$ is finite and

$$\#Aut(X,\sigma)/\langle\sigma\rangle \leq \liminf_n \frac{p_X(n)}{n}.$$

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Example.

- Sturmian subshifts: $p_X(n) = n + 1$ for all n (Olli 2013).
- Coding of minimal Interval Exchange Transformations.
- Pisot substitution (Salo-Törmä 2013)
- Linearly recurrent subshift (substitutive, ...).

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Example. This includes also

- Subshifts with subexponential complexity $p_X(n) \geq g(n)$ i.o. where $\lim_n g(n)/\alpha^n = 0$ for any $\alpha > 1$.

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Result is sharp. Host-Parreau, Lemańczyk-Mentzen (1989): for any finite group G there exists a minimal subshift (X, σ) with

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$\hat{\phi}_g : \mathcal{L}(X_\tau) \to \mathcal{L}(X_\tau)$ $\qquad$ $\phi_g\colon X_\tau \to X_\tau$.

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Theorem (Donoso-Durand-Maass & P., Cyr & Kra (15))

Let (X,σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $\#Aut(X,\sigma)/\langle\sigma\rangle \leq \liminf_n \frac{p_X(n)}{n}$.

Result is sharp. Salo (14), DDMP (16): $\forall \epsilon > 0$, there exists a Toeplitz subshift with complexity $O(n^{1+\epsilon})$ with a non finitely generated automorphism group.

Main Ideas

A word $w \in \mathcal{L}(X)$ is right special if there are two letters $a, b \in A$ s.t. wa and wb are words of X.

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Two sequences $x = (x_n)_{n\in\mathbb{Z}}$, $y = (y_n)_{n\in\mathbb{Z}} \in X$ are asymptotics if there is a $n_0 \in \mathbb{Z}$

$$x_n = y_n \;\; \forall n < n_0 \text{ and } x_{n_0} \neq y_{n_0}.$$

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$$x_n = y_n \ \ \forall n < n_0 \text{ and } x_{n_0} \neq y_{n_0}.$$

This defines an equivalence relation on σ-orbits.
Non trivial class are asymptotic pairs.

$$\lim_n d(\sigma^n(x), \sigma^n(y)) = 0.$$

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Proposition

Let (X, σ) be a subshift with $\liminf_n p_X(n)/n = K < \infty$. Then there is at most K asymptotic pairs.

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claim $\Rightarrow$ unif. bound on special words.

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Let (X,σ) *be a subshift with* $\liminf_n p_X(n)/n = K < \infty$.
Then there is at most K *asymptotic pairs.*

Proof. claim: $p_X(n+1) - p_X(n) < K+1$ i.o.

claim $\Rightarrow$ unif. bound on special words.

By contradiction: $\forall n \geq m$ big enough

$$
\begin{aligned}
p_X(n) - p_X(m) &= \sum_{i=m}^{n-1} p_X(i+1) - p_X(i) \geq (n-m)(K+1) \\
p_X(n) &\geq (n-m)(K+1) + p_X(m)
\end{aligned}
$$

Main Ideas

Proposition

Let (X, σ) be a subshift with $\liminf_n p_X(n)/n = K < \infty$. Then there is at most K asymptotic pairs.

Corollary

Let (X, σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X, \sigma)/\langle \sigma \rangle$ is finite.

Toeplitz sequences

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A Toeplitz sequence, *i.e.* $(x_n)_n$ is s.t.

$$\forall n \in \mathbb{Z}, \exists \Gamma = p\mathbb{Z} < \mathbb{Z} \quad x_n = x_{n+\gamma} \quad \forall \gamma \in \Gamma.$$

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$p_1 = 2$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$
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Free to choose:

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- to fill part of the gaps.

	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
$p_1 = 2$	0	*	0	*	0	*	0	*	0	*	0	*	0	*	0	*	0	*
$p_2 = 6$	0	**1**	0	*	0	*	0	**1**	0	*	0	*	0	**1**	0	*	0	*
$p_3 = 12$	0	1	0	**0**	0	*	0	1	0	*	0	*	0	1	0	**0**	0	*
$p_4 = 60$	0	1	0	0	0	**1**	0	1	0	**1**	0	**1**	0	1	0	0	0	**1**

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$p_1 = 2$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$
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A Toeplitz subshift is the subshift X generated by a Toeplitz sequence $x = (x_n)_n$.

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- non uniquely ergodic minimal subshift (Williams, 84)
- an arbitrary entropy $h \geq 0$ (Williams, 84)
- complexity in $\Theta\left(n^{\alpha_0}(\log n)^{\alpha_1}(\log\log n)^{\alpha_2}\cdots(\log_{(k)} n)^{\alpha_k}\right)$, $\alpha_0 > 1, \alpha_1, \cdots, \alpha_k \in \mathbb{R}$.

  (Goyon, Cassaigne)

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Dynamically:

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Dynamically:
For $x \in X$ a Toeplitz sequence, for any open set $U \subset X$, the return times of x in U:

$$\{n \in \mathbb{Z} : \sigma^n(x) \in U\},$$

contains a subgroup of $\mathbb{Z}$.

Adding machine or odometer

Given a sequence of periods $(p_n)_{n\geq 0}$, with $p_n | p_{n+1}$.
The odometer

$$\mathbb{Z}_{(p_n)} = \{(x_n)_{n\geq 0} \in \prod_{n=0}^{\infty} \mathbb{Z}/p_n\mathbb{Z} : x_{n+1} \equiv x_n \mod p_n \ \forall n\}.$$

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Any minimal equicontinuous system on a Cantor set is conjugated to an odometer.

Toeplitz subshift

Theorem (Williams)

Any Toeplitz subshift (X, σ) *is an extension of an odometer* $(\mathbb{Z}_{(p_n)}, +\mathbf{1})$.
Moreover the factor map $\pi \colon X \to \mathbb{Z}_{(p_n)}$ *is injective on a* G_δ *dense set.*

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Converse true (Downarowicz, Lacroix)

Toeplitz subshift

Lemma

If $\pi : (X, \sigma) \to (\mathbb{Z}_{(p_n)}, +\mathbf{1})$ *is an almost one-to-one extension. Then*

$$\pi(x) = \pi(y) \Leftrightarrow \liminf_n d(\sigma^n(x), \sigma^n(y)) = 0.$$

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Corollary

For a Toeplitz subshift (X,σ)*.*

$$\mathrm{Aut}(X,\sigma) \subset \mathrm{Aut}(\mathbb{Z}_{(p_n)}, +\mathbf{1}) \simeq \mathbb{Z}_{(p_n)}.$$

In particular $\mathrm{Aut}(X,\sigma)$ *is abelian and residually finite.*

Toeplitz subshift

Corollary

For a Toeplitz subshift (X, σ).

$$\mathrm{Aut}(X, \sigma) \subset \mathrm{Aut}(\mathbb{Z}_{(p_n)}, +\mathbf{1}) \simeq \mathbb{Z}_{(p_n)}.$$

Consequences:

$\mathbb{Q} \not\subset \mathrm{Aut}(X, \sigma)$.

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Lemma

The torsion group of $\mathbb{Z}_{(p_n)}$ is isomorphic to $\bigoplus_p \mathbb{Z}/p^k\mathbb{Z}$, where the sum is taken over all the prime numbers p such that $\lim_{n\to\infty} v_p(p_n) = k$ is positive and finite.

If X is a Toeplitz subshift, any f.g. torsion subgroup is cyclic.

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Lemma

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If X is a Toeplitz subshift, any f.g. torsion subgroup is cyclic.
If X has periods $(p_n) = (p^n)$ for some prime p.
Then $\mathrm{Aut}(X,\sigma)$ has no torsion element.

Toeplitz subshift

Corollary

For a Toeplitz subshift (X, σ).

$$\mathrm{Aut}(X, \sigma) \subset \mathrm{Aut}(\mathbb{Z}_{(p_n)}, +\mathbf{1}) \simeq \mathbb{Z}_{(p_n)}.$$

Corollary

If X is a Toeplitz subshift $\liminf_n p_X(n)/n < +\infty$, *then*

$$\mathrm{Aut}(X, \sigma) \simeq \mathbb{Z} \text{ or } \mathbb{Z} \times \mathbb{Z}/N\mathbb{Z},$$

for some N.

See Coven, Quas,Yassawi (2016).

Toeplitz subshift

Examples of Toeplitz subshifts with:

- complexity $\mathrm{O}(n^{1+\epsilon})$ and $\mathrm{Aut}(X, \sigma)$ not f.g.

(Salo, DDMP)

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- positive entropy and $\mathrm{Aut}(X, \sigma) = \langle \sigma \rangle$.

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  (Bulatek, Kwiatkowski, Downarowicz, 90's)
- positive entropy and $\mathrm{Aut}(X, \sigma) = \langle \sigma \rangle \oplus G$ for an arbitrarily f.g. abelian group G.

  (DDMP)

Open pb: realize any countable subgroup of $\mathbb{Z}_{(p_n)}$ as $\mathrm{Aut}(X, \sigma)$?

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