Lecture 2: The Domino problem on groups, part II.

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Introduction

Objectives of this talk...

- Give basic and inheritance properties about **DP**
- Describe classes and examples of groups with undecidable **DP**
- Formulate a conjecture on the characterization of groups with decidable **DP**

Yesterday

- **DP** undecidable on $\mathbb{Z}^2$
- hierarchy of arbitrary big grids + encode Turing machines
- encode the orbits of some $f : \mathbb{R}^2 \to \mathbb{R}^2$

Outline of the talk.

Reminder

Fix G a f.g. group and S a generating set for G.

Domino problem on G

Input: A finite set of Wang tiles τ on S
Output: **Yes** if there exists a valid tiling by τ, **No** otherwise.

Remark: Decidability of **DP** does not depend on the choice of S.

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Question

Which f.g. groups have decidable Domino Problem ?

Domino problem vs. Word problem (I)

Fix G a f.g. group and S a generating set for G.

$$WP(G) = \left\{ w \in \left(S \cup S^{-1}\right)^* \mid w =_G 1_G \right\}.$$

Word problem on G

Input: A finite word w on the alphabet $S \cup S^{-1}$
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Remark: The Word problem on G is decidable iff the language $WP(G)$ is recursive.

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Word Problem vs. Domino Problem (II)

Property

Let G be a f.g. group with decidable **DP**, then G has decidable **WP**.

Sketch of the proof:

- Suppose that S generates G.
- Consider a word $w \in \left(S \cup S^{-1}\right)^*$ s.t. $w =_G g$.
- Define the SFT $X_{\mathcal{F}}$ on A ($|A| \geq 3$) by forbidden patterns

$$\mathcal{F} = \{p_a\}_{a \in A}$$

  where p_a has support $\{1_G, g\}$ s.t. $(p_a)_{1_G} = (p_a)_g = a$.
- Lemma: $w =_G 1_G \Leftrightarrow X_{\mathcal{F}} = \emptyset$.

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Property

If G has undecidable **WP**, then G has undecidable **DP**.

DP and subgroups

Property (stability by subgroup)

If $H \leq G$ is f.g. and H has undecidable **DP**, then G has undecidable **DP**.

Sketch of the proof:

- A set F of forbidden patterns on H is seen as F' on G.
- $X_F \subset A^H \neq \emptyset \Leftrightarrow X_{F'} \subset A^G \neq \emptyset$.

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Corollary

If $\mathbb{Z}^2$ embeds into G, then G has undecidable **DP**.

Examples: $\mathbb{Z}^n$ for $n \geq 3$, discrete Heisenberg group have undecidable **DP**.

DP and quotient, subgroup of finite index

Proposition (stability by quotient)

If $H \trianglelefteq G$ is a f.g. normal subgroup and G/H has undecidable **DP**, then G has undecidable **DP**.

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Proposition

(Un)Decidability of **DP** is an invariant of commensurability.

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Virtually free groups

Proposition

Free groups have decidable **DP**.

Proof: Direct algorithm that solves **DP**.

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Polycyclic groups

A group G is **polycyclic** if there exists subgroups $(G_i)_{i=0\ldots n}$ s.t.

$$\{1\} = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_0 = G$$

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Examples:

Nice closure properties:

Proposition

Quotients and subgroups of polycyclic groups are polycyclic.

In particular, subgroups of polycyclic groups are always f.g. groups.

Polycyclic groups: Hirsch number

The **Hirsch number** $h(G)$ of a polycyclic group G is the number of infinite factors in a series with cyclic finite or finite factors.

Proposition

- If G_1 is a subgroup of G_2, then $h(G_1) \leq h(G_2)$.
- If H is a normal subgroup of G, then $h(G) = h(G/H) + h(H)$
- $h(G) = 0$ iff G is finite
- $h(G) = 1$ iff G is virtually $\mathbb{Z}$
- $h(G) = 2$ iff G is virtually $\mathbb{Z}^2$.

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Hirsch number $\Rightarrow$ proofs by induction on polycyclic groups.

Polycyclic groups and **DP**

Theorem (Jeandel, 2015)

Let G be a polycyclic group. Then G has undecidable **DP** iff G is not virtually cyclic (i.e. $h(g) \geq 2$).

Proof: By induction on the Hirsch number of the group.

- If $h(G) \in \{0, 1, 2\}$, OK.

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- Every polycyclic group admits a nontrivial normal torsion-free abelian subgroup (Hirsch, 1938). Take H such a subgroup.
- If $H = \mathbb{Z}^n$ for some $n > 2$, then H has undecidable **DP**, and G has undecidable **DP** (stability by subgroup).
- Otherwise $H = \mathbb{Z}$, and G/H is a polycyclic subgroup of Hirsch number $n \geq 2$. By induction hypothesis, G/H has undecidable **DP**. By stability by quotient, G has undecidable **DP**.

Why Baumslag-Solitar groups ?

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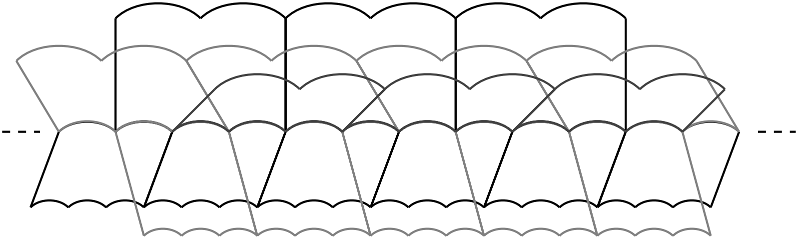

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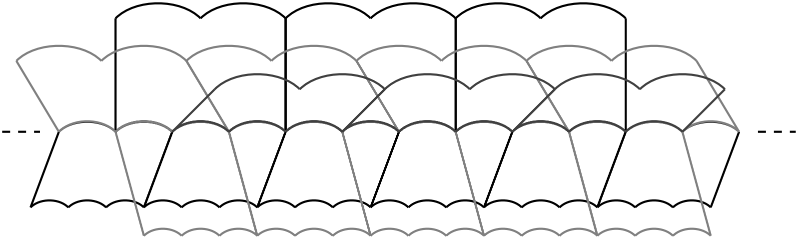

Baumslag-Solitar groups have decidable **WP**, are not virtually free, do not contain $\mathbb{Z}^2$ for $m = 1$ and $n \geq 2$.

Partial localization in BS(m, n)

Let $A = \{a, a^{-1}, b, b^{-1}\}$. Define $\psi_{m,n} : A^* \to \mathbb{R}$ by induction

$$\begin{cases} \psi_{m,n}(\varepsilon) = 0 \text{ where } \varepsilon \text{ is the empty word} \\ \psi_{m,n}(w.b) = \psi_{m,n}(w.b^{-1}) = \psi_{m,n}(w) \\ \psi_{m,n}(w.a) = \psi_{m,n}(w) + \left(\frac{m}{n}\right)^{\|w\|_b} \\ \psi_{m,n}(w.a^{-1}) = \psi_{m,n}(w) - \left(\frac{m}{n}\right)^{\|w\|_b} \end{cases}$$

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Partial localization in BS(m, n)

Define a function $\Phi_{m,n} : \mathrm{BS}(m,n) \to \mathbb{R}^2$ by

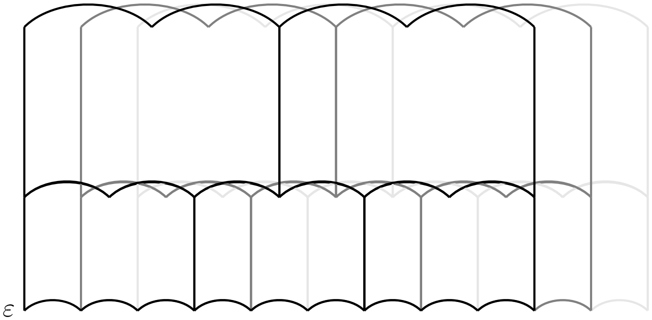
$$\Phi_{m,n}(g) = \left(\psi_{m,n}(w), \| \, w \, \|_{b^{-1}}\right),$$

where w is any writing of g.

Partial localization in BS(m, n)

Property

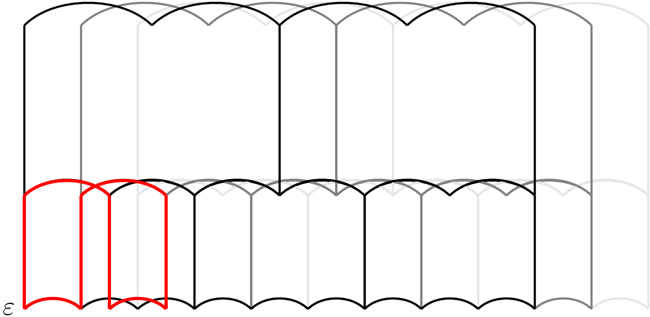
$\Phi_{m,n}$ is well-defined



Partial localization in BS(m, n)

Property

$\Phi_{m,n}$ is well-defined, but is not injective.



$$\Phi_{3,2}(\varepsilon) = \Phi_{3,2}(abab^{-1}a^{-1}ba^{-1}b^{-1})$$

DP on Baumslag-Solitar groups

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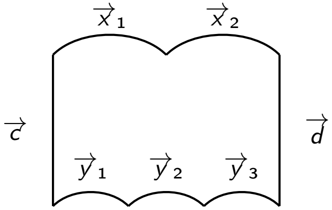

The tile **computes** the function f if the relation

$$\frac{f\left(\vec{x}_1 + \vec{x}_2\right)}{2} + \vec{c} = \frac{\vec{y}_1 + \vec{y}_2 + \vec{y}_3}{3} + \vec{d}$$

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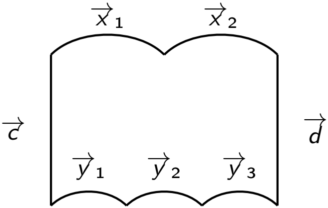

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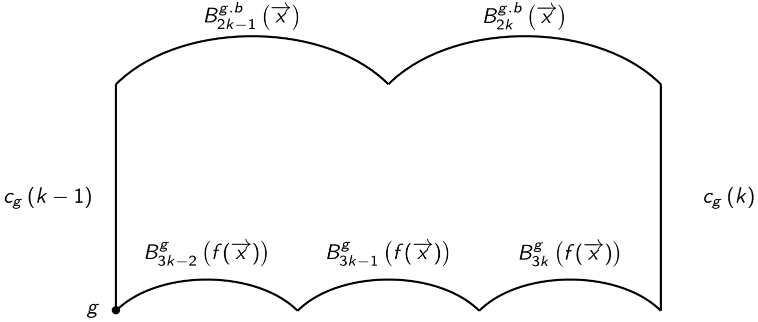
which leads to

$$f(\vec{x}) + \frac{\vec{c}_1}{k} = \vec{y} + \frac{\vec{d}_k}{k}$$

on a finite row.

DP on Baumslag-Solitar groups

Let $f(\overrightarrow{x}) = M\overrightarrow{x} + \overrightarrow{b}$, M and $\overrightarrow{b}$ with rational coefficient and integer corners.



with $c_g(k) = \frac{1}{2} f\left(\left\lfloor\left(\left(\frac{3}{2}\right)^{\beta-1}\alpha + 2k\right)\overrightarrow{x}\right\rfloor\right)$
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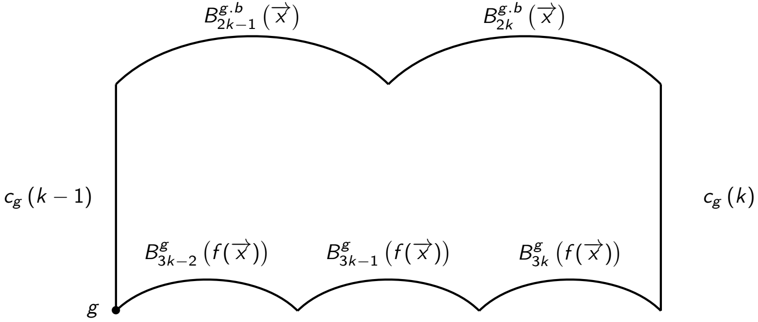

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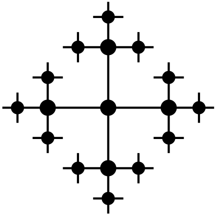
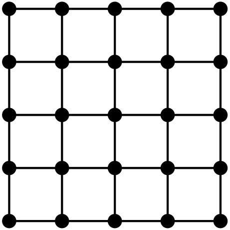
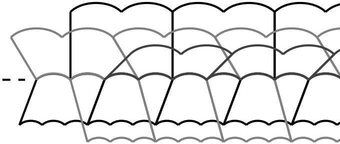
Theorem (A. & Kari, 2013)

The Domino problem is undecidable on Baumslag-Solitar groups.

Covering a group by disjoint bi-infinite paths







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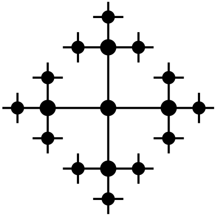

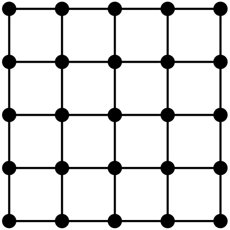

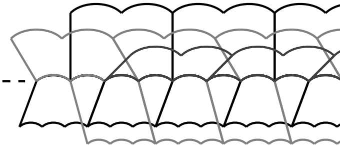

What about torsion groups ?

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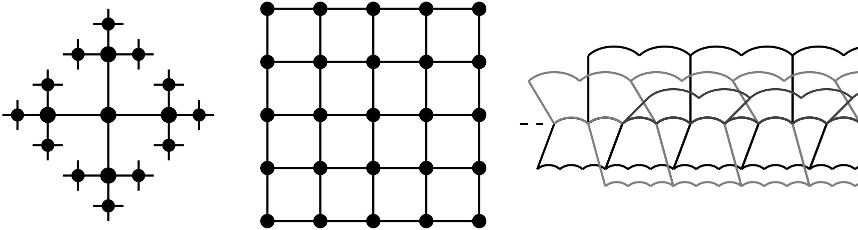

What about torsion groups ?

Theorem (Seward, 2015)

Let G be an infinite f.g. group. Then there exists a finite set S s.t. the Cayley graph $\Gamma(G, S)$ of G with generating set S can be covered by disjoint bi-infinite paths.

Seward's Theorem inside an SFT ?

Choose S as in the previous theorem. Assume S is symmetrical ($S^{-1} \subset S$).

Idea: each group element knowns the next and previous elements of its bi-infinite path.
Realization: SFT on the alphabet $S \times S$, given by

$$x \in (S \times S)^G \text{ is in } G \text{ iff}$$

$$\forall g \in G, \forall s \in S : \begin{array}{lcl} (x_g)_1 = s & \Rightarrow & (x_{gs})_2 = s^{-1} \\ (x_g)_2 = s & \Rightarrow & (x_{gs})_1 = s^{-1} \end{array}$$

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But. . . we cannot avoid cycles !!

- Configurations of X are partitions of $\Gamma(G, S)$ into cycles and bi-infinite paths.
- By Seward's result, there exist one configuration in X with no cycle.

Domino problem on $G_1 \times G_2$ groups

Theorem (Jeandel, 2015)

Let G_1 and G_2 be infinite f.g. groups. Then $G_1 \times G_2$ has undecidable **DP**.

Sketch of the proof:

- **Idea:** encode an SFT Y on $\mathbb{Z}^2$ inside an SFT Z on $G_1 \times G_2$.

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- Define $Z \subset (S_1 \times S_1 \times S_2 \times S_2 \times A)^{G_1 \times G_2}$ as follows

$$g \in Z \text{ iff } z \in X \times A^{G_1 \times G_2} \text{ and} \forall g \in G_1 \times G_2 : \begin{array}{l} \left((z_g)_5, (z_{(z_g)_1 g})_5\right) \notin F_H \\ \left((z_g)_5, (z_{(z_g)_3 g})_5\right) \notin F_V \end{array}$$

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Corollary

Grigorchuk group has undecidable **DP**.

Outline of the talk.

Conjecture (I)

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- **Why ?** Explicit algorithm for free groups + stability by subgroup of finite index.

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- **Why ?**
  - **DP** can be expressed in MSO logic (Wang, 1961)
  - a group is virtually free if and only if it has finite tree-width (Muller & Schupp, 1985)
  - graphs with finite tree-width are exactly those with decidable MSO (Kuske & Lohrey, 2005)

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Theorem (using Robertson & Seymour, 1986)

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- Remember Robinson's construction...
- Can we use these grids as computation zones for Turing machines ?
- But we do not know where this grids appear !
- And even if we knew, how to code them inside an SFT ?

Conclusion

- **DP** has good *structural* properties.
- Seems hard to adapt existing proofs on $\mathbb{Z}$ to the general case.
- Several characterizations of virtually free groups.

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Thank you for your attention !!

Domino Problem as a Markov property

A property of f.p. groups is a **Markov property** if

(i) there exists a f.p. group with this property,

(ii) there exists a f.p. group that cannot be embedded in any f.p. group with the property.

Examples: being trivial, abelian, nilpotent, solvable, free, torsion-free. . . are Markov properties.

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Proposition

The group property *G has decidable domino problem* is a Markov property.