The Černý conjecture and 1-contracting automata

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Radboud University Nijmegen

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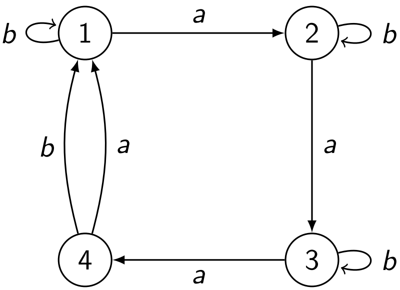

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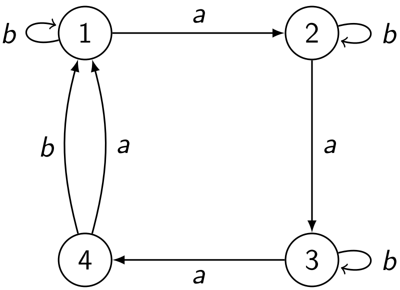

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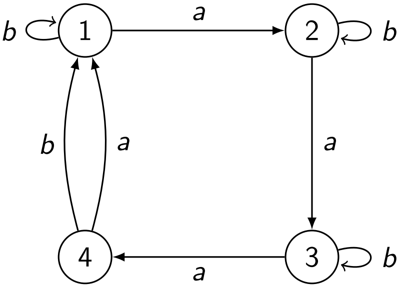

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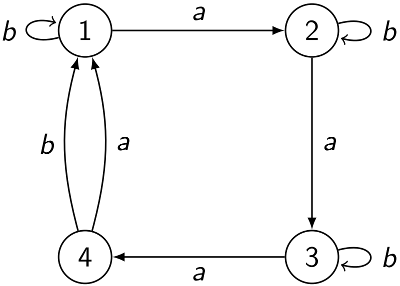

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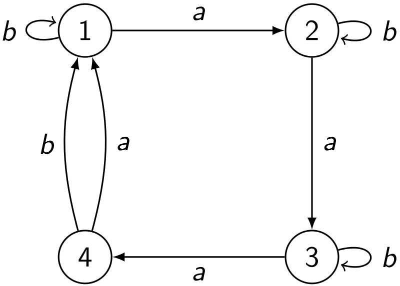

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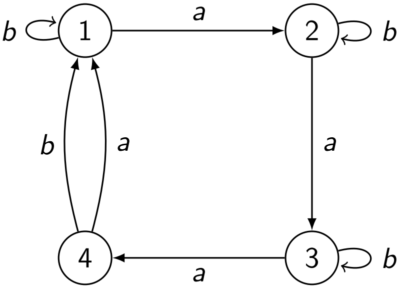

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- Set of symbols: $\Sigma = \{a, b\}$. Words: e.g. *abba*.
- Transitions: e.g. $1a = 2$, $1abba = 3$ and $\{1, 4\}\, b = 1$.

Definition (Synchronizing word)

- A DFA is *synchronizing* if there exists a word $w \in \Sigma^*$ and a state $q_s \in Q$ such that $qw = q_s$ for all $q \in Q$.
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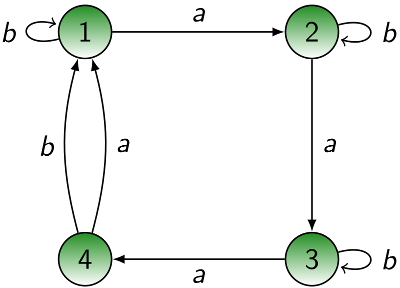
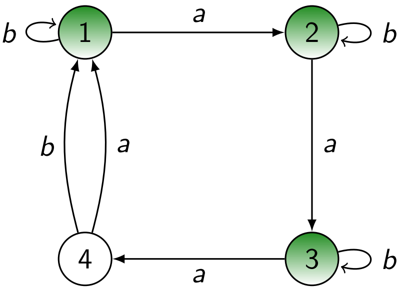
Idea: wherever you start, following the path labelled by w leads to the fixed state q_s. Word w acts as reset button.



- Claim: $w = baaabaaab$ is synchronizing.
- Let's check: read w starting from all states simultaneously. Green means occupied.

Example by Černý: $\mathscr{C}_4$



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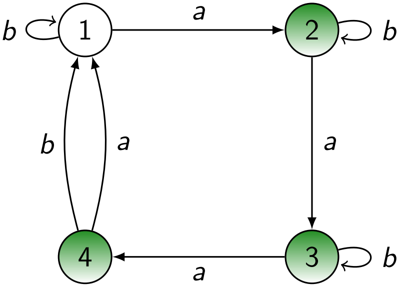

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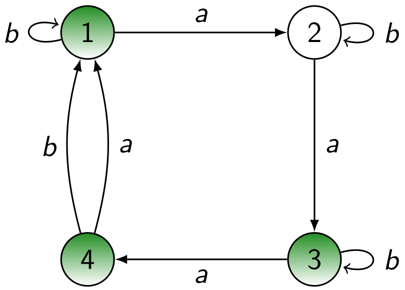

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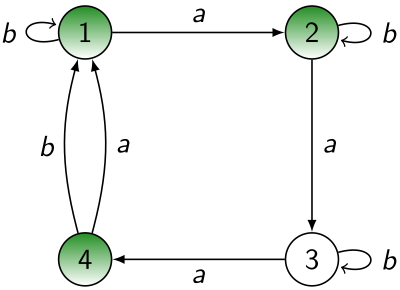

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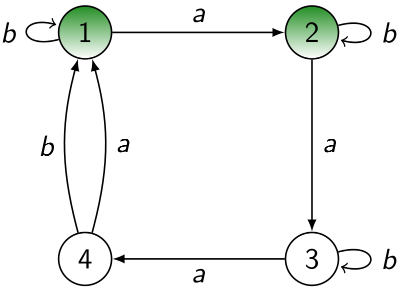

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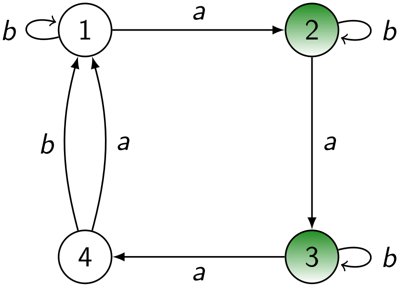

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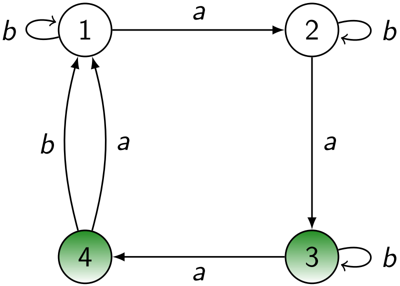

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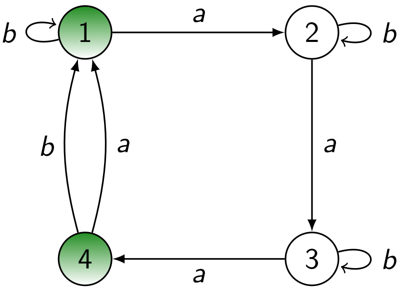

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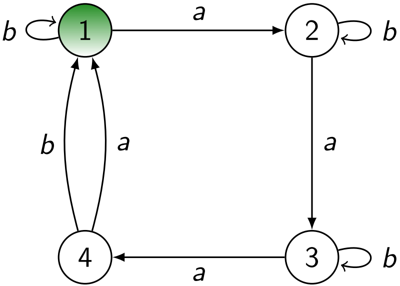

- Synchronizing word: $w = baaabaaab$, since $qw = 1$ for all $q = 1, 2, 3, 4$.

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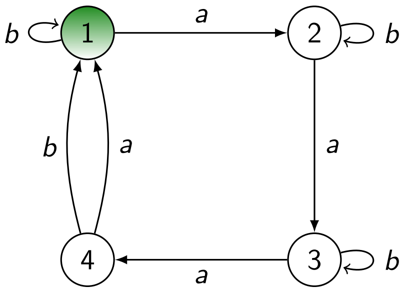
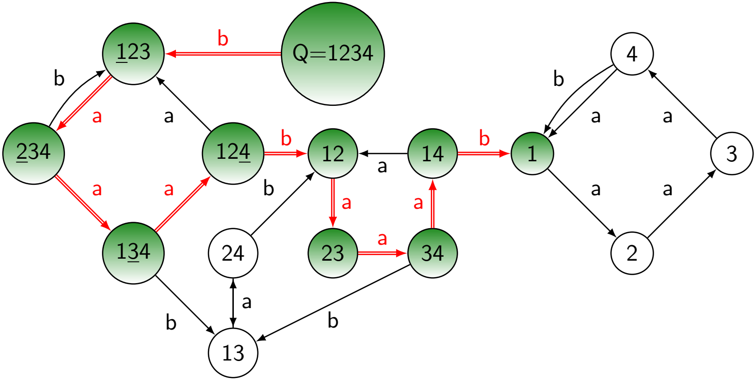
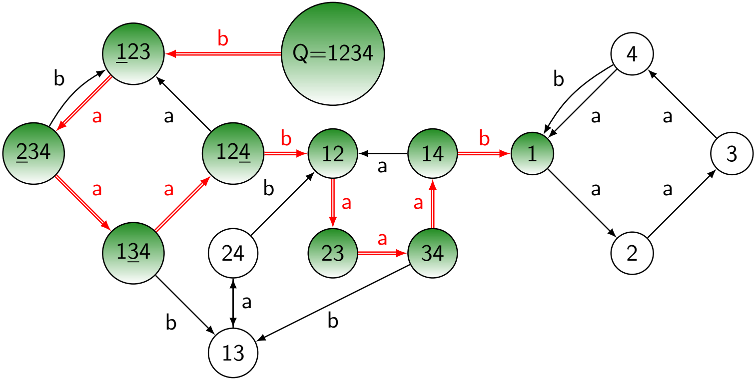


- Synchronizing word: $w = baaabaaab$, since $qw = 1$ for all $q = 1, 2, 3, 4$.
- Is there a shorter synchronizing word?


Q=1234
b
123
b
a
a
234
124
b
12
a
14
b
1
a
a
a
b
a
a
134
24
23
a
34
a
a
13
b
b
4
b
a
a
3
a
a
2



Path from Q to singleton corresponds to synchronizing word.
Shortest synchronizing word: *baaabaaab*. Length: 9.

Question: Worst case? How long can a shortest synchronizing word be?

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Conjecture (Černý, 1964)

If a DFA is synchronizing, then the shortest synchronizing word has length at most $(n-1)^2$.

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Conjecture (Černý, 1964)

If a DFA is synchronizing, then the shortest synchronizing word has length at most $(n-1)^2$.

Conjecture still open. Best upper bound is cubic in n.

Partial results:

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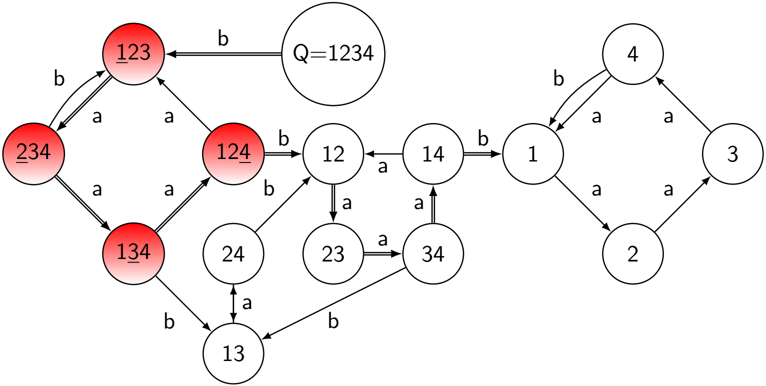
- Easy upper bound: $2^n - n - 1$ (Černý, 1964).
- Best known upper bound: $\frac{n^3-n}{6}$ (Pin, Frankl, 1983).
- Sequence $\mathscr{C}_n$ reaches $(n-1)^2$.
- Conjecture settled for certain subclasses, e.g. cyclic automata.



- All sets of size $n-1$ are reachable.
- 1-deficient words: b, ba, baa, $baaa$, ... and many more.

Definition (1-deficient word)

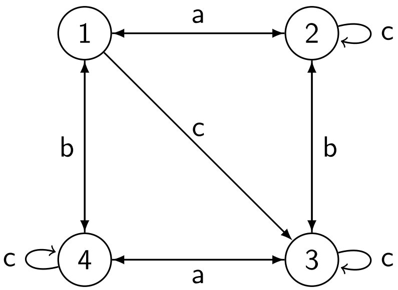
A word is called *1-deficient* if $|Qw| = n - 1$.

Definition (1-contracting automaton)

A DFA is called *1-contracting* if for all $q \in Q$ there exists $w_q \in \Sigma^*$ such that $Qw_q = Q \setminus \{q\}$.

Equivalent: all $(n-1)$-sets are reachable. Note: w_q can be chosen to have length at most n.

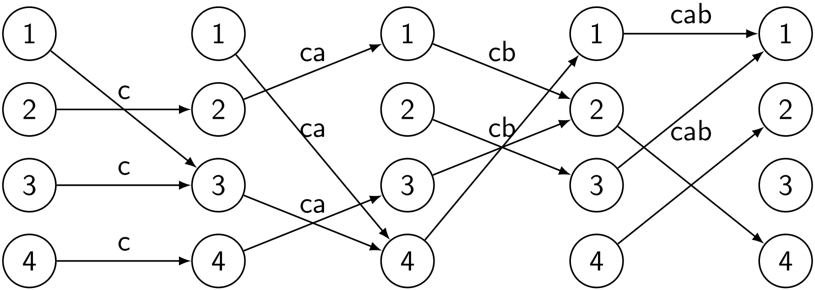
Example:



This DFA is 1-contracting:

- $Qc = \{2, 3, 4\}$
- $Qca = \{1, 3, 4\}$
- $Qcb = \{1, 2, 3\}$
- $Qcab = \{1, 2, 4\}$

Another way to see it is 1-contracting:



- $Qc = Q \setminus \{1\}$
- $Qca = Q \setminus \{2\}$
- $Qcb = Q \setminus \{4\}$
- $Qcab = Q \setminus \{3\}$

- Are 1-contracting automata synchronizing?
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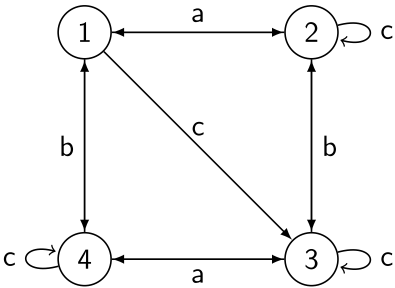

This DFA is 1-contracting, as we have seen.

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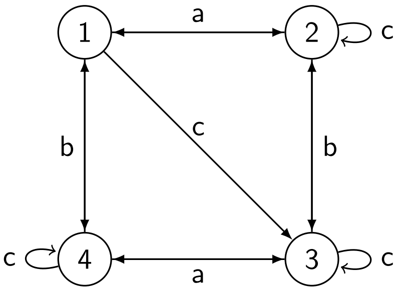
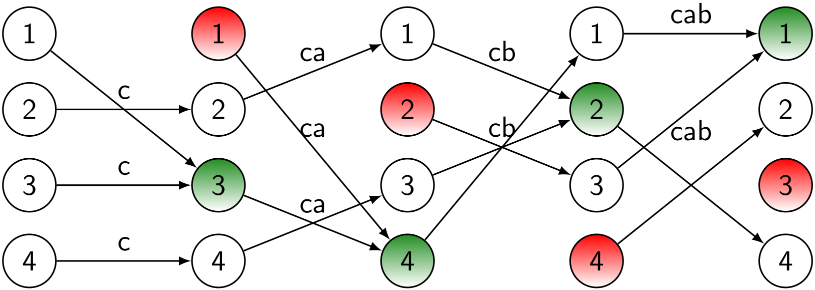


This DFA is 1-contracting, as we have seen.

Not synchronizing! For instance $\{1, 2\}$ can not be synchronized.

Properties of 1-deficient word w:

- Unique excluded state: *not reached* by w.
- Unique contracting state: *reached twice* by w.



Additional structure needed for synchronization.

Definition (1-contracting collection of words)

A collection of 1-deficient words W is called 1-contracting if for every $q \in Q$ there is exactly one $w_q \in W$ excluding q.

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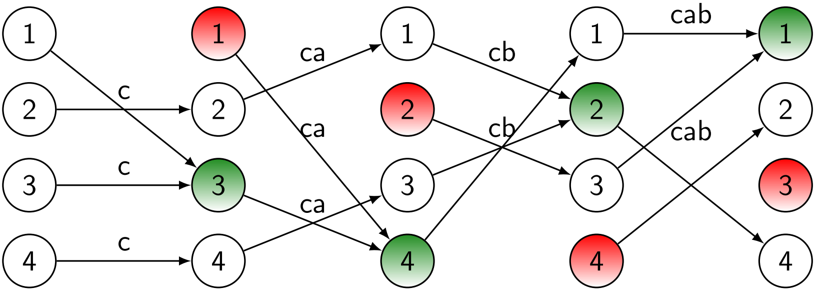
Now the following map is well-defined:

Definition (state map)

Let W be a 1-contracting collection. The *state map* $\sigma_W : Q \to Q$ induced by W is defined by

$$\sigma_W(q) = q_c$$

if q_c is the contracting state for the word excluding q.



$W = \{c, ca, cb, cab\}$ is a 1-contracting collection. State map:

- $\sigma_W(1) = 3$
- $\sigma_W(2) = 4$
- $\sigma_W(3) = 1$
- $\sigma_W(4) = 2.$

Theorem

Let $\mathscr{A} = (Q, \Sigma, \delta)$ *be a DFA with 1-contracting collection* W*. If* σ_W *is a cyclic permutation on* Q*, then*

1. $\mathscr{A}$ *is completely reachable.*
2. *In particular,* $\mathscr{A}$ *is synchronizing.*

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- In fact, if S is a subset of Q of size k, then S is reachable by concatenating $n - k$ words from W.

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- In fact, if S is a subset of Q of size k, then S is reachable by concatenating $n - k$ words from W.
- In the previous example, the state map clearly is *not* a cyclic permutation.

Theorem

Let $\mathscr{A} = (Q, \Sigma, \delta)$ be a DFA with 1-contracting collection W. If σ_W is a cyclic permutation on Q and $|w| \leq n$ for all $w \in W$, then

1. *If $S \subseteq Q$ has size $1 \leq k \leq n$, then S is reachable by a word of length at most $n(n-k)$.*
2. *$\mathscr{A}$ has a synchronizing word of length at most $(n-1)^2$.*

Open questions:

- Words in a 1-contracting collection can always be chosen to have length at most n. Is this still true if we require the state map to be a cyclic permutation?

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Conjecture

Let $\mathscr{A} = (Q, \Sigma, \delta)$ be an n-state DFA. If $S \subseteq Q$, $|S| = k$ and there exists a word $w \in \Sigma^*$ such that $Qw = S$, then there exists a word with this property of length at most $n(n-k)$.

This would imply Černý's conjecture.

J. Černý.
Poznámka k homogénnym experimentom s konečnými automatmi.
Matematicko-fyzikálny časopis, Slovensk. Akad. Vied, Vol. 14, No. 3, 208–216, 1964.

H. Don.
The Černý conjecture and 1-contracting automata.
Electronic Journal of Combinatorics 23 (3), 2016.

J.-E. Pin.
On two combinatorial problems arising from automata theory.
Annals of Discrete Mathematics, 17, 535–548, 1983.

E.A. Bondar and M.V. Volkov.
Completely reachable automata. arXiv:1607.00554, 2016.