A Fast Contour-Integral Eigensolver for Non-Hermitian Matrices and the Approximation Accuracy

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Overview

- Introduction
- Fast contour-integral eigenvalue solution for non-Hermitian matrices: filter function, eigenvalue count
- Eigenvalue accuracy subject to certain matrix approximations
- Superfast divide-and-conquer eigenvalue decomposition for structured Hermitian matrices
- Numerical tests

Eigenvalue solutions

$$Aq = \lambda q, \quad A : n \times n$$

- Hermitian and non-Hermitian eigenvalue solutions

  Power/inverse/subspace iterations, Jacobi's method, QR iterations, Rayleigh-ritz iterations, bisection and inverse iterations, divide-and-conquer, ...

- Major operations
  - Matrix transformations and reductions
  - Matrix factorizations
  - Matrix-matrix and matrix-vector multiplications
  - Linear system solutions

- Costs
  - Typically $O(n^3)$ flops
  - $O(n^2)$ or even $O(n)$ possible

Structured eigenvalue problems

- Selected examples
  - Companion matrix (polynomial roots) & related ($O(n^2)$ cost) [Benner, Bini, Chandrasekaran, Eidelman, Gemignani, Gohberg, Gu, Kailath, Mastronardi, Olshevsky, Pan, Van Barel, Van Dooren, Vandebril, Watkins, Xia, et al.]
  - Toeplitz ($\geq O(n^2)$ cost)
  - Rank-1 updated eigenproblem (since [Golub, 1973]), Hermitian tridiagonal (since [Cupen, 1981]), stablilization [Gu, Eisenstat]
    $O(n)$ (eigenvalues only), $O(n^2)$ (eigendecomposition, $O(n)$ possible)

- Our results
  - $O(n^2)$ cost contour-integral eigensolver for non-Hermitian rank structured dense/sparse matrices
  - Nearly $O(n)$ cost (superfast) eigendecomposition for Hermitian matrices with small off-diagonal ranks
  - $O(n^{1.5})$ and $O(n^2)$ cost for 2D and 3D discretized sparse Hermitian A, resp.
  - Eigenvalue accuracy after structured approximations

Contour-integral based eigensolvers

- SS method [Sakurai, Sugiura] (generalized eigenvalue problem to Hankel eigenvalue problem), CIRR method [Sakurai, Tadano] (stable version)
- Hermitian FEAST [Polizzi]
- Non-Hermitian FEAST [Kestyn, Polizzi, Tang], [Yin, Chan, Yeung]

$$\phi(z) = \frac{1}{2\pi\mathbf{i}} \int_\Gamma \frac{1}{\mu - z} \mathrm{d}\mu \qquad (z \notin \Gamma)$$

  - Projector for eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ inside Γ

$$\Phi = \frac{1}{2\pi\mathbf{i}} \int_\Gamma (\mu I - A)^{-1} \mathrm{d}\mu = Q \left(\frac{1}{2\pi\mathbf{i}} \int_\Gamma (\mu I - \Lambda)^{-1} \mathrm{d}\nu \right) Q^{-1} = Q \begin{pmatrix} I_s & \\ & 0 \end{pmatrix} Q^{-1}$$

  - Projected subspace iteration

$$\Phi Y \approx \tilde{\Phi} Y \equiv \frac{1}{2\pi\mathbf{i}} \Sigma_j w_j (\mu_j I - A)^{-1} Y \implies Q \implies Y$$

Remark: related – spectrum slicing with polynomial and rational filters [Saad, et al.]

Filter function and quadrature rules

With $\Gamma \equiv C_1(0)$ (unit circle),

$$\phi(z) = \frac{1}{2}\int_{-1}^{1} \frac{e^{i\pi t}}{e^{i\pi t} - z} \mathrm{d}t \approx$$

$$\tilde{\phi}(z) = \frac{1}{2}\sum_{j=1}^{q} w_j \frac{e^{\mathbf{i}\pi t_j}}{e^{\mathbf{i}\pi t_j} - z} \equiv \frac{1}{2}\sum_{j=1}^{q} \frac{w_j z_j}{z_j - z} \equiv \frac{f(z)}{g(z)}$$

Theorem. With any interpolatory quadrature rule, for $z \in \mathbb{C}$,

- $\tilde{\phi}(0) = 1$;
- $|\tilde{\phi}(z)| > \frac{1}{2}$ when $|z| < 1$;
- $|\tilde{\phi}(z)| < \frac{1}{\delta}$ when $|z_j - z| > \delta > 0,\ j = 1, 2, \ldots, q$.

Proof. When $|z| < 1$, by $|z_j\overline{z} + \overline{z_j}z| \le 2|z| < 1 + |z|^2 < 2$,

$$\mathrm{Re}(\tilde{\phi}(z)) = \frac{1}{4}\sum_{j=1}^{q}\left(\frac{w_j z_j}{z_j - z} + \frac{w_j \overline{z_j}}{\overline{z_j} - \overline{z}}\right) = \frac{1}{4}\sum_{j=1}^{q} w_j \frac{2 - (z_j\overline{z} + \overline{z_j}z)}{1 + |z|^2 - (z_j\overline{z} + \overline{z_j}z)} > \frac{1}{2}$$

Filter function with optimal decay property

With $\Gamma \equiv C_1(0)$ (unit circle),

$$\phi(z) = \frac{1}{2}\int_{-1}^{1} \frac{e^{\mathbf{i}\pi t}}{e^{\mathbf{i}\pi t} - z} \mathrm{d}t \approx$$

$$\tilde{\phi}(z) = \frac{1}{2}\sum_{j=1}^{q} w_j \frac{e^{\mathbf{i}\pi t_j}}{e^{\mathbf{i}\pi t_j} - z} \equiv \frac{1}{2}\sum_{j=1}^{q} \frac{w_j z_j}{z_j - z} \equiv \frac{f(z)}{g(z)}$$

On the decay property away from Γ, noticing $\deg f \geq 0$:

Theorem. $f(z)$ in $\tilde{\phi}(z) = \frac{f(z)}{g(z)} = \frac{f(z)}{\prod_{j=1}^{q}(z-z_j)}$ satisfies

$$\begin{cases} \deg(f) = 0 \text{ (in fact, } f = (-1)^{q+1}), & \text{with Trapezoidal rule} \\ \deg(f) \geq 1, & \text{with Gauss-Legendre rule} \end{cases}$$

Remark. Optimization/least squares strategies for filter function design [Van Barel], [Xi, Saad]; numerical observations for the Trapezoidal filter [Tang, et al.]

Filter function with optimal decay property

Sketch of Proof. Trapezoidal rule:

$$g(z) = z^q - (-1)^q$$

$$f(z) = -\frac{1}{2}\sum_{j=1}^{q} w_j z_j \prod_{i\neq j}(z - z_i) \equiv \sum_{j=1}^{q} C_{q-k} z^{q-k}$$

$$C_{q-k} = \frac{k}{q}\left[(-1)^k \sum_{1\leq i_1<i_2<\cdots<i_k\leq q} z_{i_1} z_{i_2}\cdots z_{i_k}\right] = \begin{cases} 0\,, & 1\leq k\leq q-1 \\ (-1)^{q+1}, & k=q \end{cases}$$

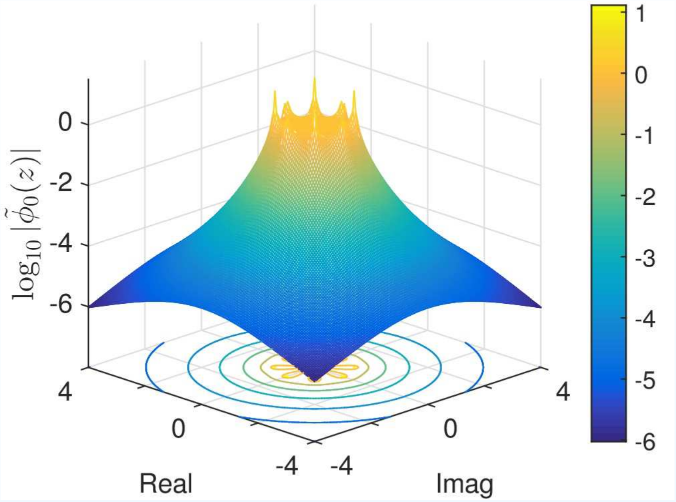
Gauss-Legendre rule, by contradiction:

$$\mathcal{S}_k = \{(i_1, i_2, \ldots, i_k) : 1\leq i_1 < i_2 < \cdots < i_k \leq q\}$$

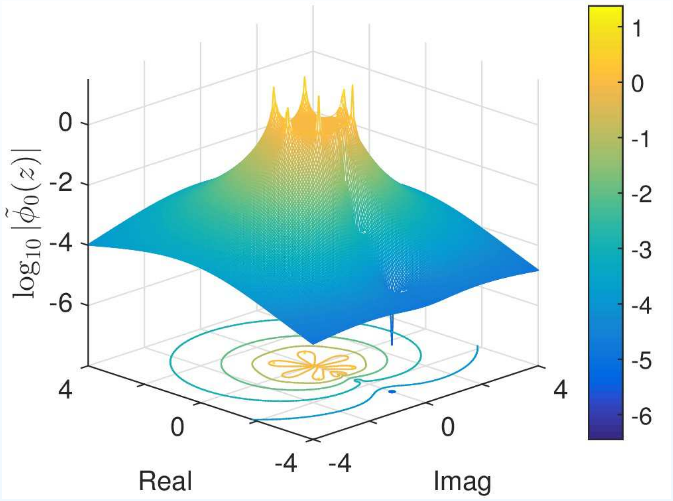
$$C_k = \left((-1)^{q-k} \sum_{(i_1,i_2,\ldots,i_k)\in\mathcal{S}_k} z_{i_1} z_{i_2}\cdots z_{i_k}\right) - (-1)^{q-2k} C_{q-k}$$

$$\prod_{j=1}^{q} z_j = 1, \quad \sum_{(i_1,,\ldots,i_k)\in\mathcal{S}_k} z_{i_1}\cdots z_{i_k} = 0, \quad 1\leq k\leq q-1 \implies z_j^q + (-1)^q = 0$$

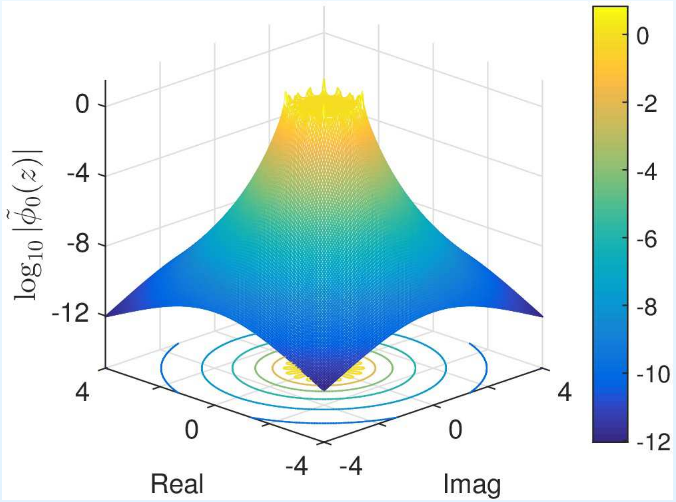
Filter function with optimal decay property



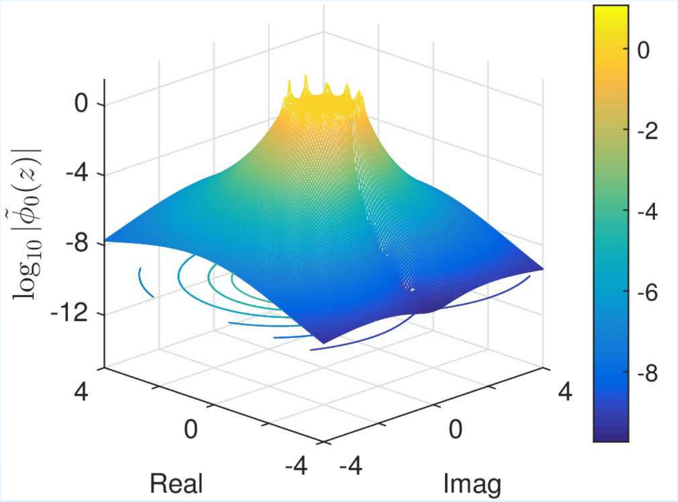
8-point Trapezoidal



8-point Gauss-Legendre



16-point Trapezoidal



16-point Gauss-Legendre

Contour-integral eigenvalue solution with subspace iteration

$$\Phi \equiv \phi(A) = \frac{1}{2\pi\mathbf{i}} \int_\Gamma (\mu I - A)^{-1} \mathrm{d}\mu$$

- Subspace iteration with projection (SS, FEAST)
  - Decide eigenvalue count $k = \#_\Lambda(A, \Gamma)$ inside a region enclosed by Γ
  - Y: random $n \times k$
  - Repeat until convergence
    - $\Phi Y \approx \sum_{j=1}^{q} c_j (z_j I - A)^{-1} Y$ orthonormalized
    - Project and solve reduced eigenvalue problem
    - $Y \leftarrow$ recovered approximate eigenvectors
- Inexact eigenvalue count
  - Oversampling is suggested [Polizzi, Tang, et al.]
  - Randomized trace estimation $\frac{1}{m}\,\mathrm{trace}(\tilde{Y}^T \Phi \tilde{Y})$ [Hutchinson]
- Linear solutions for multiple shifts and multiple right-hand sides

Eigenvalue count after low-accuracy matrix approximation

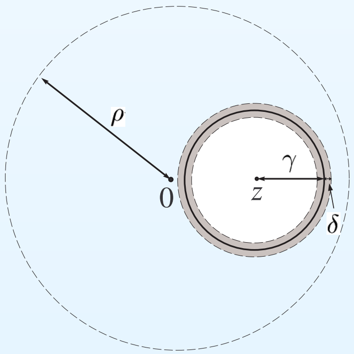
Lemma. A: Hermitian. $\tilde{A} \approx A$ with accuracy δ. If s (shift) $\in [\lambda_{i+1}, \lambda_i]$, $\lambda_i - \lambda_{i+1} > 2\delta$, then

$$|n_-(A - sI) - n_-(\tilde{A} - sI)| = 0 \text{ or } 1$$

and it may be 1 only if $|\lambda_i - s| \leq \delta$ or $|\lambda_{i+1} - s| \leq \delta$

Theorem. A: Non-Hermitian. $\tilde{A} \approx A$ with accuracy τ, $\delta \equiv \max_i \kappa(\lambda_i)\tau < \rho$, $|\lambda_i(A)| < \rho$. For any $0 < \gamma < \rho$ and $z \in \mathbb{C}$, let

$$\mathcal{A}_{\gamma,\delta}(z) = \{\omega \in \mathbb{C} : \gamma - \delta < |\omega - z| < \gamma + \delta\}$$



1. If A has no eigenvalue inside $\mathcal{A}_{\gamma,\delta}(z)$, then

$$\#_\Lambda(A, \mathcal{C}_\gamma(z)) = \#_\Lambda(\tilde{A}, \mathcal{C}_\gamma(z))$$

2. If $|\#_\Lambda(A, \mathcal{C}_\gamma(z)) - \#_\Lambda(\tilde{A}, \mathcal{C}_\gamma(z))| \geq \alpha$ for an integer $\alpha > 0$, then there must be at least α eigenvalues of A inside $\mathcal{A}_{\gamma,\delta}(z)$

Probability estimates and approximation accuracy

$$\mathcal{A}_{\gamma,\delta}(z) = \{\omega \in \mathbb{C} : \gamma - \delta < |\omega - z| < \gamma + \delta\}$$

Lemma. (Randomly distributed eigenvalues) Let

$$\mathcal{D}_\rho(0) = \{z : |z| < \rho\}$$

Suppose the eigenvalues λ of A are uniformly i.i.d. in $\mathcal{D}_\rho(0)$. Then for any fixed $z \in \mathbb{C}$ and $\gamma, \delta \in (0, \rho)$, the probability for any λ to lie inside $\mathcal{A}_{\gamma,\delta}(z)$ is

$$\Pr\{\lambda \in \mathcal{A}_{\gamma,\delta}(z)\} \le \mathcal{P} \equiv \frac{4\delta \max(\gamma, \delta)}{\rho^2}$$

Lemma. (Randomly placed circles) Suppose λ is a fixed point in the complex plane, z is uniformly i.i.d. in $\mathcal{D}_\rho(0)$, γ is random and uniformly distributed on $(0, \rho)$, and z and γ are independent. Then for any $\delta \in (0, \rho)$,

$$\Pr\{\lambda \in \mathcal{A}_{\gamma,\delta}(z)\} < 2\frac{\delta}{\rho} + \frac{1}{3}\left(\frac{\delta}{\rho}\right)^3$$

Probability of miscounting eigenvalues with low-accuray matrix approximation

Theorem. Let
$\tilde{A}$: approximation with accuracy bounded by $\delta < \rho$
λ: uniformly i.i.d. in $\mathcal{D}_\rho(0)$; $\mathcal{P} \equiv \dfrac{4\delta \max(\gamma, \delta)}{\rho^2}$
Then for any integer $\alpha \geq n\mathcal{P}$ and fixed $z \in \mathbb{C}$ and $\gamma \in (0, \rho)$,

$$\begin{aligned}&\Pr\{|\#_\Lambda(A, \mathcal{C}_\gamma(z)) - \#_\Lambda(\tilde{A}, \mathcal{C}_\gamma(z))| \geq \alpha\} \\ &\leq \frac{(\alpha+1)}{\alpha+1-(n+1)\mathcal{P}} \binom{n}{\alpha} \mathcal{P}^\alpha (1-\mathcal{P})^{n-\alpha+1}\end{aligned}$$

γ	δ	Bound for $\Pr\{\lvert\#_\Lambda(A, \mathcal{C}_\gamma(z)) - \#_\Lambda(\tilde{A}, \mathcal{C}_\gamma(z))\rvert \geq \alpha\}$				
		$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
	$1e-1$	$3.99e-3$	$7.97e-6$	$1.06e-8$	$1.06e-11$	$8.45e-15$
100	$1e-2$	$4.00e-4$	$7.99e-8$	$1.06e-11$	$1.06e-15$	$8.48e-20$
	$1e-3$	$4.00e-5$	$7.99e-10$	$1.06e-14$	$1.06e-19$	$8.48e-25$

A Cauchy-like example with $n = 1600,\ \rho = 4000$

Fast eigenvalue decomposition for structured matrices

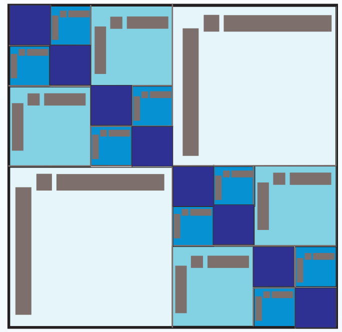
- Rank structured (HSS) matrices
  - Approximation accuracy conveniently controlled by off-diagonal compression
  - Shifted ULV factorization update (saving: $40\% \sim 60\%$)

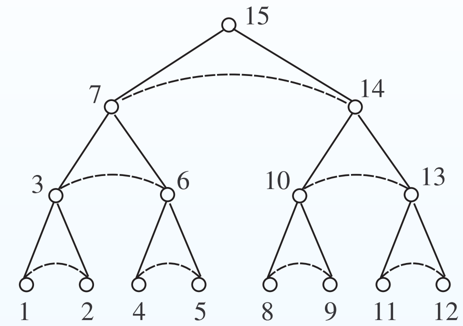
$$A = ULV^* \implies \mu I - A = \tilde{U}\tilde{L}\tilde{V}^*$$

- Quadsection/eigenvalue count stage
  - Find $\tilde{A} \approx A$ with low-accuracy and pre-factorize $\tilde{A}$
  - Apply Trapezoidal rule $\tilde{Z} = \frac{1}{2\pi \mathrm{i}} \sum_j w_j (\mu_j I - \tilde{A})^{-1} Y$ (Y: skinny and random)
  - If $\#_\Lambda(A, \Gamma) \approx \frac{1}{m} \mathrm{trace}(Y^T \tilde{Z}) \ll K$ (threshold), quadsect and repeat; otherwise, mark as target subregion
- Eigenvalue solution stage via projected subspace iterations
  - Shifted factorization update
  - Deflation/locking

Proposition. For structured matrices with off-diag rank r, optimal threshold: $K = O(r)$, complexity: $O(rn^2)$

Hermitian eigensolver: superfast HSS divide-and-conquer





$$D_{\mathbf{i}} = \begin{pmatrix} D_{\mathbf{c}_1} & \\ & D_{\mathbf{c}_2} \end{pmatrix} + \begin{pmatrix} U_{\mathbf{c}_1} & \\ & U_{\mathbf{c}_2} \end{pmatrix} \begin{pmatrix} & B_{\mathbf{c}_1} \\ B_{\mathbf{c}_1}^T & \end{pmatrix} \begin{pmatrix} U_{\mathbf{c}_1}^T & \\ & U_{\mathbf{c}_2}^T \end{pmatrix}$$

— rank-$2r$ update

$$= \begin{pmatrix} D_{\mathbf{c}_1} - U_{\mathbf{c}_1} U_{\mathbf{c}_1}^T & \\ & D_{\mathbf{c}_2} - U_{\mathbf{c}_2} B_{\mathbf{c}_1}^T B_{\mathbf{c}_1} U_{\mathbf{c}_2}^T \end{pmatrix} + \begin{pmatrix} U_{\mathbf{c}_1} \\ U_{\mathbf{c}_2} B_{\mathbf{c}_1}^T \end{pmatrix} \begin{pmatrix} U_{\mathbf{c}_1} & U_{\mathbf{c}_2} B_{\mathbf{c}_1}^T \end{pmatrix}$$

$$\equiv \begin{pmatrix} \tilde{D}_{\mathbf{c}_1} & \\ & \tilde{D}_{\mathbf{c}_2} \end{pmatrix} + Z Z^T$$

— rank-r update

HSS form of $\tilde{D}_{\mathbf{c}_i}$: same off-diagonal basis (structure fully preserved)

Rank structured (HSS) divide-and-conquer – conquering

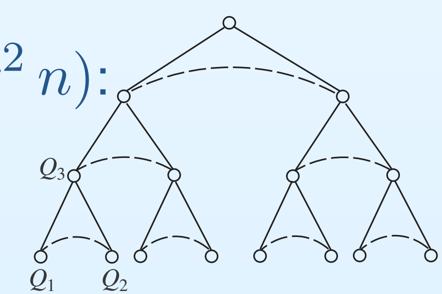
$$
\begin{aligned}
&\operatorname{diag}(\tilde{D}_{\mathbf{c}_1}, \tilde{D}_{\mathbf{c}_2}) + ZZ^T \\
= &\begin{pmatrix} \tilde{Q}_{\mathbf{c}_1} & \\ & \tilde{Q}_{\mathbf{c}_2} \end{pmatrix} (Q^{(1)} \cdots Q^{(i-1)}) \left(\begin{pmatrix} \tilde{\Lambda}_{\mathbf{c}_1}^{(i-1)} & \\ & \tilde{\Lambda}_{\mathbf{c}_2}^{(i-1)} \end{pmatrix} + \sum_{j=1}^{i} v^{(j)} (v^{(j)})^T \right) \\
&\cdot (Q^{(1)} \cdots Q^{(i-1)})^T \begin{pmatrix} \tilde{Q}_{\mathbf{c}_1}^T & \\ & \tilde{Q}_{\mathbf{c}_2}^T \end{pmatrix} = Q_{\mathbf{i}} \Lambda_{\mathbf{i}} Q_{\mathbf{i}}^T
\end{aligned}
$$

Fast and stable eigenvalue/eigenvector computations

- Modified Newton's method with the Middle Way [R.-C. Li], and fast multipole method (FMM) acceleration with kernel $\phi(x) = \frac{1}{x}$ or $\frac{1}{x^2}$ [Gu, Eisenstat]
- Stable computation of eigenvectors with Löwner's formula, and FMM acceleration with kernel $\phi(x) = \log x$ [Gu, Eisenstat]

Theorem. The off-diagonal numerical rank is bounded by $O(r \log^2 n)$:

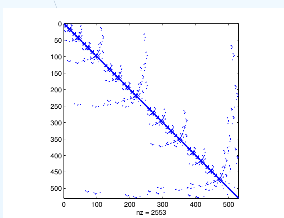
$$Q\text{:}\ \{Q_{\mathbf{1}}, \ldots, Q_{\mathbf{k}}\}, \quad Q_{\mathbf{i}} = Q_{\mathbf{i}}^{(1)} \cdots Q_{\mathbf{i}}^{(r)}$$



Complexity and generalizations

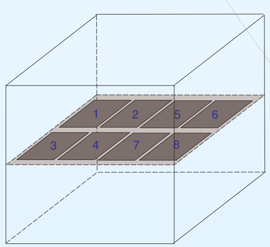
	General symmetric HSS	Banded (finite bw)	Toeplitz
Eigendecomposition	$O(r^2 n \log n) + O(rn \log^2 n)$	$O(n \log^2 n)$	$O(n \log^3 n)$
Eigenmatrix-vec	$O(rn \log n)$	$O(n \log n)$	$O(n \log^2 n)$
Storage	$O(rn \log n)$	$O(n \log n)$	$O(n \log^2 n)$

- *General* Hermitian discretized matrices *(attractive for shifted factorization)*
  - Sparse discretized matrices



	Eigendecomposition	Eigenmatrix-vec	Storage
2D	$O(n^{3/2} \log^2 n)$	$O(n^{3/2} \log n)$	$O(n^{3/2} \log n)$
3D	$O(n^{5/3} \log^2 n)$	$O(n^{5/3} \log n)$	$O(n^{5/3} \log n)$

  - Dense discretized matrices



	Eigendecomposition	Eigenmatrix-vec	Storage
2D	$O(n^2 \log n)$	$O(n^{3/2} \log n)$	$O(n^{3/2} \log n)$
3D	$O(n^{7/3} \log n)$	$O(n^{5/3} \log n)$	$O(n^{5/3} \log n)$

- Superfast SVD for non-Hermitian HSS, solution of separable PDEs

Eigenvalue accuracy after rank structured approximations

Theorem. l-level HSS approximation $\tilde{A} \approx A$ via off-diagonal truncation at each level:

$$A_{ij} = U_i B_i V_j^T + E \approx U_i B_i V_j^T, \quad ||E||_2 \le \tau$$

Then

$$||A - \tilde{A}\,\text{(HSS)}||_2 \le l\tau \text{ (attainable)}$$

$$|\lambda_i - \tilde{\lambda}_i| \le \begin{cases} l\tau, & A\text{: Hermitian} \\ \kappa(\lambda_i) l\tau + O((l\tau)^2), & \text{otherwise} \end{cases}$$

Proof. Direct summation. To attain the error bound

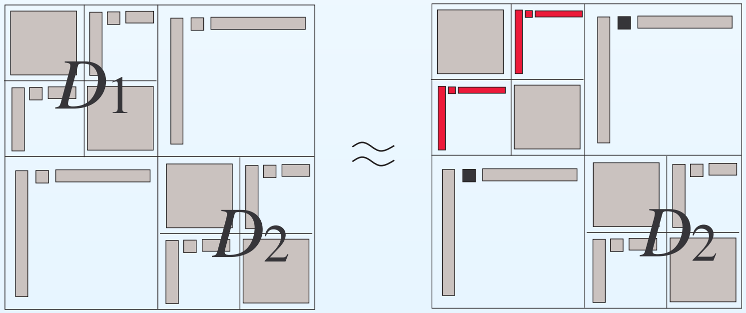
$$E^{(l)} \equiv \sum_{\tilde{l}=1}^{l} \operatorname{diag}\left(\begin{pmatrix} 0 & \tau I \\ \tau I & 0 \end{pmatrix}, \quad \mathbf{i}\text{: all nodes at level } \tilde{l}\right)$$

$$\lambda(E^{(l)}) = \begin{cases} \pm\tau, \pm 3\tau, \ldots, \pm l\tau, & \text{if } l \text{ is odd} \\ 0, \pm 2\tau, \ldots, \pm l\tau, & \text{otherwise} \end{cases}$$

Remark. With hierarchial off-diagonal compression, the matrix approximation error bounds become $O(\tau\sqrt{rn}\log n)$ [Xi, Xia, et al.]

Bounds for selected eigenvalues

- Perturbation along certain eigenspace for well-separated eigenvalues
  Corollary. A: Hermitian. l-level HSS approximation $\tilde{A} \approx A$. If $|\lambda_i - \lambda_{i\pm1}| > 2l\tau$, then
  $$|\lambda_i - \tilde{\lambda}_i| \leq ||Eq_i||_2$$
  (Based on [Ipsen]. Related results in [R.-C. Li and C.-K. Li])
- Error isolation effect (based on [Paige])
  A: Hermitian. If D_1 and D_2 have disjoint spectra, then to approximate eigenvalues originating from D_2, low-accuracy structured approximations can be applied to D_1



- Perturbation in the direction of certain eigenvectors
  Theorem. [Ding, Zhou] A: non-Hermitian. If $Aq_1 = \lambda_1 q_1$, then
  $$\lambda_i(A + q_1 v^T) = \lambda_1 + v^T q_1, \lambda_2, \ldots, \lambda_n$$

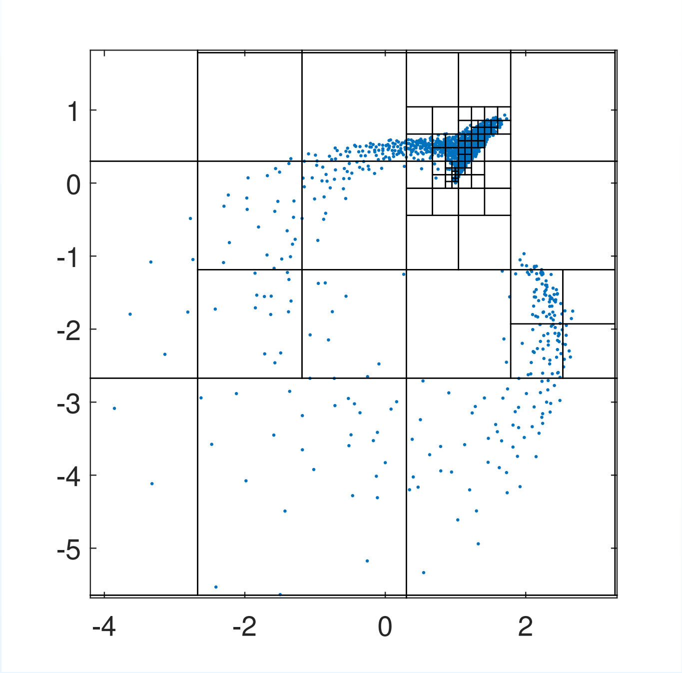
Eigenvalue count with low-accuracy structured approx.

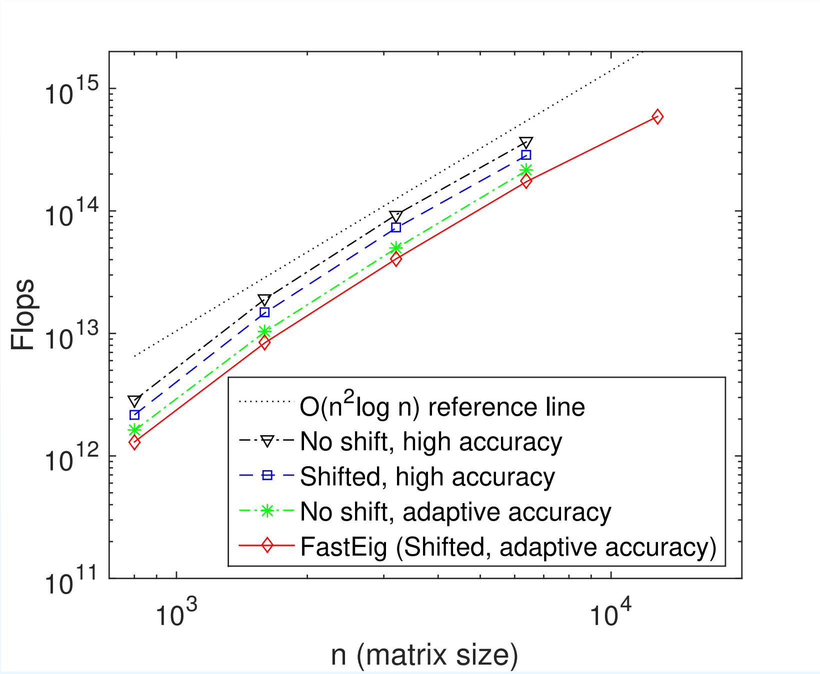
z	γ	$\#_\Lambda(A, \mathcal{C}_\gamma(z))$	$\lvert\#_\Lambda(A, \mathcal{C}_\gamma(z)) - \#_\Lambda(\tilde{A}, \mathcal{C}_\gamma(z))\rvert$				
			$\tau = 10^{-1}$	10^{-2}	10^{-3}	10^{-4}	10^{-5}
			$r = 4$	7	9	11	14
$976.8517 - 596.6716\mathbf{i}$	109.5545	2	0	0	0	0	0
$122.4701 + 395.7090\mathbf{i}$	221.7331	42	1	0	0	0	0
$-250.9437 + 91.2499\mathbf{i}$	395.2032	147	1	0	0	0	0
$-1029.6903 - 1599.1273\mathbf{i}$	986.0082	127	1	1	0	0	0
$1646.1010 + 2850.7448\mathbf{i}$	1315.6815	10	0	0	0	0	0
$-493.2565 + 1022.0571\mathbf{i}$	1526.3885	865	0	0	0	0	0
$115.6055 - 2472.7009\mathbf{i}$	2063.6158	400	2	0	0	0	0
$-1014.5968 + 1995.9028\mathbf{i}$	3004.7346	1220	1	0	0	0	0
$660.5523 + 507.5861\mathbf{i}$	3954.0531	1596	0	0	0	0	0

A: a 1600×1600 Cauchy-like matrix with $\rho \approx 4000$

A non-Hermitian discretized matrix from Foldy-Lax formulation for studying scattering effects

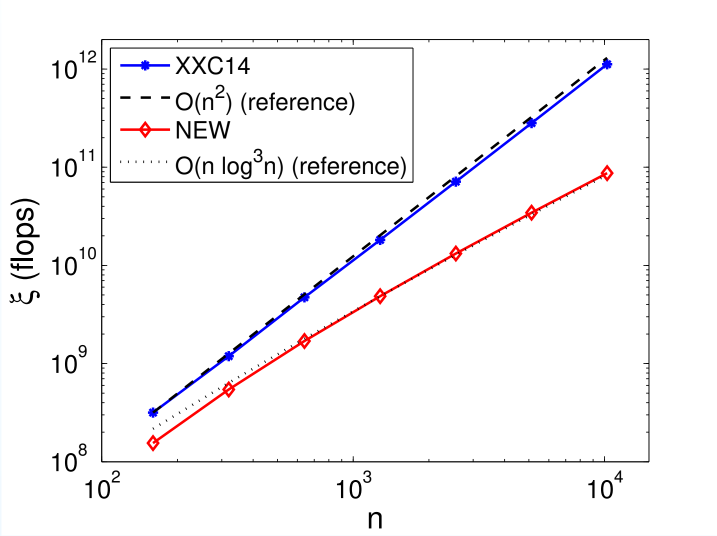
$$A_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -G(s_i, t_j)\sigma_j, & \text{otherwise,} \end{cases} \quad G(x,y) = \frac{e^{4\pi \mathbf{i}|s-t|}}{4\pi|s-t|} \ (s \neq t), \ \sigma_j\text{: random}$$



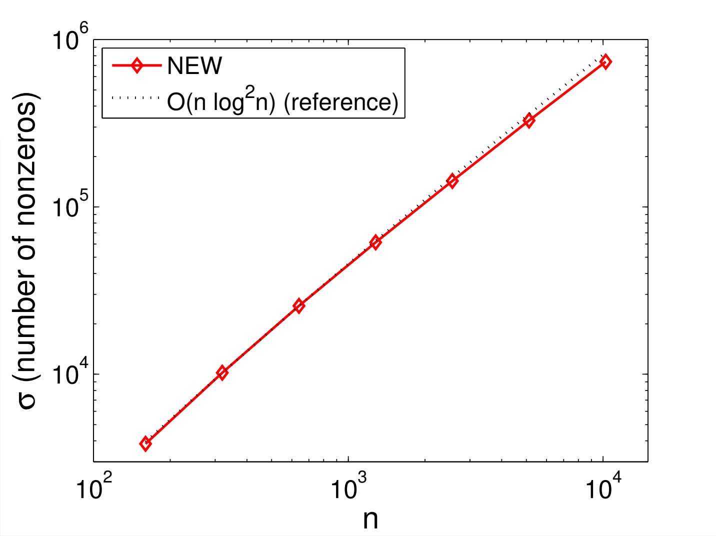


n (matrix size)	800	1,600	3,200	6,400	12,800
$\max(\mathbf{e}_i)$	$9.33e-7$	$3.08e-6$	$3.00e-5$	$1.04e-5$	$1.55e-6$
$\mathrm{mean}(\mathbf{e}_i)$	$1.76e-9$	$8.53e-10$	$1.52e-9$	$1.27e-9$	$7.07e-10$
$\max(\mathbf{r}_i)$	$8.24e-6$	$3.88e-6$	$4.45e-5$	$6.65e-6$	$1.88e-6$
$\mathrm{mean}(\mathbf{r}_i)$	$8.55e-9$	$6.39e-9$	$1.37e-8$	$1.60e-8$	$1.28e-8$

Hermitian Toeplitz eigenvalue solution



Eigenvalue solution cost ξ



Structured eigenmatrix storage σ

	n	160	320	640	1280	2560
XXC14	e	$2.40e-10$	$1.02e-10$	$5.80e-11$	$4.39e-11$	$3.84e-11$
NEW ($\tau \approx 10^{-10}$)	e	$1.00e-09$	$1.07e-10$	$1.47e-10$	$9.32e-11$	$8.45e-11$
	γ	$3.49e-09$	$1.49e-09$	$7.38e-10$	$2.53e-10$	$9.99e-11$
	θ	$1.79e-16$	$3.69e-16$	$7.94e-16$	$6.56e-16$	$8.53e-16$
NEW ($\tau \approx 10^{-15}$)	e	$9.64e-16$	$1.01e-15$	$1.27e-15$	$1.07e-15$	$1.31e-15$
	γ	$4.14e-15$	$4.40e-15$	$6.69e-15$	$7.62e-15$	$6.26e-15$
	θ	$4.25e-16$	$5.33e-16$	$7.24e-16$	$9.37e-16$	$7.18e-16$

Conclusions

Summary

- $O(n^2)$ complexity for non-Hermitian structured problems
- $\approx O(n)$ complexity eigendecomposition for Hermitian HSS matrices
- Accuracy analysis, flexible accuracy control, generalizations to PDE sol and SVD

Ongoing research directions

- Additional accuracy analysis for well-separated eigenvalues
- Eigenspace accuracy and preconditioning
- Multi-rank update Hermitian eigenvalue solution

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