Number of ergodic lifts for finite-to-one factor maps between shifts of finite type

(joint with *Jisang Yoo (Seoul National Univ.)*)

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Shift spaces and codes

Let X be a shift of finite type (SFT).

- A *word* is a finite sequence of symbols from an alphabet. Denote by $\mathcal{B}_n(X)$ the set of words of length n occurring in X and $\mathcal{B}(X) = \bigcup_n \mathcal{B}_n(X)$.
- The *entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

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$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

- X is *irreducible* if, given $u, v \in \mathcal{B}(X)$, there is $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$.
- A point $x \in X$ is *doubly transitive* if every word in X appears in x infinitely often to the right and the left.
- A shift space X is irreducible if and only if it has a right (or left) transitive point. If X is irreducible, the set of doubly transitive points in X is residual.

Degree of a code

Let X be an SFT and $\pi : X \to Y$ be a factor code onto a subshift Y.

- Let X be irreducible. Then the following are equivalent.
  1. $h(X) = h(Y)$.
  2. For each $y \in Y$, we have $|\pi^{-1}(y)| < \infty$.
  3. For each $y \in Y$, the set $|\pi^{-1}(y)|$ is at most countable.
  4. π is *finite-to-one*, i.e., there is $M \in \mathbb{N}$ such that $|\pi^{-1}(y)| \leq M$ for each $y \in Y$.

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- The *degree* of π is defined to be the minimum number of π-preimages of points in Y, i.e., degree of $\pi = \inf\{|\pi^{-1}(y)| : y \in Y\}$.

- (Welch; Hedlund; Coven and Paul) If π is finite-to-one and d is the degree of π, then every doubly transitive point of Y has exactly d preimages.

Structure and the existence of factor maps

Let X and Y be irreducible SFTs.

	$h(X) = h(Y)$	$h(X) > h(Y)$
Structure	bounded preimages finite-to-one x is transitive iff $\phi(x)$ is. same number of preimages a.e. preimages are unif. separated a.e. degree represented combinatorically $\exists$ permutation properties ...	unbounded preimages infinite-to-one
Existence	generally hard only known for closing maps K-theoretical invariants	solved completely (Boyle, '83) known for many maps simple iff condition ... and many generalizations (Boyle,Tuncel,Thomsen,J.)

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Structure	bounded preimages finite-to-one *x is transitive iff $\phi(x)$ is.* *same number of preimages a.e.* *preimages are unif. separated a.e.* *degree represented combinatorically* *$\exists$ permutation properties* ...	unbounded preimages infinite-to-one *can be extended* *using class degree* (Allahbakhshi, Quas, Hong, J.)
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Induced mapping on the space of measures

- For a shift space X, let $\mathcal{M}(X)$ be the set of σ-invariant Borel probability measures on X.
  - $\mathcal{M}(X)$ is compact and convex.
  - $\mathcal{M}(X) \ni$ Bernoulli measures, Markov measures, Gibbs measures, $\cdots$
- A factor code $\pi : X \to Y$ induces a surjective map $\bar{\pi} : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $(\bar{\pi}(\mu))(B) = \mu(\pi^{-1}(B))$ for $\mu \in \mathcal{M}(X)$ and a Borel set B of Y.
- The map $x \mapsto \delta_x$ is a natural embedding from X into the set of Borel probability measures on X, hence $\bar{\pi}$ is a restriction of an extension of π to $\mathcal{M}(X)$; We will use π again instead of $\bar{\pi}$.

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- The map $x \mapsto \delta_x$ is a natural embedding from X into the set of Borel probability measures on X, hence $\bar{\pi}$ is a restriction of an extension of π to $\mathcal{M}(X)$; We will use π again instead of $\bar{\pi}$.
- If π is finite-to-one, then there is $d \in \mathbb{N}$, the degree of π, with (1) $|\pi^{-1}(y)| \geq d$ and (2) $|\pi^{-1}(y)| = d$ for almost all $y \in Y$.
- What are the numbers of preimages of this induced map $\pi : \mathcal{M}(X) \to \mathcal{M}(Y)$?

Ergodic measures

- **Question**: What are the numbers of preimages of $\pi : \mathcal{M}(X) \to \mathcal{M}(Y)$?
- π is affine: If $\pi(\mu_1) = \pi(\mu_2) = \nu$, then $\pi(p\mu_1 + (1-p)\mu_2) = \nu$;
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- A measure $\mu \in \mathcal{M}(X)$ is *ergodic* if every μ-invariant set has measure 0 or 1, equivalently, if μ is an extreme point of a convex set $\mathcal{M}(X)$.
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- **Question'**: What are the numbers of *ergodic* preimages of an *ergodic* measure under $\pi : \mathcal{M}(X) \to \mathcal{M}(Y)$?
- (Folklore) For each fully supported ergodic measure $\nu \in \mathcal{M}(Y)$, $\pi^{-1}(\nu)$ contains at most d ergodic measures.

Inverse stucture of Markov measures

Let X be an irreducible SFT and $\pi : X \to Y$ be a finite-to-one factor code onto Y.

- For $\mu \in \mathcal{M}(X)$ and $w \in \mathcal{B}(X)$, let $\mu(w) = \mu(\{x \in X : x_1 \cdots x_n = w\})$.
- A measure $\mu \in \mathcal{M}(X)$ is a *Markov measure* if

$$\mu(uvw|uv) = \mu(vw|v) \text{ for all } v \in \mathcal{B}_1(X) \text{ and } uvw \in \mathcal{B}(X).$$

- (Boyle and Tuncel) Let $v \in \mathcal{M}(Y)$ be a *fully supported* Markov measure. Then $|\pi^{-1}(\nu)| = 1$.
  - Since X and Y are intrinsically ergodic, this is clear for maximal measure in Y.
  - Inverses of Markov measures in Y have the same structure.
- Markov measures on Y are dense in $\mathcal{M}(Y)$. Hence $|\pi^{-1}(\nu)| = 1$ on a dense (but not residual) set.

Number of ergodic lifts

Theorem (J and Yoo)

Let X be an SFT and $\pi : X \to Y$ be a finite-to-one factor code onto a subshift Y. Let d be the degree of π. Then there is a fully supported measure $\nu \in \mathcal{M}(Y)$ with exactly d ergodic preimages (i.e., $\pi^{-1}(\nu)$ contains exactly d ergodic measures).

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- $\pi^{-1}(\nu)$ is the $d-1$ dimensional simplex in $\mathcal{M}(X)$.
- ν cannot be a Markov measure. Indeed, ν cannot be a Gibbs measure with nice potential function.
- There is a factor code π such that $\pi^{-1}(\nu)$ contains only 1 or d ergodic measures.

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- ν cannot be a Markov measure. Indeed, ν cannot be a Gibbs measure with nice potential function.
- There is a factor code π such that $\pi^{-1}(\nu)$ contains only 1 or d ergodic measures.
- Such ν's are dense in $\mathcal{M}(Y)$. It is open whether (1) such ν's are residual in $\mathcal{M}(Y)$ (2) the metric entropies of such ν's are dense in $(0, h(Y))$.
- It is open whether the analogous statement holds for the infinite-to-one case, i.e., the case where $h(X) > h(Y)$.

Sketch of the proof I

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- 1. Construct a periodic point $y \in Y$ so that $\pi^{-1}(y)$ consists of d periodic points whose orbits are all distinct.
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  - heavily depends on the structure of *magic word* and *permutation property* for factor maps in symbolic dynamics
- 2. Consider an atomic measure

$$\nu = \frac{1}{\mathrm{per}(y)} \sum_{i=1}^{\mathrm{per}(y)} \bar{\sigma}^i \delta_y = \frac{1}{\mathrm{per}(y)} \sum_{i=1}^{\mathrm{per}(y)} \delta_y \circ \sigma^- i.$$

$\nu \in \mathcal{M}(Y)$ has exactly d ergodic preimages, which are atomic measures concentrated on the orbits of elements in $\pi^{-1}(y)$.

Sketch of the proof II

- Let $x_1, \cdots, x_d = \pi^{-1}(y)$ and $\mu_1, \cdots, \mu_d$ be the ergodic measures over ν. Each μ_i is a CO-measure generated by x_i.
- 3. Construct a d-fold joining λ^* of $\mu_1, \cdots$ and μ_d.
    - λ^* is ergodic, mutually separated, has different margins, but their margins don't have full support.

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- 3. Construct a d-fold joining λ^* of $\mu_1, \cdots$ and μ_d.
  - λ^* is ergodic, mutually separated, has different margins, but their margins don't have full support.
- 4. Modify λ^* to obtain another d-fold joining λ, which is ergodic, mutually separated and has different margins, and *all margins having full support*.
  - Choose an invariant measure η on $\{0,1\}^{\mathbb{Z}}$ with high probability of the symbol 1 and positive probability for each arbitrary long $00\ldots00$.
  - For η-a.e. s, we construct a point $(z^{(1)}, \ldots, z^{(d)}) \in X^d$ that copies from $(x^{(1)}, \ldots, x^{(d)})$ for regions of 1 in s.
  - For regions of 0 in s, fill in a way that ensures full support margins.

Number of ergodic lifts

Theorem (J and Yoo)

Let X be an SFT and $\pi : X \to Y$ be a finite-to-one factor code onto a subshift Y. Let d be the degree of π. Then there is a fully supported measure $\nu \in \mathcal{M}(Y)$ with exactly d ergodic preimages (i.e., $\pi^{-1}(\nu)$ contains exactly d ergodic measures).

- Is it true whether (1) such ν's are dense in $\mathcal{M}(Y)$? (2) the metric entropies of such ν's are dense in $(0, h(Y))$?.
- What is the condition on a factor code of degree d to have $\nu \in \mathcal{M}(Y)$ such that $\pi^{-1}(\nu)$ contains exactly k ergodic measures for $1 < k < d$?
- Is it true whether the analogous statement holds for the infinite-to-one case, i.e., the case where $h(X) > h(Y)$.

Ergodic measures, revisited

- $\mathcal{M}(X)$ is the set of σ-invariant Borel probability measures on X.
- Let $\mathcal{E}(X) \subset \mathcal{M}(X)$ be the set of ergodic measures on X, i.e. the set of extreme points of the convex set (Choquet simplex) $\mathcal{M}(X)$.
- $\{$ CO-measures $\} \subset \{$ ergodic Markov measures $\} \subset \mathcal{E}(X)$.
- $\mathcal{E}(X)$ is a G_δ set. The set of fully supported measures is $\emptyset$ or residual.
- For SFT X, CO-measures are dense in $\mathcal{M}(X)$. Hence $\mathcal{E}(X)$ is residual and the closure of $\mathcal{E}(X)$ is $\mathcal{M}(X)$ (i.e. $\mathcal{M}(X)$ is a Poulsen simplex).

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- (Lindenstrauss, Olsen, Sternfeld '78) $\mathcal{E}(X)$ is path-connected.
- (Sigmund '78) Let μ_1 and μ_2 be CO-measures in $\mathcal{E}(X)$. Then there is a path in $\mathcal{E}(X)$ consisting of *fully supported Markov measures*.

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Theorem (J and Yoo)

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- 1. Let $\Sigma \subset X^d$ be a d-fold fibered product of $\pi : X \to Y$:
  $\Sigma = \{(x_1, \cdots, x_d) \in X^d : \pi(x_1) = \cdots = \pi(x_d)\}$.

  and $\Sigma_0 \subset \Sigma$ be an irreducible component of mutual separated part of Σ.
  - Entropy argument gives $h(\Sigma_0) = h(X) = h(Y)$.
- 2. Given $\mu \in \mathcal{M}(\Sigma_0)$, find a CO-measure $\lambda \in \mathcal{M}(\Sigma_0)$ near μ.
  - The support of CO-measure is a d-pair of disjoint periodic points with the same image.
  - λ is ergodic, mutually separated, has different margins.

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- 3. Find a path from λ to another CO-measure in $\mathcal{M}(\Sigma_0)$, which gives a *fully supported measure* λ^* which is close to λ.
  - full supportedness of λ^* follows from Sigmund.
  - Since λ^* and λ are close, λ^* has d-margins.

Transition between points

Let X be an SFT and $\phi : X \to Y$ be a factor code onto a subshift Y.

- Let $x, \bar{x} \in X$. We say that there is a *transition* from x to $\bar{x}$ and denote it by $x \to \bar{x}$ if, for each $n \in \mathbb{Z}$, there is a point $z \in Z$ such that
  1. $\phi(z) = \phi(x) = \phi(\bar{x})$,
  2. $z_{(-\infty,n]} = x_{(-\infty,n]}$, and
  3. z and $\bar{x}$ are right asymptotic.

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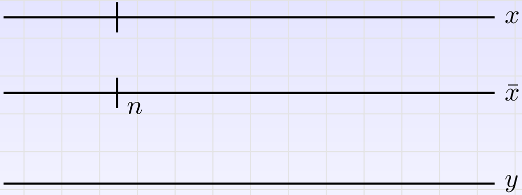

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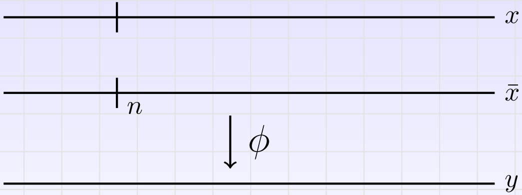

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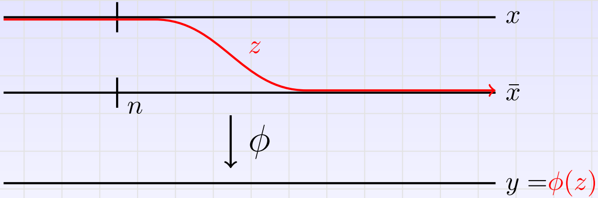

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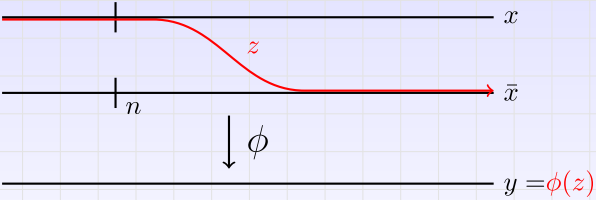

- Write $x \sim \bar{x}$ if $x \to \bar{x}$ and $\bar{x} \to x$. The relation $\sim$ is an equivalence relation.

Class degrees

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- For $x, \bar{x} \in X$, write $x \sim \bar{x}$ if $x \to \bar{x}$ and $\bar{x} \to x$. An equivalence class over $\sim$ is called a *transition class*.
- For $y \in Y$, denote by $\mathcal{C}(y)$ the set of transition classes in X over y.
- The minimal number of transition classes over points of Y is called the *class degree* of ϕ. The class degree is a conjugacy invariant.

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Theorem (Allahbakhshi and Quas, Allahbakhshi and J. and Hong)

Let X be irreducible and d the class degree of ϕ. Then the following holds.

1. *There are d transition classes over every right transitive point of Y.*
2. *If ϕ is finite-to-one, then the degree $= d =$ class degree.*
3. *For a typical $y \in Y$, their transition classes are mutually separated.*
4. *Let ν be a fully supported ergodic measure on Y. Then the number of ergodic measures of relative maximal entropy over ν is at most d.*

Thank You!