Stagnation of block GMRES and its relationship to block FOM

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Krylov and block Krylov subspaces

- Consider: $\mathbf{AX} = \mathbf{B} \in \mathbb{C}^{n\times L}$
- For $L = 1$ with $\mathbf{x}_0 \in \mathbb{C}^n$ and $\mathbf{f}_0 = \mathbf{B} - \mathbf{A}\mathbf{x}_0$, we have the **Krylov subspace**

$$\mathcal{K}_j(\mathbf{A}, \mathbf{f}_0) = \operatorname{span}\left\{\mathbf{f}_0, \mathbf{A}\mathbf{f}_0, \ldots, \mathbf{A}^{j-1}\mathbf{f}_0\right\}$$

- For $L > 1$ let $\mathbf{X}_0 \in \mathbb{C}^{n\times L}$ and

$$\mathbf{F}_0 = \mathbf{B} - \mathbf{A}\mathbf{X}_0 = \begin{bmatrix} \mathbf{f}_0^{(1)} & \mathbf{f}_0^{(2)} & \mathbf{f}_0^{(3)} & \cdots & \mathbf{f}_0^{(L)} \end{bmatrix} \in \mathbb{C}^{n\times L}.$$

- Then we have the **block Krylov subspace**

$$\mathbb{K}_j(\mathbf{A}, \mathbf{F}_0) = \mathcal{K}_j(\mathbf{A}, \mathbf{f}_0^{(1)}) + \mathcal{K}_j(\mathbf{A}, \mathbf{f}_0^{(2)}) + \cdots + \mathcal{K}_j(\mathbf{A}, \mathbf{f}_0^{(L)}).$$

- $L > 1$ is a generalization of the case $L = 1$.
- $\dim \mathbb{K}_j(\mathbf{A}, \mathbf{F}_0) = jL$ except when otherwise noted.

(Block) Krylov Subspace Methods

- Let $\mathbf{F}_0 = \mathbf{V}_1 \mathbf{S}_0$ be a skinny QR-factorization.
- The **block Arnoldi process** at step j produces $\mathbf{V}_{j+1} \in \mathbb{C}^{n \times L}$ with orthonormal columns
- $\mathbf{W}_j = \begin{bmatrix} \mathbf{V}_1, & \ldots, & \mathbf{V}_j \end{bmatrix} \in \mathbb{C}^{n \times jL}$ has orthonormal columns spanning $\mathbb{K}_j(\mathbf{A}, \mathbf{F}_0)$
- Arnoldi relation: $\mathbf{A}\mathbf{W}_j = \mathbf{W}_{j+1}\overline{\mathbf{H}}_j$, $\overline{\mathbf{H}}_j \in \mathbb{C}^{(j+1)L \times jL}$ is (block) upper Hessenberg
- $\overline{\mathbf{H}}_j = (\mathbf{H}_{ik})_{ik} \in \mathbb{C}^{(j+1)L \times jL}$ is block upper Hessenberg
- For $\blacksquare$, $\text{◥} \in \mathbb{C}^{L \times L}$ and ◥ upper triangular

$$\overline{\mathbf{H}}_j = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots & \blacksquare \\ \text{◥} & \blacksquare & \blacksquare & \blacksquare & \cdots & \blacksquare \\ & \text{◥} & \blacksquare & \blacksquare & \cdots & \blacksquare \\ & & \text{◥} & \blacksquare & \cdots & \blacksquare \\ & & & \text{◥} & & \blacksquare \\ & & & & \ddots & \vdots \\ & & & & & \text{◥} \end{bmatrix}$$

Restarted Block GMRES and Block FOM

Block GMRES and Block FOM valid for all $L \geq 1$

- Build an orthonormal basis for $\mathbb{K}_m(\mathbf{A}, \mathbf{F}_0)$
- For block GMRES
  - Compute $\mathbf{Y}_m^{(G)} = \underset{Y \in \mathbb{C}^{mL \times L}}{\operatorname{argmin}} \left\| \overline{\mathbf{H}}_m \mathbf{Y} - \mathbf{E}_1^{(m+1} \mathbf{S}_0 \right\|_F$ [a]
  - Set $\mathbf{X}_m^{(G)} = \mathbf{X}_0 + \mathbf{W}_m \mathbf{Y}_m^{(G)}$, $\mathbf{F}_m^{(G)} = \mathbf{B} - \mathbf{A}\mathbf{X}_m$
- For block FOM
  - Compute $\mathbf{Y}_m^{(F)} = \mathbf{H}_m^{-1} \mathbf{E}_1^{[m]} \mathbf{S}_0$ [b]
  - Set $\mathbf{X}_m^{(F)} = \mathbf{X}_0 + \mathbf{W}_m \mathbf{Y}_m^{(F)}$, $\mathbf{F}_m^{(F)} = \mathbf{B} - \mathbf{A}\mathbf{X}_m^{(F)}$
- Overwrite $\mathbf{X}_0$ and $\mathbf{F}_0$ with the mth approximation and residual.
- Repeat until convergence

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[a] $\mathbf{E}_1^{(m+1)} \in \mathbb{R}^{(m+1)L \times L}$ has appropriate columns of an identity matrix

[b] $\mathbf{E}_1^{[m]} \in \mathbb{R}^{mL \times L}$ has appropriate columns of an identity matrix

When FOM breaks down (for all L)

If $\mathbf{H}_j$ is singular, then the (block) FOM solution at iteration j does not exist.

The generalized (block) FOM approximation

$$\widetilde{\mathbf{X}}_j^{(F)} = \mathbf{X}_0 + \widetilde{\mathbf{T}}_j^{(F)} \quad \text{where} \quad \widetilde{\mathbf{T}}_j^{(F)} = \mathbf{V}_j \widetilde{\mathbf{Y}}_j^{(F)} \quad \text{and} \quad \widetilde{\mathbf{Y}}_j^{(F)} = \mathbf{H}_j^{\dagger} \left(\mathbf{E}_1^{[m]} \mathbf{S}_0 \right),$$

- Always exists
- $\widetilde{\mathbf{X}}_j^{(F)} = \mathbf{X}_j^{(F)}$ in the case (block) FOM does not breakdown
- Used in [Brown '91] to prove certain results.

The QR-factorizations of $\overline{\mathbf{H}}_j$ and $\mathbf{H}_j$ for $L = 1$

- Since they are upper Hessenberg, we use Givens rotations to annihilate lower subdiagonal.
- Let c_j and s_j be the Givens cosine and sine for column j of $\overline{\mathbf{H}}_j$.
- If $\mathbf{H}_j$ is singular, then $c_j = 0$ and $s_j = 1$.
- The first $j - 1$ rotations triangularize $\mathbf{H}_j$.
- Thus, $\mathbf{R}_j = \mathbf{Q}_j^T \mathbf{H}_j$ and $\overline{\mathbf{R}}_j = \overline{\mathbf{Q}}_j^T \overline{\mathbf{H}}_j$ have the same upper-left $(j-1) \times (j-1)$ submatrices.

Stagnation of GMRES and breakdown of FOM for $L = 1$

Theorem in [Brown '91]

The matrix $\mathbf{H}_j$ is singular (and thus $\mathbf{X}_j^{(F)}$ does not exist) if and only if GMRES stagnates at iteration j with $\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)}$. Furthermore, in the case that $\mathbf{H}_j$ is singular, we have $\widetilde{\mathbf{X}}_j^{(F)} = \mathbf{X}_j^{(G)}$.

Relationship of GMRES and FOM for $L = 1$

Proposition shown in [Saad '03] based on results in [Cullum and Greenbaum '96]

We can write $\mathbf{X}_j^{(G)}$ as the following convex combination,

$$\mathbf{X}_j^{(G)} = c_j^2 \mathbf{X}_j^{(F)} + s_j^2 \mathbf{X}_{j-1}^{(G)}$$

where s_j and c_j are the jth Givens sine and cosine, respectively, as defined earlier.

- Derived from relationship between $\mathbf{R}_j$ and $\overline{\mathbf{R}}_j$.
- We can replace $\mathbf{X}_j^{(F)}$ with $\widetilde{\mathbf{X}}_j^{(F)}$ to make this compatible with the FOM breakdown case.
- If FOM breaks down, then $c_j = 0$, $s_j = 1$, and we recover the stagnation result $\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)}$

Selected previous work analyzing $L = 1$ and $L > 1$ cases

The relationship between GMRES and FOM for $L = 1$

- **Relationship of FOM/GMRES convergence**: [Walker '95], [Zhou and Walker '94], [Brown '91], [Saad '03]
- **Galerkin/norm minimizing pairs of methods (e.g., BiCG/QMR)**: [Cullum '95], [Cullum and Greenbaum '96]
- **Geometric analysis**: [Eiermann and Ernst '01]

Analysis of block Krylov methods $L > 1$

- [O'Leary '80] [Simoncini '96] [Simoncini and Gallopoulos '96], [Robbé and Sadkane '06], [Gutknecht and Schmelzer '09], [Elbouyahyaoui, Messaoudi, and Sadok '09]
- **What we saw this morning:** [Frommer, Lund-Nguyen, Szyld '16 (on website of D.B. Szyld in two weeks)]

Generalizing to the case $L > 1$

Questions:

- How to characterize of stagnation to the block GMRES?
  - Is $\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)}$ a generalization of stagnation for $L = 1$? **YES!**
  - If $< L$ columns of $\mathbf{X}_j^{(G)}$ stagnate, is this the between-case to analyse? **NO! only a special case**
- How to generalize FOM breakdown to the block case?
  - Is $\operatorname{rank} \mathbf{H}_j = (j-1)L$ a generalization of stagnation for $L = 1$? **YES!**
  - What happens when $\operatorname{rank} \mathbf{H}_j = (j-1)L + r$ for $0 < r \leq L$?
- How to relate $\mathbf{X}_j^{(G)}$ and $\mathbf{X}_j^{(F)}$ using sines and cosines?

Key observation: we can write

$$\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)} + \mathbf{S}_j^{(G)} \quad \text{and} \quad \mathbf{X}_j^{(F)} = \mathbf{X}_{j-1}^{(G)} + \mathbf{S}_j^{(F)};$$

the column spaces of $\mathbf{S}_j^{(G)}$ and $\mathbf{S}_j^{(F)}$ characterize everything.

FOM and GMRES as twin minimization problems

Let $\widetilde{\mathbf{X}}_j$ be the generalized block FOM approximation. We can write

$$\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)} + \mathbf{S}_j^{(G)} \quad \text{and} \quad \widetilde{\mathbf{X}}_j^{(F)} = \mathbf{X}_{j-1}^{(G)} + \widetilde{\mathbf{S}}_j^{(F)}$$

with

$$\mathbf{S}_j^{(G)} = \mathbf{W}_j \mathbf{Y}_{\mathbf{S}_j}^{(G)} \quad \text{and} \quad \widetilde{\mathbf{S}}_j^{(F)} = \mathbf{W}_j \widetilde{\mathbf{Y}}_{\mathbf{S}_j}^{(F)},$$

both with columns in $\mathbb{K}_j(\mathbf{A}, \mathbf{F}_0)$.

Block FOM and GMRES minimizations, similar to [Brown '91]

$$\begin{aligned} \mathbf{Y}_{\mathbf{S}_j}^{(G)} &= \operatorname*{argmin}_{\mathbf{Y}\in\mathbb{C}^n} \left\| \begin{bmatrix} \mathbf{E}_1^{[jL]}\mathbf{S}_0 - \overline{\mathbf{H}}_{j-1}\mathbf{Y}_{j-1}^{(G)} \\ \mathbf{0} \end{bmatrix} - \overline{\mathbf{H}}_j \mathbf{Y} \right\| \quad \text{and} \\ \widetilde{\mathbf{Y}}_{\mathbf{S}_j}^{(F)} &= \operatorname*{argmin}_{\mathbf{Y}\in\mathbb{C}^n} \left\| \mathbf{E}_1^{[jL]}\mathbf{S}_0 - \overline{\mathbf{H}}_{j-1}\mathbf{Y}_{j-1}^{(G)} - \mathbf{H}_j \mathbf{Y} \right\|. \end{aligned}$$

Triangularization of $\overline{\mathbf{H}}_j$ and $\mathbf{H}_j$

- We can progressively triangularize $\overline{\mathbf{H}}_j$ and $\mathbf{H}_j$:

$$\mathbf{Q}_{j-1}^{(j+1)} \cdots \mathbf{Q}_1^{(j+1)} \overline{\mathbf{H}}_j = \begin{bmatrix} \mathbf{R}_{j-1} & \mathbf{Z}_j \\ & \widehat{\mathbf{H}}_{j,j} \\ & \mathbf{H}_{j+1,j} \end{bmatrix} \quad \text{and}$$

$$\mathbf{Q}_{j-1}^{(j)} \cdots \mathbf{Q}_1^{(j)} \mathbf{H}_j = \begin{bmatrix} \mathbf{R}_{j-1} & \mathbf{Z}_j \\ & \widehat{\mathbf{H}}_{j,j} \end{bmatrix}.$$

- $\mathbf{R}_{j-1} \in \mathbb{C}^{(j-1)L \times (j-1)L}$ u.pper triangular from QR-factorization at step $j-1$
- $\mathbf{Q}_i^{(j+1)} \in \mathbb{C}^{(j+1)L \times (j+1)L}$ and $\mathbf{Q}_i^{(j)} \in \mathbb{C}^{jL \times jL}$ use the same transformation in different sized matrices.
- $\mathbf{Q}_j^{(j+1)}$ and $\mathbf{Q}_j^{(j)}$ complete the triangularizations, respectively

Two core problems

Using the QR-factorizations, we reformulate the block GMRES and FOM subproblems.

$$\begin{bmatrix} \mathbf{R}_{j-1} & \mathbf{Z}_j \\ & \mathbf{N}_j \end{bmatrix} \mathbf{Y}_j^{(G)} = \mathbf{G}_j^{(G)} = \begin{bmatrix} \mathbf{G}_{j-1}^{(G)} \\ \mathbf{C}_j \end{bmatrix}$$

$$\text{and} \qquad \begin{bmatrix} \mathbf{R}_{j-1} & \mathbf{Z}_j \\ & \widehat{\mathbf{N}}_j \end{bmatrix} \mathbf{Y}_j^{(F)} = \mathbf{G}_j^{(F)} = \begin{bmatrix} \mathbf{G}_{j-1}^{(G)} \\ \widehat{\mathbf{C}}_j \end{bmatrix},$$

with $\mathbf{N}_j$ and $\widehat{\mathbf{N}}_j$ upper triangular, where we have that

$$\mathbf{G}_j^{(G)} = \left(\mathbf{Q}_j^{(j+1)} \begin{bmatrix} \mathbf{G}_{j-1}^{(G)} \\ \widetilde{\mathbf{C}}_j \\ \mathbf{0} \end{bmatrix} \right)_{1:jL} = \left(\begin{bmatrix} \mathbf{G}_{j-1}^{(G)} \\ \mathbf{C}_j \\ * \end{bmatrix} \right)_{1:jL}.$$

Behavior of Block GMRES and $\mathrm{rank}\ \mathbf{H}_{\mathrm{j}}$

Theorem

The matrix $\mathbf{H}_j$ *is singular with* $\mathrm{rank}\ \mathbf{H}_{\mathrm{j}} = (\mathrm{j}-1)\mathrm{L}+\mathrm{r}$ *with* $r < L$ *if and only for the* j*th block GMRES update* $\mathbf{S}_j^{(G)}$,

$$\dim\left(\mathcal{R}\left(\mathbf{S}_j^{(G)}\right)\cap\mathcal{R}\left(\mathbf{V}_j\right)\right) = r$$

In other words

There exists $\mathcal{Z}\subset\mathcal{R}(\mathbf{V}_j)$ with $\dim\mathcal{Z} = L - r$ such that

$$\mathcal{R}\left(\mathbf{S}_j^{(G)}\right)\perp\mathcal{Z}\quad\text{and}$$
$$\mathcal{R}\left(\mathbf{S}_j^{(G)}\right)\not\perp\left(\mathcal{R}(\mathbf{V}_j)\setminus\mathcal{Z}\right).$$

Full stagnation of Block GMRES and $\mathrm{rank}\ \mathbf{H}_{\mathrm{j}}$

Corollary

The matrix $\mathbf{H}_j$ *is singular with* $\mathrm{rank}\ \mathbf{H}_{\mathrm{j}} = (\mathrm{j}-1)\mathrm{L}$ *if and only if block GMRES has totally stagnated with* $\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)}$.

Relationship of block GMRES and FOM Updates

Theorem

The span of the columns of $\widetilde{\mathbf{S}}_j^{(F)}$ has a non-trivial intersection with exactly an r-dimensional subspace of $\mathcal{R}(\mathbf{V}_j)$ if and only if the same is true of $\mathbf{S}_j^{(G)}$. In this case, we also have $\operatorname{rank} \mathbf{C}_j = r$.

The is the **correct third case** between no stagnation and total block stagnation for $L > 1$ that one should analyse.

Total stagnation of block GMRES leads to block FOM breakdown

Corollary

Block GMRES at iteration j totally stagnates with $\mathbf{X}_j^{(G)} = \mathbf{X}_{j-1}^{(G)}$ if and only if $\widetilde{\mathbf{X}}_j^{(F)} = \mathbf{X}_{j-1}^{(G)}$.

Characterization of partial stagnation

Definition

Partial stagnation refers to the case that certain columns of the GMRES iterate stagnate but $\mathbf{X}_j^{(G)} \neq \mathbf{X}_{j-1}^{(G)}$.

Theorem

Block GMRES suffers a partial stagnation at iteration j if and only if $0 < \mathrm{rank}\ \mathbf{C}_\mathrm{j} \leq \tilde{\mathrm{r}}$ where $\tilde{r}$ is the number of stagnating columns such that if the ith column of $\mathbf{X}_j^{(G)}$ stagnates, then ith column of $\mathbf{C}_j$ is the zero vector.

Block GMRES/FOM connection using CS-decomposition

Theorem

Let $\mathbf{H}_j$ be nonsingular. Then we have,

$$\mathbf{X}_j^{(G)} = \mathbf{X}_j^{(F)} \left(\widehat{\mathbf{C}}_j^{-1} \mathcal{Q} \mathcal{C}^2 \mathcal{Q}^* \widehat{\mathbf{C}}_j \right) + \mathbf{X}_{j-1}^{(G)} \left(\widehat{\mathbf{C}}_j^{-1} \mathcal{Q} \mathcal{S}^2 \mathcal{Q}^* \widehat{\mathbf{C}}_j \right) \quad \textit{where}$$

- $\mathcal{Q} \in \mathbb{C}^{L \times L}$ *is a unitary matrix*
- $\mathbf{C}$ *and* $\mathbf{S}$ *are* $L \times L$ *diagonal matrices with entries* ≤ 1
- $\mathbf{C}^2 + \mathbf{S}^2 = \mathbf{I}$ *(sines and cosines)*
- $\mathbf{C}$, $\mathbf{S}$ *come from CS-decomposition of non-identity block of*

$$\mathbf{Q}_j^{(j+1)} = \begin{bmatrix} \mathbf{I}_{(j-2)L} & & \\ & \blacksquare & \blacksquare \\ & \blacksquare & \blacksquare \end{bmatrix}$$

In case of dependent Arnoldi vectors: stay calm

- Block Arnoldi vectors might *not* be linearly independent
- For stability, one can, e.g.:
  - reduce block size
  - replace dependent vector with random one
- Block size reduction
  - → less elegant analysis with non-square matrix blocks
  - → weaker versions of many results
  - → some results not valid in this setting
- Block size maintained with random vector replacement
  - → square block structure maintained
  - → $\widehat{\mathbf{C}}_j$, $\widetilde{\mathbf{C}}_j$, $\mathbf{C}_j$ become rank deficient
  - → Only results depending on full rank of these matices effected.

Modified relationship between block GMRES and block FOM

Assumptions:

- p dependent basis vectors generated at iteration j.
- No single system has converged.
- No dependent vectors generated subsequently.

Theorem

With the above assumptions, at iteration $j+k$, if $\mathbf{H}_{j+k}$ is nonsingular, then we have that

$$\widetilde{\mathbf{X}}_{j+k}^{(F)} - \mathbf{X}_{j+k}^{(G)} = \mathbf{W}_{j+k} \begin{bmatrix} \mathbf{R}_{j+k-1}^{-1} \mathbf{Z}_{j+k} \\ \mathbf{I} \end{bmatrix} \widehat{\mathbf{Y}}_2^{-1} \mathcal{Q} \mathcal{S}^2 \mathcal{Q}^* \widehat{\mathbf{C}}_j.$$

Principal angles between $\mathcal{R}\left(\mathbf{F}_{j-1}^{(G)}\right)$ and $\mathbf{A}\mathbb{K}_j(\mathbf{A}, \mathbf{F}_0)$

Theorem

The angles represented by the sines and cosines from the CS-decomposition are the principal angles between $\mathcal{R}\left(\mathbf{F}_{j-1}^{(G)}\right)$ and $\mathbf{A}\mathbb{K}_j(\mathbf{A}, \mathbf{F}_0)$.

- The smaller the angles are, the greater the decrease in residual norm from the minimum residual projection at this iteration.
- Using implementation strategy of [Gutknecht and Schmelzer '08] angles available at the cost of two $L \times L$ SVD computations per iteration.

Numerical example: partial stagnation of block GMRES

- We use the coordinate shift matrix $\mathbf{A}_{st}$ but with $n = 30$.
- The right-hand sides are $\mathbf{e}_1^{[30]}$ and $\mathbf{e}_{25}^{[30]}$.
- At iteration 5, we have convergence for the first right-hand side.
- The code used performs a replacement of the dependent vector.
- Near-stagnation of the other system until iteration $j = 15$

Numerical example: partial stagnation of block GMRES

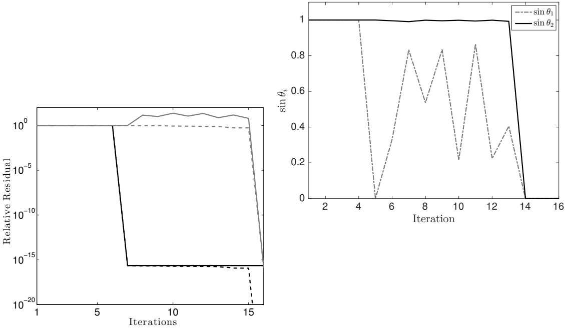

Figure : The **black solid** and **dashed** curves correspond respectively to the block FOM and GMRES residuals for the first right-hand side. Similarly, the **gray solid** and **dashed** curves, respectively, correspond to the second right-hand side.

Numerical example: fast and slow phases of block GMRES convergence

- Build a diagonal matrix with two blocks:
  - the `sherman5` matrix[1]
  - the 200×200 coordinate shift matrix $\mathbf{A}_{st}$
- Two right-hand sides chosen to produce perfect stagnation in the $\mathbf{A}_{st}$ block but convergence in the `sherman4` block.
- subvectors of the right-hand sides for $\mathbf{A}_{st}$ block were $\mathbf{e}_{50}^{[200]}$ and $\mathbf{e}_{150}^{[200]}$.
- subvectors of the right-hand sides `sherman5` block were a vector packaged with `sherman5` and a random vector scaled to have norm on the order of 10^7.[2].

[1] Univ. Florid Sparse Matrix Library

[2] to exaggerate convergence prior to stagnation

Numerical example: fast and slow phases of convergence

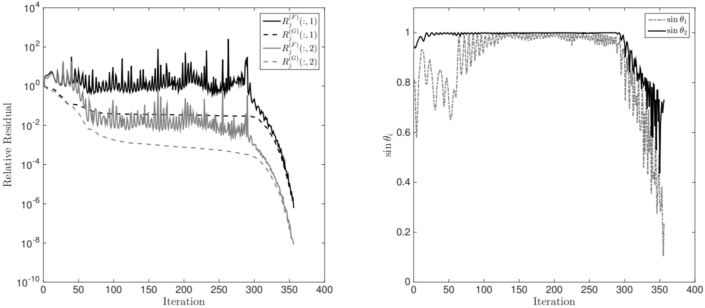

Figure : **Left-hand figure**: the 2-norm residual curves of block GMRES and FOM. **Right-hand figure**: squares of the sines $\left\{ s_1^2, s_2^2 \right\}$ from the CS-decomposition of the orthogonal transformation in the our analysis.

Conclusions and further work

Conclusions:

- $\mathbf{X}_j^{(F)}$ may not exist but $\widetilde{\mathbf{X}}_j^{(F)}$ is a possible improvement of $\mathbf{X}_{j-1}^{(G)}$.
- Block FOM breakdown and block GMRES "stagnation" are characterized by the intersection of the column spaces of $\mathbf{S}_j^{(G)}$ and $\widetilde{\mathbf{S}}_j^{(F)}$ with that of $\mathbf{V}_j$.
- "Convex combination" relationship between $\mathbf{X}_{j-1}^{(G)}$ and $\widetilde{\mathbf{X}}_j^{(F)}$.
- Connected QR-factorization of $\overline{\mathbf{H}}_j$ with principal angles between $\mathcal{R}\left(\mathbf{F}_{j-1}^{(G)}\right)$ and $\mathbf{A}\mathbb{K}_j(\mathbf{A}, \mathbf{F}_0)$.

Future work:

- Extend to block-type methods for Sylvester equations (perhaps other matrix equations or matrix functions?).
- Use the angles for deflation/recycling schemes (see, e.g., [de Sturler '99])

Thank you!

To learn more (about me, my work, my interests,...), visit `http://math.soodhalter.com`.

1. S. *Stagnation of block GMRES and its relationship to block FOM*. Submitted and available on arXiv (linked from my website).
2. S. *Block Krylov subspace recycling for shifted systems with unrelated right-hand sides*. SISC Volume 38 and available on arXiv (linked from my website).