Automatic sequences, generalised polynomials, and nilmanifolds

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Sketch of a proof: We prove that the sequence

$$f(n) = \lfloor \alpha n + \beta \rfloor \bmod m, \quad m \geq 2,$$

is not k-automatic if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The rotation by $2\pi\alpha$ on the unit circle is ergodic if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. This gives a contradiction.

Generalised Polynomials

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$$f(n) = n\lfloor \sqrt{2}n^2 + 3\lfloor \sqrt{3}n\rfloor^2\rfloor.$$

The class of generalised polynomials is closed under the operations:

- $f(n) \bmod 1 = f(n) - \lfloor f(n)\rfloor$.
- $\langle\langle f(n)\rangle\rangle = \lfloor f(n) + 1/2\rfloor$, the nearest integer to $f(n)$.

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We call a set $E \subset \mathbb{N}$ generalised polynomial if its characteristic function is generalised polynomial.

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Weyl Equidistribution Theorem, 1914

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Examples (Bergelson-Leibman):

- If α, β are $\mathbb{Q}$-independent irrational numbers, then

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- The sequence

$$(-\sqrt{2}n\lfloor\sqrt{2}n\rfloor \bmod 1)$$

is uniformly distributed on $[0,1]$ with respect to the measure $\frac{dx}{2\sqrt{2x}}$ on $[0,1/2]$ and $\frac{dx}{2\sqrt{2x-1}}$ on $[1/2,1]$.

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Distribution of generalised polynomials III

There are general equidistribution results. However, we are interested in sparse general polynomials that take value 1 on a set of density zero. Not much is known about those.

There are non-trivial examples:

- The set of Fibonacci numbers is generalised polynomial.
- We conjecture that the set of powers of 2 is not generalised polynomial.

Nilmanifolds

A nilmanifold is a homogenous space $X = G/\Gamma$, where G is a nilpotent Lie group and Γ is a discrete cocompact subgroup, together with the action of G on X via left translations.

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We consider nilmanifolds as dynamical systems under left translation by $g \in G$.

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Theorem (Bergelson–Leibman, 2006)

1. If $X = G/\Gamma$ is a nilmanifold, $g \in G$ acts on X by left translations, $p\colon X \to \mathbb{R}$ is a piecewise polynomial map, and $x \in X$, then $u\colon \mathbb{Z} \to \mathbb{R}$ given by $u(n) = p(g^n x)$ is a bounded generalised polynomial.

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2. If $u \colon \mathbb{Z} \to \mathbb{R}$ is a bounded generalised polynomial, then there exists a nilmanifold $X = G/\Gamma$, $g \in G$ acting on X by left translations in such a way that the action is ergodic, a piecewise polynomial map $p \colon X \to \mathbb{R}$, and $x \in X$ such that $u(n) = p(g^n x)$, $n \in \mathbb{Z}$.

Bergelson-Leibman Theorem: Example

Let $G = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ and $g = \begin{pmatrix} 1 & -a & 1 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & ab \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

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Let $\Gamma = G \cap GL_4(\mathbb{Z})$, $X = G/\Gamma$.

Then for a certain choice of a function $p \colon X \to \mathbb{R}$ we have

$$p(g^n\Gamma) = \langle\!\langle an\lfloor bn \rfloor \rangle\!\rangle.$$

In fact, $p(g\Gamma)$ is the $(4,1)$-coordinate of the unique representative of $g\Gamma$ with all the coordinates in $[0,1)$.

Bergelson-Leibman Theorem: Example II

Let $G = \left\{ \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$ and $g = \begin{pmatrix} 1 & 1 & 0 & \dots & b_d \\ 0 & 1 & 1 & \dots & b_{d-1} \\ 0 & 0 & 1 & \dots & b_{d-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

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Let $\Gamma = G \cap GL_d(\mathbb{Z})$, $X = G/\Gamma$.

Then the nilsystem (X, m_g) is isomorphic with the skew product map on a torus. For a polynomial $q \in \mathbb{R}[X]$, we can find find a function $p \colon X \to \mathbb{R}$ so that

$$p(g^n\Gamma) = \lfloor q(n) \rfloor.$$

IP sets and IPS sets

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A set $E \subset \mathbb{N}$ is called an IP set if it contains a set of the form

$$\mathrm{FS}(n_i) = \{n_{i_1} + \ldots + n_{i_k} \mid i_1 < i_2 < \ldots < i_k\}.$$

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We consider a more general class of IPS sets. These are "shifted" IP sets. The can be described as sets belonging to ultrafilters of the form $r + p$ with p idempotent.

Sparse generalised polynomials and IPS sets

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Let $\varphi = \frac{1+\sqrt{5}}{2}$ and let

$$E = \{n \in \mathbb{N} \mid \|n\varphi\| < 1/2n\} = \{n \in \mathbb{N} \mid \lfloor 2n \, \|n\varphi\| \rfloor = 0\},$$

where $\|n\alpha\| = |\alpha - \langle\langle \alpha \rangle\rangle|$. Then E coincides with the set of Fibonacci numbers (up to a finite set).

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This can be generalised to sets of the form

$$E = \{\langle\!\langle \beta^i \rangle\!\rangle \mid i \in \mathbb{N}\},$$

where β is a quadratic irrational of norm ± 1.

Cubic linear recurrence sets

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Theorem

Let a, b be integers such that either ($a \geq 0$ and $0 \leq b \leq a + 1$) or ($a \geq 2$ and $b = -1$). Assume further that there is a unique real root β of the equation

$$\beta^3 = a\beta^2 + b\beta + 1$$

and that $\beta > 1$. Then the set $\{\langle\langle \beta^i \rangle\rangle \mid i \in \mathbb{N}\}$ is generalised polynomial.

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The reason: the sequence of best approximations of the point $\theta = (\beta^{-1}, \beta^{-2}) \in \mathbb{R}^2$ satisfies the same linear recurrence as β^n.

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Theorem

There exists a constant $C > 0$ such that for any sequence $(n_i)_{i \geq 0}$ such that $n_0 \geq 2$ and $n_{i+1} \geq n_i^C$ for all $i \geq 0$, the set $E = \{n_i \mid i \in \mathbb{N}\}$ is generalised polynomial.

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For this reason, it seems unlikely that a comprehensive understanding of sparse generalised polynomials is possible.

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Example (Thue-Morse): The sequence $(t_n)_{n\geq 0}$ is given by $t_n = 1$ if $s_2(n)$ is odd and $t_n = 0$ if $s_2(n)$ is even, where $s_2(n)$ is the sum of digits in base 2 expansion of n.)

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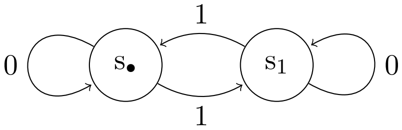

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Alternative interpretations of automatic sequences

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- Sequences $a = (a_n)_{n \geq 0}$ with finite kernel

$$N(a) = \{(a_{k^m n + l})_{n \geq 0} \mid m \geq 0, 0 \leq l < k^m\}.$$

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- If $a_n \in \mathbb{F}_p$, then (a_n) is p-automatic if and only if the power series $\sum_{n \geq 0} a_n X^n$ is algebraic over $\mathbb{F}_p(X)$.

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Automatic sequences have been generalised to a class of sequences admitting possibly infinitely many values (the so-called k-regular sequences of Allouche and Shallit).

Can automatic sequences be generalised polynomials?

Conjecture

Suppose that a sequence is simultaneously automatic and generalised polynomial. Then it is ultimately periodic.

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Theorem

Suppose that a sequence f is automatic and generalised polynomial. Then the sequence is periodic except possibly on a set of (upper Banach) density zero. In fact, we have a stronger bound on the growth of the set of possible exceptions Z:

$$|Z \cap [0, N-1]| = O((\log N)^k), \quad k \geq 0.$$

Main results for automatic generalised polynomials

Theorem

Let $k \geq 2$ be an integer and let $(a_n)_n$ be a $\{0,1\}$-valued k-automatic sequence. Then one of the following statements holds:

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1. either the set $\{n \mid a_n = 1\}$ is an IPS set; or
2. the set $\{n \mid a_n = 1\}$ is a finite union of sets of the form

$$E = \left\{ [w_0 u_1^{l_1} w_1 u_2^{l_2} \dots u_r^{l_r} w_r]_k \;\middle|\; l_1, \dots, l_r \in \mathbb{N}_0 \right\}.$$

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Let $k \geq 2$ be an integer. Then one of the following statements holds:

1. either all sequences which are simultaneously k-automatic and generalised polynomial are ultimately periodic; or
2. the characteristic sequence g_k of powers of k is generalised polynomial.

Questions

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For which $\lambda > 1$ is the set $\mathrm{E}_\lambda := \left\{ \langle\langle \lambda^{\mathrm{t}} \rangle\rangle \;\middle|\; \mathrm{t} \in \mathbb{N} \right\}$ generalised polynomial?

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Question

Let $\lambda > 1$ be a Pisot number. Assume the set $E_\lambda := \left\{ \langle\langle \lambda^t \rangle\rangle \;\middle|\; t \in \mathbb{N} \right\}$ is generalised polynomial. Is it then true that the norm of λ is ± 1? Does the converse hold?

Questions II

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Assume that a sequence is both regular and generalised polynomial. Is it then true that it is ultimately a quasi-polynomial?

Selected references

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