Block Kronecker Linearizations of Matrix Polynomials and their Backward Errors

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joint work with Froilán Dopico (UC3Madrid, Sp),
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and dedicated to Miroslav Fiedler (1926-2015)

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Polynomial eigenvalue problems

- Polynomial matrices with $m \times n$ matrix coefficients P_i

$$P(\lambda) = \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0$$

- Want to compute the complete eigenstructure
  - Finite elementary divisors
  - Infinite elementary divisors
  - Left and right null space structure
- Many application areas:
  - Vibrating systems
  - Electrical circuits
  - Dynamical systems
  - Differential algebraic equations

What I'll talk about

- Companion and Fiedler matrices
- Block-Kronecker pencils (Fiedler-like)
- A new and simple proof
- Dual minimal bases
- Structure preserving backward stability
- Conclusions and extensions

Companion and Fiedler matrices

- The eigenvalues of a companion matrix

$$\lambda I_d - C; \quad C =: \begin{bmatrix} -p_{d-1} & \dots & -p_1 & -p_0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

  are the roots of the monic polynomial (Frobenius)

$$p(\lambda) = \lambda^d + \lambda^{d-1} p_{d-1} + \ldots + \lambda p_1 + p_0$$

- Fiedler extended this to a new family of matrices with the same elements, but rearranged in a strange way

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Companion matrix proof

Since

$$\lambda I_d - C = \begin{bmatrix} \lambda + p_{d-1} & \dots & p_1 & p_0 \\ -1 & \lambda & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \end{bmatrix}$$

we also have

$$(\lambda I_d - C) \begin{bmatrix} \lambda^{d-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} p(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

from which it follows that $p(\lambda) = \det(\lambda I_d - C)$

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Companion and Fiedler matrices

- The so-called Fiedler matrices can be constructed from products of elementary factors of the type

$$A_k = \begin{bmatrix} I_{d-k-1} & & \\ & C_k & \\ & & I_{k-1} \end{bmatrix}, \quad C_k = \begin{bmatrix} -p_k & -1 \\ 1 & 0 \end{bmatrix}.$$

- and typically contain a staircase of elements p_k, e.g. $(d = 4)$:

$$\lambda I_4 - F := \begin{bmatrix} \lambda + p_3 & -1 & 0 & 0 \\ p_2 & \lambda & p_1 & p_0 \\ -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}.$$

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Permuted Fiedler matrices

- If we permute the staircase to the top left corner, and scale we can obtain the following block anti-triangular form

$$\lambda B + A := \left[\begin{array}{ccc|c} \lambda + p_3 & & 0 & 1 \\ p_2 & p_1 & p_0 & -\lambda \\ \hline 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \end{array}\right],$$

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Block Kronecker pencils

- We start from the definitions

$$L_k(\lambda) = \left.\begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix}\right\} k, \quad \Pi_k(\lambda) = \begin{bmatrix} \lambda^k \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \left.\vphantom{\begin{bmatrix} \lambda^k \\ \vdots \\ \lambda \\ 1 \end{bmatrix}}\right\} k+1$$

  and the equation $L_k(\lambda)\Pi_k(\lambda) = 0$, implying that the rows of $L_k(\lambda)$ are dual to the columns of $\Pi_k(\lambda)$.

- A general block Kronecker pencil with $d = \epsilon + \eta + 1$ is then is of the form

$$\lambda B + A = \left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta^T(\lambda) \otimes I_m \\ \hline L_\epsilon(\lambda) \otimes I_n & 0 \end{array}\right] \begin{array}{l} \}\, (\eta+1)m \\ \}\, \epsilon n \end{array}$$

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A simple transformation

- $L_k(\lambda)$ can be embedded in a unimodular matrix

$$U_k(\lambda) := \left.\begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix}\right\} k+1$$

- with unimodular inverse $V_k(\lambda)$ whose last column is $\Pi_k(\lambda)$

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Now apply this to $\lambda B + A$ (scalar case for simplicity):

$$\left[\begin{array}{c|c} V_\eta^T(\lambda)\otimes I_m & \\ \hline 0 & I_{\epsilon n}\end{array}\right]\left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta^T(\lambda)\otimes I_m \\ \hline L_\epsilon(\lambda)\otimes I_n & 0\end{array}\right]\left[\begin{array}{c|c} V_\epsilon(\lambda)\otimes I_n & \\ \hline 0 & I_{\eta m}\end{array}\right]$$

$$= \left[\begin{array}{c|c|c} X(\lambda) & Y(\lambda) & I_{\eta m} \\ \hline Z(\lambda) & P(\lambda) & 0 \\ \hline I_{\epsilon n} & 0 & 0\end{array}\right]$$

where $P(\lambda) = (\Pi_\eta^T(\lambda)\otimes I_m)(\lambda M_1 + M_0)(\Pi_\epsilon(\lambda)\otimes I_n)$ provided

$$\lambda M_1 + M_0 = \begin{bmatrix} \lambda P_d & \iddots & \iddots \\ \iddots & \iddots & \iddots \\ \iddots & \iddots & P_0 \end{bmatrix},$$

$$\sum_{i+j=d-k+2}[M_1]_{i,j} + \sum_{i+j=d-k+1}[M_0]_{i,j} = P_k,\ k\in 0:d,\ i\in 1:\epsilon+1,\ j\in 1:\eta+1$$

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Example (matrix case)

Two possible linearizations for the quartic polynomial matrix

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0$$

$$\lambda B + A = \left[\begin{array}{ccc|c} \lambda P_4 & \lambda P_3 + P_2 & 0 & I_m \\ 0 & P_1 & P_0 & -\lambda I_m \\ \hline I_n & -\lambda I_n & 0 & 0 \\ 0 & I_n & -\lambda I_n & 0 \end{array}\right],$$

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Strong linearizations

- Results also hold for "reversed" polynomial since the reversed minimal bases satisfy similar equations

$$\left[\begin{array}{c|c} M_1 + \lambda M_0 & L^T_{rev,\eta}(\lambda) \otimes I_m \\ \hline L_{rev,\epsilon}(\lambda) \otimes I_n & 0 \end{array}\right] \approx \left[\begin{array}{c|c|c} P_{rev}(\lambda) & Z_{rev}(\lambda) & 0 \\ \hline Y_{rev}(\lambda) & X_{rev}(\lambda) & I_{\eta m} \\ \hline 0 & I_{\epsilon n} & 0 \end{array}\right]$$

Therefore, we can prove

- Theorem

Let $P(\lambda)$ be a $m \times n$ polynomial matrix of degree d. Then the block-Kronecker pencils are all strong linearizations of $P(\lambda)$ and they have the same left and right null space dimensions, provided $\sum_{i+j=d-k+2} [M_1]_{i,j} + \sum_{i+j=d-k+1} [M_0]_{i,j} = P_k\,, \quad 0 \le k \le d.$

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Polynomial backward stability

- We will assume that the matrices were scaled such that

$$\|(A, B)\| := \max(\|A\|_2, \|B\|_2) \approx 1 , \ \|P(\cdot)\| := \max_i(\|P_i\|_2) \approx 1$$

- The QZ algorithm applied to a block-Kronecker pencil $\lambda B + A$ perturbs the pencil as follows $\|(\Delta A, \Delta B)\| \approx \varepsilon$;

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- then there exists an *equivalent* $\|\delta P(\cdot)\| = O(\varepsilon)$ such that

$$M_\Delta(\lambda)\left[(\lambda B + A) + (\lambda \Delta B + \Delta A)\right] N_\Delta(\lambda) = \begin{bmatrix} P(\lambda) + \delta P(\lambda) & 0 \\ 0 & I \end{bmatrix}$$

This shows that the Kronecker pencil approach is *structurally* backward stable (whenever $P(\lambda)$ was scaled s.t. $\|P(\cdot)\| \approx 1$)

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Polynomial backward stability

We show backward stability to a nearby polynomial matrix $P(\lambda) + \delta P(\lambda)$ provided we scaled the coefficient matrix to 1 :

First we restore the anti-triangular structure by strict equivalence

$$\begin{bmatrix} I_{(\eta+1)m} & 0 \\ C & I_{\epsilon n} \end{bmatrix} \left(\left[\begin{array}{c|c} \lambda M_1 + M_0 & L_\eta^T(\lambda) \otimes I_m \\ \hline L_\epsilon(\lambda) \otimes I_n & 0 \end{array} \right] + \left[\begin{array}{c|c} \lambda \Delta B_{11} + \Delta A_{11} & \lambda \Delta B_{12} + \Delta A_{12} \\ \hline \lambda \Delta B_{21} + \Delta A_{21} & \lambda \Delta B_{22} + \Delta A_{22} \end{array} \right] \right) \begin{bmatrix} I_{(\epsilon+1)n} & D \\ 0 & I_{\eta m} \end{bmatrix} =$$

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This is a nonlinear system of equations $(C, D) = f(A, B, \Delta A, \Delta B)$, but has a solution of norm $\|(C, D)\| = O(\varepsilon)$.

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Polynomial backward stability

Second, we construct the perturbed dual minimal bases

$$\left(L_\epsilon(\lambda) \otimes I_n + \lambda \widetilde{B}_{21} + \Delta \widetilde{A}_{21}\right)\left(\Pi_\epsilon(\lambda) \otimes I_n + \delta R \epsilon(\lambda)\right) = 0$$

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By bounding $\|(C, D)\|$, $\|\delta R_\epsilon(\cdot)\|$ and $\|\delta R_\eta(\cdot)\|$, we can then prove

Theorem

Let $\lambda B + A$ be a regular block Kronecker linearization of a regular polynomial matrix $P(\lambda)$ and let $\lambda \Delta A + \Delta B$ be the backward error induced by the QZ algorithm. Then the corresponding backward error $\delta P(\cdot)$ has a norm satisfying

$$\|\delta P(\cdot)\| \leq d^2 \|(\Delta A, \Delta B)\| \approx d^2 \varepsilon$$

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- A simple proof is provided based on dual minimal bases
- We can prove "polynomial backward stability" for the QZ algorithm
- This extends to the singular case with the staircase algorithm
- We can sometimes preserve certain symmetries of $P(\lambda)$

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Block-Kronecker ℓ-ifications

For $\ell = 2$ we can obtain "quadratifications" of an even polynomial matrix as follows

$$P(\lambda) = \lambda^6 P_6 + \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \,.$$

$$\lambda^2 C + \lambda B + A = \left[\begin{array}{cc|c} \lambda^2 P_6 + \lambda P_5 + P_4 & \lambda P_3/2 & I_m \\ \lambda P_3/2 & \lambda^2 P_2 + \lambda P_1 + P_0 & -\lambda^2 I_m \\ \hline I_n & -\lambda^2 I_n & 0 \end{array}\right].$$

This is also a symmetric "quadratification" if $P(\lambda)$ is symmetric

Proof is very analogous (but not for backward errors...)

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