A Framework for Structured Linearizations of Matrix Polynomials in Various Bases

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Luminy, October, 2016

Outline

Introduction

Main Theorem

Other Bases

Product Bases: Beyond Two

Sums of Polynomials and Rational Functions

Preserving Even, Odd, and Palindromic Structures

Conclusions

Problem Statement

- Given a matrix polynomial (P_i's square matrices)

$$P(\lambda) = \sum_{i=0}^{n} P_i \lambda^i.$$

- We want to compute its eigenvalues, i.e., the λ for which

$$\det(P(\lambda)) = 0.$$

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- See, e.g., Mackey, Mehl, Mehrmann, Higham, Delvaux, Fassbender, De Terán, Tisseur, Noferini, Bini, Dopico, Pérez, Hook, Pestana, Van Barel, Eidelman, Lawrence, Gemignani, Boito, Frederix, Van Dooren, ...

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- In this talk we construct linear matrix polynomials with the same eigenvalues.
- For simplicity we omit strongness (results in the paper).
- We will freely swap between (scalar) polynomials and matrix polynomials.

Fiedler-like Linearizations

Inspiration

- We were inspired by
- F. Dopico, J. Pérez, P. Lawrence, and P. Van Dooren, *Block Kronecker Linearizations of Matrix Polynomials and their Backward Errors*, 2016.




Monomial Basis

Companion Matrix

- Given $p(\lambda) = p_d\lambda^d + p_{d-1}\lambda^{d-1} + \cdots + p_1\lambda + p_0$.
- Matrix polynomial case similar.
- The classical Frobenius linearization (Companion) looks like

$$\lambda B - A = \begin{bmatrix} \lambda p_d + p_{d-1} & p_{d-2} & \cdots & p_0 \\ -1 & \lambda & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \end{bmatrix}.$$

- M. Fiedler, *A note on Companion matrices*, provided a factorization of $\lambda B - A$ leading to many other linearizations.

Monomial Basis

Extension of the Fiedler-like Linearizations

- F. Dopico, J. Pérez, P. Lawrence, and P. Van Dooren generalized this.
- For instance for

$$p(\lambda) = p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0$$

also the following is possible

$$\left[\begin{array}{ccc|c} \lambda p_4 + p_3 & p_2/2 & p_1/2 & -1 \\ p_2/2 & p_1/2 & p_0 & \lambda \\ \hline -1 & \lambda & 0 & 0 \\ 0 & -1 & \lambda & 0 \end{array}\right].$$

- The coefficients are distributed in a Hankel form.
- Coefficients can be interpreted as “convolution” coefficients.

Outline

So Leonardo and I sat together

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and asked ourselves the following...

- Can we do it for a basis different than the monomials?
- Can we do it for different basis left and right?

Preliminaries: Product Families

Definition (Product Family)

- Two families of polynomials (basis):
  - $\{\phi_i\}$ for $0 \leq i \leq \epsilon$,
  - $\{\psi_j\}$ for $0 \leq j \leq \eta$.
- Let

$$\phi \otimes \psi := \{\phi_i(\lambda)\psi_j(\lambda),\ i = 0, \ldots, \epsilon, \quad j = 0, \ldots, \eta\},$$

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Definition (Vector of Basis)

- We often identify the basis ϕ with $\pi_{k,\phi}(\lambda) := \begin{bmatrix} \phi_k(\lambda) \\ \vdots \\ \phi_0(\lambda) \end{bmatrix}$.

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Definition (Vector of Basis)

- We often identify the basis ϕ with $\pi_{k,\phi}(\lambda) := \begin{bmatrix} \phi_k(\lambda) \\ \vdots \\ \phi_0(\lambda) \end{bmatrix}$.
- Note that $\phi \otimes \psi$ identifies with $\pi_{\epsilon,\phi}(\lambda) \otimes \pi_{\eta,\psi}(\lambda)$.

Preliminaries: Dual Basis

Definition (Polynomial Basis)

- Matrix polynomial
- is a polynomial basis
- if its rows span the polynomial n-tuples.

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Definition (Dual basis)

- Polynomial bases $G(\lambda)$ and $H(\lambda)$ are
- dual if

$$G(\lambda)H(\lambda)^T = 0$$

.

Preliminaries: Dual Linear Basis of Full Row-Rank

Definition (Full row-rank linear dual basis)

- A matrix polynomial $L_{k,\phi}(\lambda)$ is a
- full row-rank linear dual basis to $\pi_{k,\phi}(\lambda)$
- if
  - duality $L_{k,\phi}(\lambda)\pi_{k,\phi}(\lambda) = 0$, holds,
  - $L_{k,\phi}(\lambda)$ is linear,
  - and $L_{k,\phi}(\lambda)$ has full row rank.
- We will say dual basis for short.
- An alternative (often correct) viewpoint: matrix of recurrences.

Main Theorem in the Scalar Setting

Theorem (Linearizations for Various Bases: Scalar Case)

- *Under mild assumptions (no common divisor in the basis family).*
- *Let* $L_{k,\phi}(\lambda), L_{k,\psi}(\lambda)$ *be dual bases for* $\{\phi_i\}$ *and* $\{\psi_i\}$.
- *Then*

$$\mathcal{L}(\lambda) := \begin{bmatrix} \lambda M_1 + M_0 & L_{\eta,\phi}(\lambda)^T \\ L_{\epsilon,\psi}(\lambda) & 0 \end{bmatrix}$$

is a linearization for $P(\lambda) = \pi_{\eta,\phi}(\lambda)^T(\lambda M_1 + M_0)\pi_{\epsilon,\psi}(\lambda)$.

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More detail

- Written out tells how to distribute the coefficients:

$$P(\lambda) = [\phi_\eta, \ldots, \phi_1, \phi_0] \ (\lambda M_1 + M_0) \ \begin{bmatrix} \psi_\epsilon \\ \vdots \\ \psi_1 \\ \psi_0 \end{bmatrix}.$$

Main Theorem

Theorem (Linearizations for Various Bases: Matrix Case)

- *Under mild assumptions (no common divisor in a family).*
- *Let $L_{k,\phi}(\lambda), L_{k,\psi}(\lambda)$ be dual linear bases for $\{\phi_i\}$ and $\{\psi_i\}$.*
- *Then*

$$\mathcal{L}(\lambda) := \begin{bmatrix} \lambda M_1 + M_0 & L_{\eta,\phi}(\lambda)^T \otimes I \\ L_{\epsilon,\psi}(\lambda) \otimes I & 0 \end{bmatrix}$$

is a linearization for $P(\lambda) = (\pi_{\eta,\phi}(\lambda) \otimes I)^T (\lambda M_1 + M_0)(\pi_{\epsilon,\psi}(\lambda) \otimes I)$.

Notes

- Now of course M_1 and M_0 are block matrices.
- The block values of $\lambda M_1 + M_0$ are multiplied with the basis elements

$$\pi_{\eta,\phi}(\lambda)^T \, \pi_{\epsilon,\psi}(\lambda),$$

Outline

Orthogonal Bases

Lemma

- *Let* $\{\phi_i\}$ *be degree graded satisfying a three-terms recurrence*

$$\alpha\phi_{j+1}(\lambda) = (\lambda - \beta)\phi_j(\lambda) - \gamma\phi_{j-1}(\lambda), \qquad \alpha \neq 0, \quad j > 0,$$

- *Then* $L_{k,\phi}(\lambda) = \begin{bmatrix} \alpha & (\beta-\lambda) & \gamma & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & (\beta-\lambda) & \gamma \\ & & & \phi_0(\lambda) & -\phi_1(\lambda) \end{bmatrix}$

- *since*

$$L_{k,\phi}(\lambda)\pi_{k,\phi}(\lambda) = 0, \qquad \textit{with } \pi_{k,\phi}(\lambda) := \begin{bmatrix} \phi_k(\lambda) \\ \vdots \\ \phi_0(\lambda) \end{bmatrix}.$$

Chebyshev Case

- Chebyshev basis of the first kind $\{T_i(\lambda)\}$ satisfying

$$T_{j+1}(\lambda) = 2\lambda T_j(\lambda) - T_{j-1}(\lambda).$$

- Then the dual basis is $L_{k,T}(\lambda) := \begin{bmatrix} 1 & -2\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2\lambda & 1 \\ & & & 1 & -\lambda \end{bmatrix},$

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- and $\mathcal{L}(\lambda) = \begin{bmatrix} \lambda M_1 + M_0 & L_{\eta,T}(\lambda)^T \\ L_{\epsilon,T}(\lambda) & 0 \end{bmatrix}$

- linearizes $p(\lambda) = \sum_{i=1}^{\epsilon} \sum_{j=1}^{\eta} (\lambda M_1 + M_0)_{i,j} T_i(\lambda) T_j(\lambda)$.

- Also discussed by J. Pérez and P. W. Lawrence, *Constructing strong linearizations of matrix polynomials expressed in Chebyshev bases.*

So Leonardo and Piers sat together

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and asked theirselves the following...

- can we do it for the Lagrange basis?
- can we do it for Hermite?

Interpolation Basis: Lagrange Basis

Preliminaries

- Take node sets $\sigma_1^{(1)}, \ldots, \sigma_\epsilon^{(1)}$ and $\sigma_1^{(2)}, \ldots, \sigma_\eta^{(2)}$ (not necessarily disjoint).
- Weights and Lagrange polynomials defined as

$$t_i^{(s)} := \prod_{j \neq i} (\sigma_i^{(s)} - \sigma_j^{(s)}), \qquad l_i^{(s)}(\lambda) := \frac{1}{t_i^{(s)}} \prod_{j \neq i} (\lambda - \sigma_j^{(s)}), \qquad s \in \{1, 2\}.$$

- Next, set $\phi_j(\lambda) = l_j^{(1)}(\lambda)$ and $\psi_j(\lambda) = l_j^{(2)}(\lambda)$.

Interpolation Basis: Lagrange Basis

Theorem (Lagrange)

- *The linearization for a polynomial expressed in a product family*
- *is of the form*

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda M_1 + M_0 & L_{\eta,\psi}(\lambda)^T \\ L_{\epsilon,\phi}(\lambda) & 0 \end{bmatrix}$$

- *where ($L_{k,\psi}(\lambda)$ analogously)*

$$L_{k,\phi}(\lambda) = \begin{bmatrix} t_1^{(1)}(\lambda - \sigma_1) & -t_2^{(1)}(\lambda - \sigma_2) & & \\ & \ddots & \ddots & \\ & & t_{k-1}^{(1)}(\lambda - \sigma_{k-1}) & -t_k^{(1)}(\lambda - \sigma_k) \end{bmatrix}$$

- Also for Hermite, Newton, and other bases.

Outline

So Leonardo and I stood together,

So Leonardo and I stood together,

and asked ourselves the following...

- one can do it for two different bases,
- can we do it for more than two?

Preliminaries: Product Basis

More than two?

- Consider

$$\phi^{(1)} \otimes \ldots \otimes \phi^{(j)} := \{\phi^{(1)}_{i_1} \ldots \phi^{(j)}_{i_j} \mid i_s = 0, \ldots, k_s,\ s = 1, \ldots, j\}$$

- for $\{\phi^{(s)}_i \mid i = 0, \ldots, k_s\}$ families of polynomials, $1 \leq s \leq j$.

Preliminaries: Product Basis

More than two?

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- for $\{\phi^{(s)}_i \mid i = 0, \ldots, k_s\}$ families of polynomials, $1 \le s \le j$.
- And equivalently we have

$$\pi_{k,\phi^{(1)}\otimes\ldots\otimes\phi^{(j)}}(\lambda) = \pi_{\epsilon_1,\phi^{(1)}}(\lambda) \otimes \ldots \otimes \pi_{\epsilon_j,\phi^{(j)}}(\lambda).$$

Product Dual Basis

Definition (Product Dual Basis for Two)

- Consider two dual basis namely
  - $L_{\epsilon,\phi}(\lambda)$ for $\{\phi_i\}$,
  - $L_{\eta,\psi}(\lambda)$ for $\{\psi_i\}$.
- Take w be a constant vector with $w^T \pi_{\eta,\psi}(\lambda) \neq 0$, e.g., $w = e_n^T$.
- Let A be an invertible matrix, e.g., $A = I$.

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- Let A be an invertible matrix, e.g., $A = I$.
- The matrix

$$L_{k,\phi\otimes\psi}(\lambda) = \begin{bmatrix} A \otimes L_{\eta,\psi}(\lambda) \\ L_{\epsilon,\phi}(\lambda) \otimes w^T \end{bmatrix}, \qquad k := (\epsilon + 1)(\eta + 1) - 1,$$

  is a <u>product dual basis</u> of $L_{\epsilon,\phi}(\lambda)$ and $L_{\eta,\psi}(\lambda)$.
- We denote it as $L_{\epsilon,\phi}(\lambda) \underline{\times} L_{\eta,\psi}(\lambda)$.

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- More than two: just continue...
- The form is essentially unique.

Theorem

Theorem

- *Consider the families of polynomials*
  - $\{\phi_i^{(1)}\}, \dots, \{\phi_i^{(j)}\}$,
  - $\{\psi_i^{(1)}\}, \dots, \{\psi_i^{(l)}\}$.
- *The matrix polynomial*

$$\mathcal{L}(\lambda) = \begin{bmatrix} \lambda M_1 + M_0 & (L_{\epsilon_1,\phi^{(1)}} \underline{\times} \dots \underline{\times} L_{\epsilon_j,\phi^{(j)}}(\lambda))^T \\ L_{\eta_1,\psi^{(1)}} \underline{\times} \dots \underline{\times} L_{\eta_l,\psi^{(l)}}(\lambda) & 0 \end{bmatrix}$$

- *linearizes the polynomial*

$$P(\lambda) = (\pi_{\epsilon_1,\phi^{(1)}}(\lambda) \otimes \dots \otimes \pi_{\epsilon_j,\phi^{(j)}}(\lambda))^T (\lambda M_1 + M_0)(\pi_{\eta_1,\psi^{(1)}}(\lambda) \otimes \dots \otimes \pi_{\eta_l,\psi^{(l)}}(\lambda)).$$

Example

- Let $\{\phi_i\}$ be the Chebyshev basis.
- Let $\{\psi_i\}$ be a degree graded basis.
- $L_{\epsilon,\phi} \underline{\times} L_{\eta,\psi}(\lambda)$, for $A = I$ and $w = e_{\eta+1}$ looks like

$$\left[\begin{array}{ccccc} L_{\eta,\psi}(\lambda) & & & & \\ & L_{\eta,\psi}(\lambda) & & & \\ & & \ddots & & \\ & & & L_{\eta,\psi}(\lambda) & \\ & & & & L_{\eta,\psi}(\lambda) \\ \hline 1 & -2\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2\lambda & 1 \\ & & & 1 & -\lambda \end{array}\right].$$

Example Monomial Basis

Example

- Let $p(\lambda) = \sum_{i=0}^{3} p_i \lambda^i$ a degree 3 polynomial.
- Choosing $\{\psi_i\} = \{1, \lambda, \lambda^2\}$ and $\{\phi_i\} = \{1\}$ yields Frobenius form:

$$\mathcal{L}(\lambda) = \left[\begin{array}{ccc} \lambda p_3 + p_2 & p_1 & p_0 \\ \hline 1 & -\lambda & \\ & 1 & -\lambda \end{array}\right].$$

- Choosing $\{\psi_i\} = \{\phi_i\} = \{1, \lambda\}$ yields, e.g., the symmetric

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|c} \lambda p_3 + p_2 & \frac{1}{2} p_1 & 1 \\ \frac{1}{2} p_1 & p_0 & -\lambda \\ \hline 1 & -\lambda & 0 \end{array}\right].$$

Example Monomial Basis Continued

Example

- Let $p(\lambda) = \sum_{i=0}^{3} p_i \lambda^i$ a degree 3 polynomial.
- Choosing $\{\psi_i\} = \{1, \lambda\} \otimes \{1, \lambda\}$ and $\{\phi_i\} = \{1\}$ with
- $L_{1,\psi}(\lambda) = [1 \;\; -\lambda]$ as dual basis of $\{\psi_i\}$,
- yields

$$\mathcal{L}(\lambda) = \left[\begin{array}{cccc} \lambda p_3 + p_2 & \frac{1}{2}p_1 & \frac{1}{2}p_1 & p_0 \\ \hline \multicolumn{4}{c}{I \otimes L_{1,\psi}(\lambda)} \\ \hline \multicolumn{4}{c}{L_{1,\psi}(\lambda) \otimes e_n^T} \end{array}\right] = \left[\begin{array}{cccc} \lambda p_3 + p_2 & \frac{1}{2}p_1 & \frac{1}{2}p_1 & p_0 \\ \hline 1 & -\lambda & & \\ & & 1 & -\lambda \\ \hline & 1 & & -\lambda \end{array}\right],$$

as linearization.

Outline

Intersections of Polynomials

Theorem

- *Let* $p(\lambda)$ *and* $q(\lambda)$ *be two polynomials*

$$p(\lambda) := \sum_{j=0}^{\epsilon} p_j \phi_j(\lambda), \qquad q(\lambda) := \sum_{j=0}^{\eta} q_j \psi_j(\lambda).$$

- *Let* $L_{\epsilon,\phi}(\lambda)$ *and* $L_{\eta,\psi}(\lambda)$ *be dual bases for* $\{\phi_i\}$ *and* $\{\psi_i\}$*.*
- *Then the matrix polynomial*

$$\mathcal{L}(\lambda) := \begin{bmatrix} p w_\psi^T - w_\phi q^T & L_{\epsilon,\phi}^T(\lambda) \\ L_{\eta,\psi}(\lambda) & 0 \end{bmatrix}, \qquad \star \in \{\phi, \psi\}$$

- *is a linearization for* $r(\lambda) := p(\lambda) - q(\lambda)$*.*
- $w_\star$ *contains the coefficients of the constant* 1 *in* $\{\phi_i\}$ *or* $\{\psi_i\}$*.*
- *Remark the rank 2 block in the upper left corner.*

Intersections of Rational Functions

Theorem

- *Let* $p(\lambda), q(\lambda), r(\lambda)$ *and* $s(\lambda)$ *polynomials*

$$p(\lambda) = \sum_{i=0}^{\epsilon} p_i \phi_i(\lambda), \qquad q(\lambda) = \sum_{i=0}^{\epsilon} q_i \phi_i(\lambda),$$

$$r(\lambda) = \sum_{i=0}^{\eta} q_i \psi_i(\lambda), \qquad s(\lambda) = \sum_{i=0}^{\eta} s_i \psi_i(\lambda)$$

  for polynomial bases $\{\phi_i\}$ *and* $\{\psi_i\}$.

- *Then (remark the structure in the upper left block)*

$$\mathcal{L}(\lambda) = \begin{bmatrix} ps^T + qr^T & L_{\epsilon,\phi}^T(\lambda) \\ L_{\eta,\psi}(\lambda) & 0 \end{bmatrix}$$

  is a linearization for $f(\lambda) = \frac{p(\lambda)}{q(\lambda)} + \frac{r(\lambda)}{s(\lambda)}$.

- p, q, r *and* s *are the column vectors containing the coefficients.*

Example

- Consider
$$f(\lambda) = \frac{2\lambda^2 - 1}{\lambda^2 + \lambda + 3} + \frac{T_1(\lambda) + T_0(\lambda)}{T_1(\lambda) - T_0(\lambda)}.$$
- With
$$p = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
- Then we get
$$\mathcal{L}(\lambda) = \begin{bmatrix} ps^T + qr^T & L_{\epsilon,\phi}^T(\lambda) \\ L_{\eta,\psi}(\lambda) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 1 & 1 & -\lambda & 1 \\ 2 & 4 & 0 & -\lambda \\ \hline 1 & -\lambda & 0 & 0 \end{array}\right].$$

Numerical Experiment: Setting

Example (Mix of Chebyshev and Monomial)

- Let $p_1(\lambda) = \sum_{i=0}^{n} p_{i,1}\lambda^i$ and $p_2(\lambda) = \sum_{i=0}^{n} p_{i,2} T_i(\lambda)$.
- So $p_1(x)$ in the monomial, and $p_2(x)$ in the Chebyshev basis.
- We compute the roots of $q(\lambda) = p_1(\lambda) + p_2(\lambda)$ in three ways:
  1. Convert $p_2(\lambda)$ to the monomial basis: Frobenius linearization;
  2. Convert $p_1(\lambda)$ to the Chebyshev basis: Colleague linearization;
  3. Constructing the linearization using our Theorem;

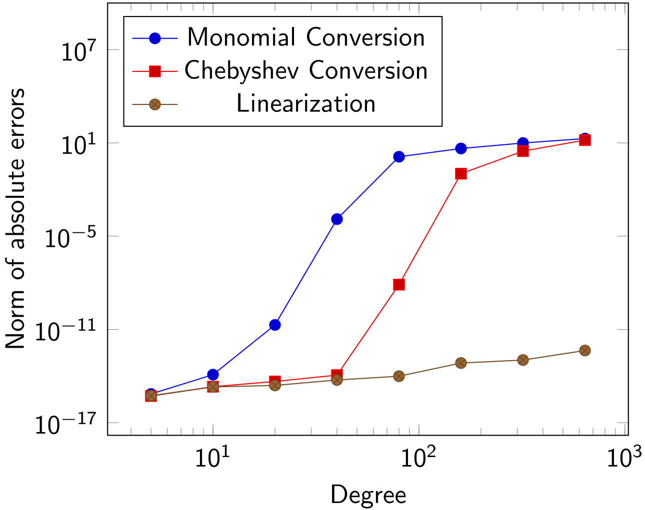
  pipe the linearization to the QZ method.
- Correct roots computed via
  - symbolic conversion to the monomial case and
  - MPSolve for the roots up to 16 accurate digits.
- Norm of the absolute forward errors is depicted.

Numerical Experiment: Results

Numerical Results (Absolute Forward Errors)

Degree	Monomial	Chebyshev	Linearization
5	7.03e-16	5.19e-16	5.44e-16
10	1.23e-14	2.07e-15	2.00e-15
20	1.95e-11	4.48e-15	2.49e-15
40	1.25e-04	1.15e-14	5.59e-15
80	1.29e+00	7.62e-09	9.76e-15
160	4.37e+00	1.05e-01	6.90e-14
320	9.85e+00	2.97e+00	1.07e-13
640	1.91e+01	1.52e+01	4.40e-13

Numerical Experiment: Results



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So Paul, Leonardo, and I sat together

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Answer: yes! But no time.

Outline

Summary of the results

We have extended the results of Dopico et al. to the following cases

- to other bases,
- to product bases,
- to sums of polynomials,
- rational functions,
- structured linearizations (e.g., $\star$-palindromic, even, odd,...) .

Closing Remarks

"My team has created a very innovative solution, but we're still looking for a problem to go with it."