Fast and backward stable computation of the eigenvalues of matrix polynomials

Leonardo Robol, KU Leuven
<leonardo.robol@cs.kuleuven.be>

joint work with Jared L. Aurentz (University of Oxford), Thomas Mach (Nazarbayev University), Raf Vandebril (KU Leuven) and David S. Watkins (WSU)

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The problem

We are interested in solving Polynomial Eigenvalue Problems, that is finding the *eigenvalues* λ such that:

$$\det P(\lambda)v = 0, \qquad P(\lambda) = \sum_{j=0}^{d} P_j \lambda^j, \qquad P_j \in \mathbb{C}^{m\times m}.$$

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- Only square ($m \times m$) matrix polynomials.
- We are mainly interested in the *eigenvalues*.
- We want a method with an optimal asymptotic cost.

The classical way

The usual strategy to solve a PEP is to build the Frobenius companion linearization:

$$W - \lambda V = \begin{bmatrix} 0_m & \dots & 0_m & -P_0 \\ I_m & & & -P_1 \\ & \ddots & & \vdots \\ & & I_m & -P_{d-1} \end{bmatrix} - \lambda \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ & & & P_d \end{bmatrix}$$

and solve it via the QZ method. The cost is cubic in the size $\Longrightarrow \mathcal{O}(m^3 d^3)$ flops.

We claim that we can do better than that!

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By combining the above remarks, we aim for $\mathcal{O}(d^2 m^3)$ flops.

Some background

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Let $p(\lambda)$ be a scalar (monic) polynomial of degree d. Its roots are the eigenvalues of the companion matrix:

$$\begin{bmatrix} & & -p_0 \\ 1 & & -p_1 \\ & \ddots & \vdots \\ & 1 & -p_{d-1} \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & \ddots & \\ & 1 & \end{bmatrix} + \begin{bmatrix} -p_0 - 1 \\ -p_1 \\ \vdots \\ -p_{d-1} \end{bmatrix} e_d^T$$

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Step 0: Enlarge the matrix by adding a row and a column:

$$\begin{bmatrix} & & & -p_0 & 1 \\ 1 & & & -p_1 & \vdots \\ & \ddots & & \vdots & \vdots \\ & & 1 & -p_{d-1} & 0 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

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Step 1: Compute its QR factorization

$$\mathcal{Q}R = \begin{bmatrix} & & & 1 & 0 \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & 0 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -p_1 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -p_{d-1} & \vdots \\ & & & -p_0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

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Step 2: Decompose the R factor as unitary plus rank 1:

$$R = Y + xy^T = \begin{bmatrix} 1 & & & 0 & 0 \\ & \ddots & & 0 & \vdots \\ & & 1 & \vdots & \vdots \\ & & & 0 & 1 \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} + \begin{bmatrix} -p_1 \\ \vdots \\ -p_{d-1} \\ -p_0 \\ -1 \end{bmatrix} e_d^T$$

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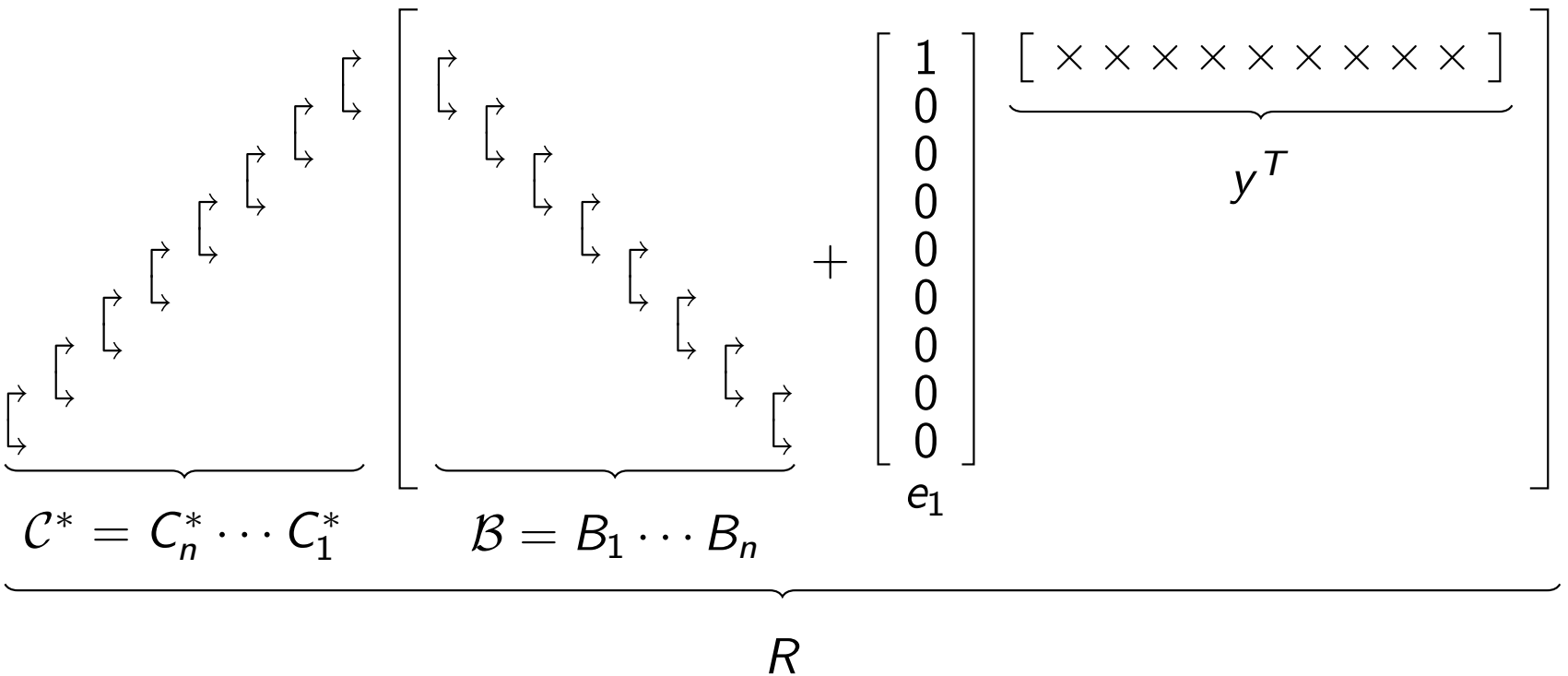
Step 3: "Roll up" the column with the coefficients

$$\mathcal{C} \begin{bmatrix} -p_1 \\ \vdots \\ -p_{d-1} \\ -p_0 \\ -1 \end{bmatrix} = \alpha e_1,$$

and set $\mathcal{B} := \mathcal{C}^* Y$, $y = \alpha e_d^T$, so $W = \mathcal{Q}\mathcal{C}^*(\mathcal{B} + e_1 y^T)$.

Core transformations

The matrices $\mathcal{Q}, \mathcal{C}$, and $\mathcal{B}$ are sequences of *core transformations*, that are essentially 2×2 matrices. We can write R as:



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In our language, that means $\mathcal{Q} = I$.

The matrix polynomial case

We know how to perform the QZ efficiently on unitary plus rank 1 matrices. What about unitary plus rank m?

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- Treating the unitary plus rank m directly is difficult, and easily leads to high complexities in m, or instabilities.
- Unitary plus rank m representations can be cumbersome to work with: it is very difficult to get efficient implementations.
- . . . so we do what mathematician always do: we go back to the previous case.

A nice factorization trick

Let W be the (block) companion of a matrix polynomial $P(\lambda)$. Assume the constant coefficient P_0 is lower triangular. Then we have:

$$W = W_1 \dots W_m,$$

where W_i are scalar companion matrices of polynomial having as coefficients (some of) the coefficients of $P(\lambda)$, appropriately permuted.

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In particular, the factorization can be computed at *no cost*!

An example

$$\begin{bmatrix} 0 & & & \alpha_1 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & \ddots & \alpha_{md-1} \\ & & 1 & \alpha_{md} \end{bmatrix} \begin{bmatrix} 0 & & & \beta_1 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & \ddots & \beta_{md-1} \\ & & 1 & 0 \end{bmatrix}$$

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A quick look at Gaussian decomposition

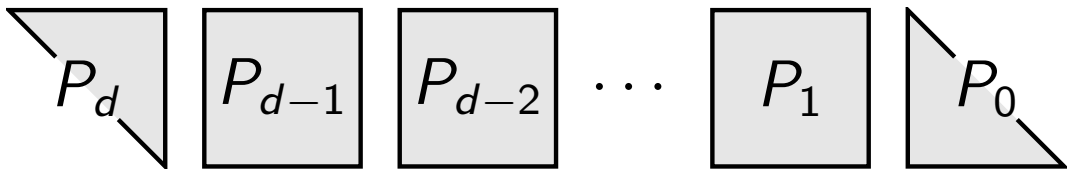
$$\begin{bmatrix} 1 & & & \alpha_1 \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & \alpha_{md} \end{bmatrix} \begin{bmatrix} 1 & & \beta_1 & \\ & \ddots & \vdots & \\ & & \beta_{md-1} & \\ & & & 1 \end{bmatrix}$$

A quick look at Gaussian decomposition

$$\begin{bmatrix} 1 & & \beta_1 & \alpha_1 \\ & \ddots & \vdots & \vdots \\ & & & \vdots \\ & & \beta_{md-1} & \\ & & & \alpha_{md} \end{bmatrix}$$

The required structure

In order to compute these decompositions we need the initial structure of the matrix polynomial to be



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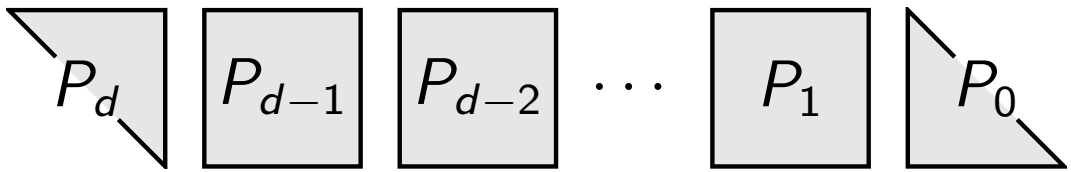

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Our pencil is now

$$W_1 \dots W_m - \lambda V_1 \dots V_m,$$

which we can solve as a product eigenvalue problem!

What's missing

Hessenberg-Triangular reduction The matrices V_j are upper triangular, but each of the W_j is upper Hessenberg, thus the product is m-Hessenberg. We need to eliminate $m-1$ subdiagonals.

Chasing We can implement the chasing passing rotations through the triangular factors.

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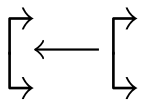
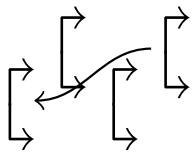
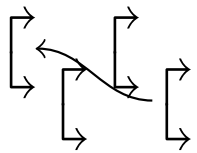
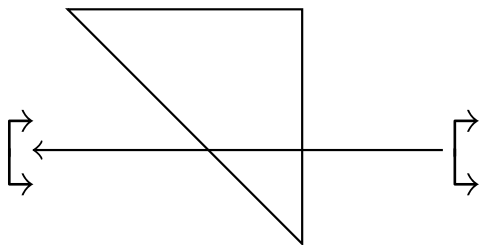
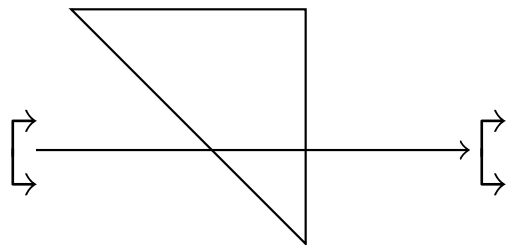
To understand how this works, we need some basic operations with core transformations.

Using core transformations

Fusion:  Turnover:  ,  .

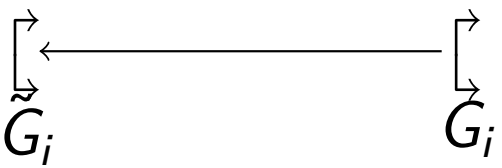
And then we need to pass rotation through upper triangular matrices:

 or 

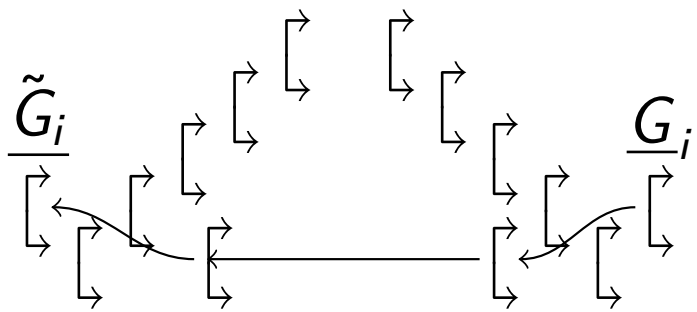
Exploiting the structure

Since we have structured upper triangular matrices, we can pass rotation through them in a fast way:

$$(\text{recall that } R = \mathcal{C}^*(\mathcal{B} + e_1 y^T))$$



is computed as



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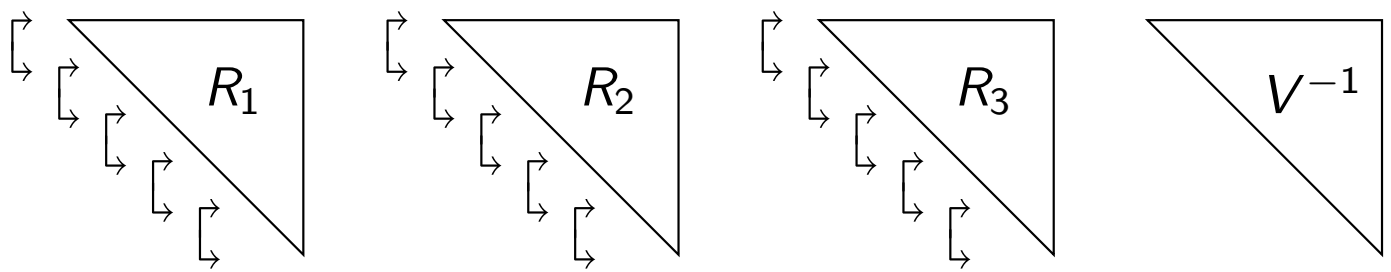

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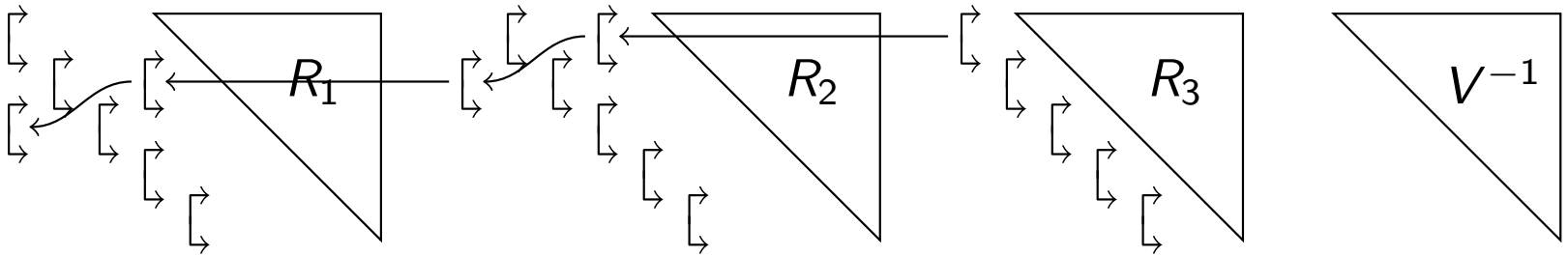

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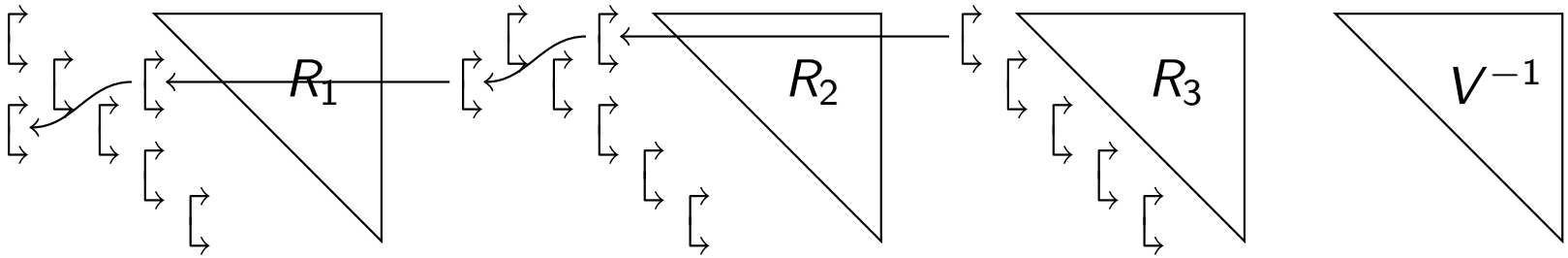

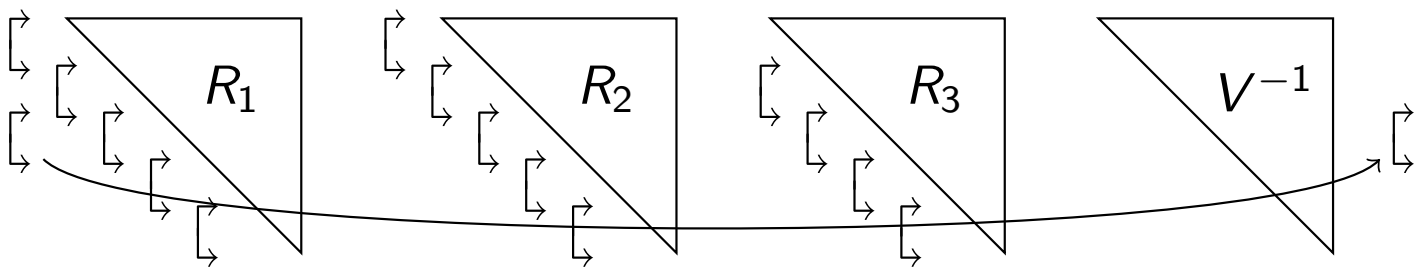

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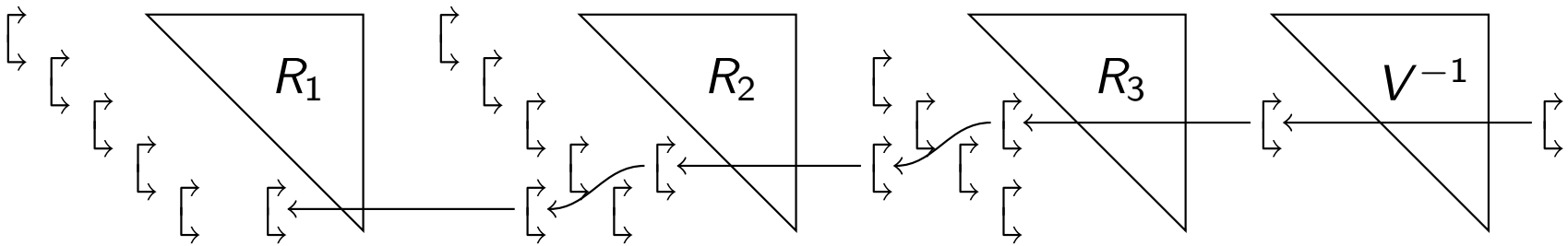

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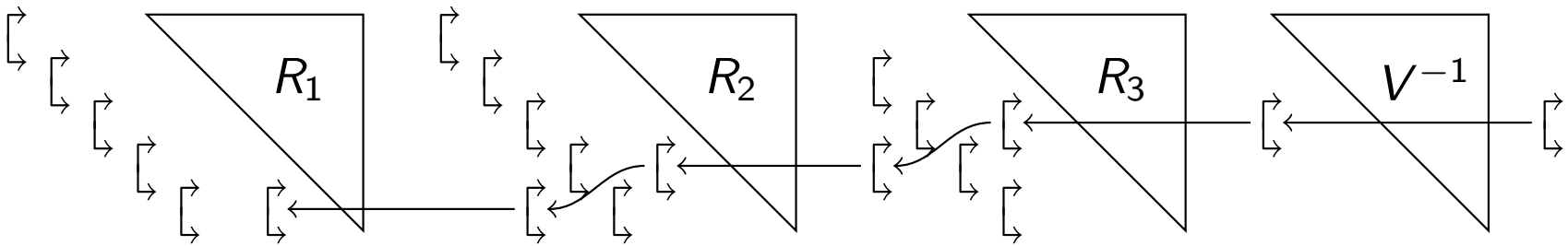

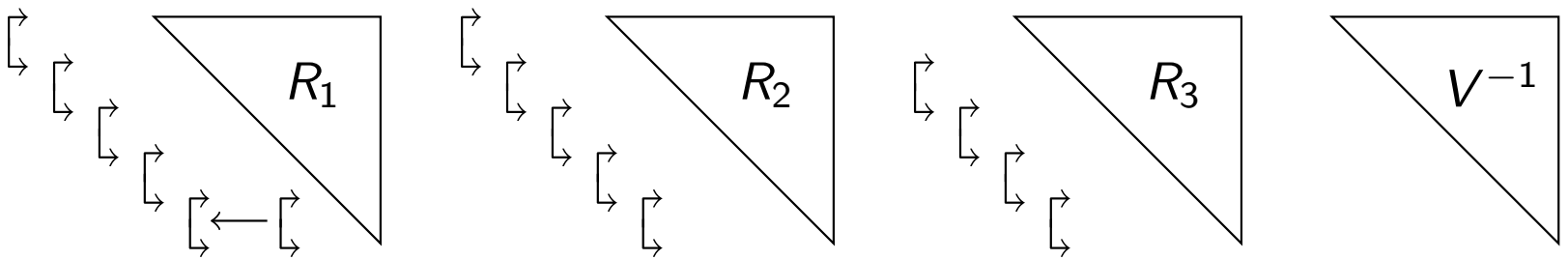

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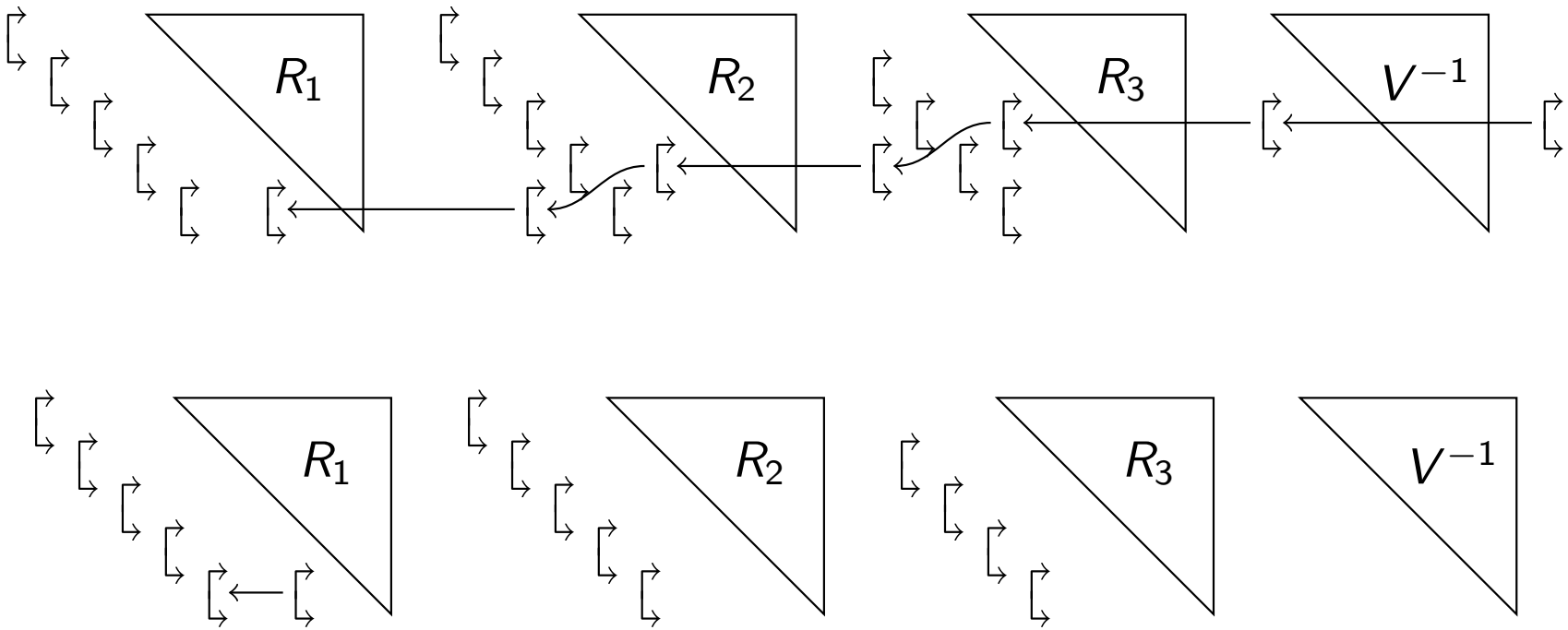

$\mathcal{O}(md)$ turnovers per core transformation and $\mathcal{O}(m^2d)$ transformations to annihilate $\implies$ $\mathcal{O}(m^3d^2)$ flops.

Chasing core transformations

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$$\tilde{W}\tilde{V}^{-1} = \underbrace{\quad}_{\tilde{\mathcal{Q}}} \underbrace{\quad}_{\tilde{R}_1\tilde{R}_2\tilde{R}_3\tilde{V}_3^{-1}\tilde{V}_2^{-1}\tilde{V}_1^{-1}} .$$

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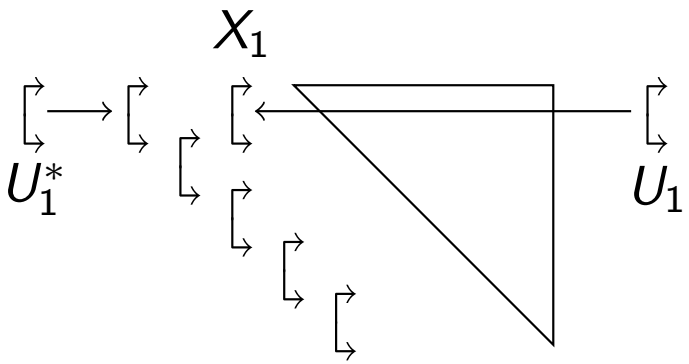
We compute the first transformation U_1:



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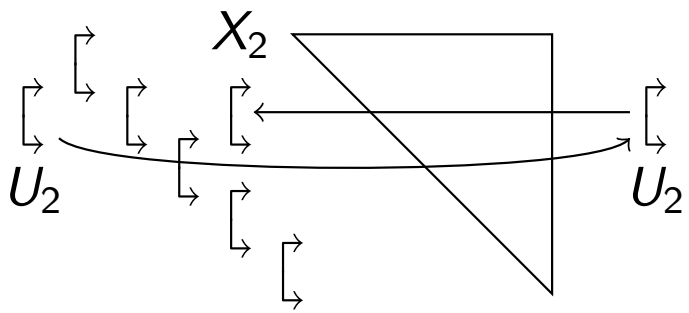
and then we chase it by executing a turnover and similarity:



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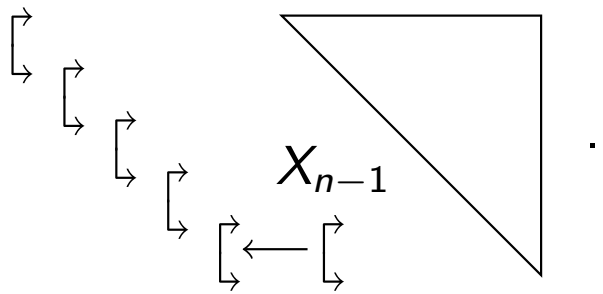
until we fuse it at the bottom:



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A quick flop count

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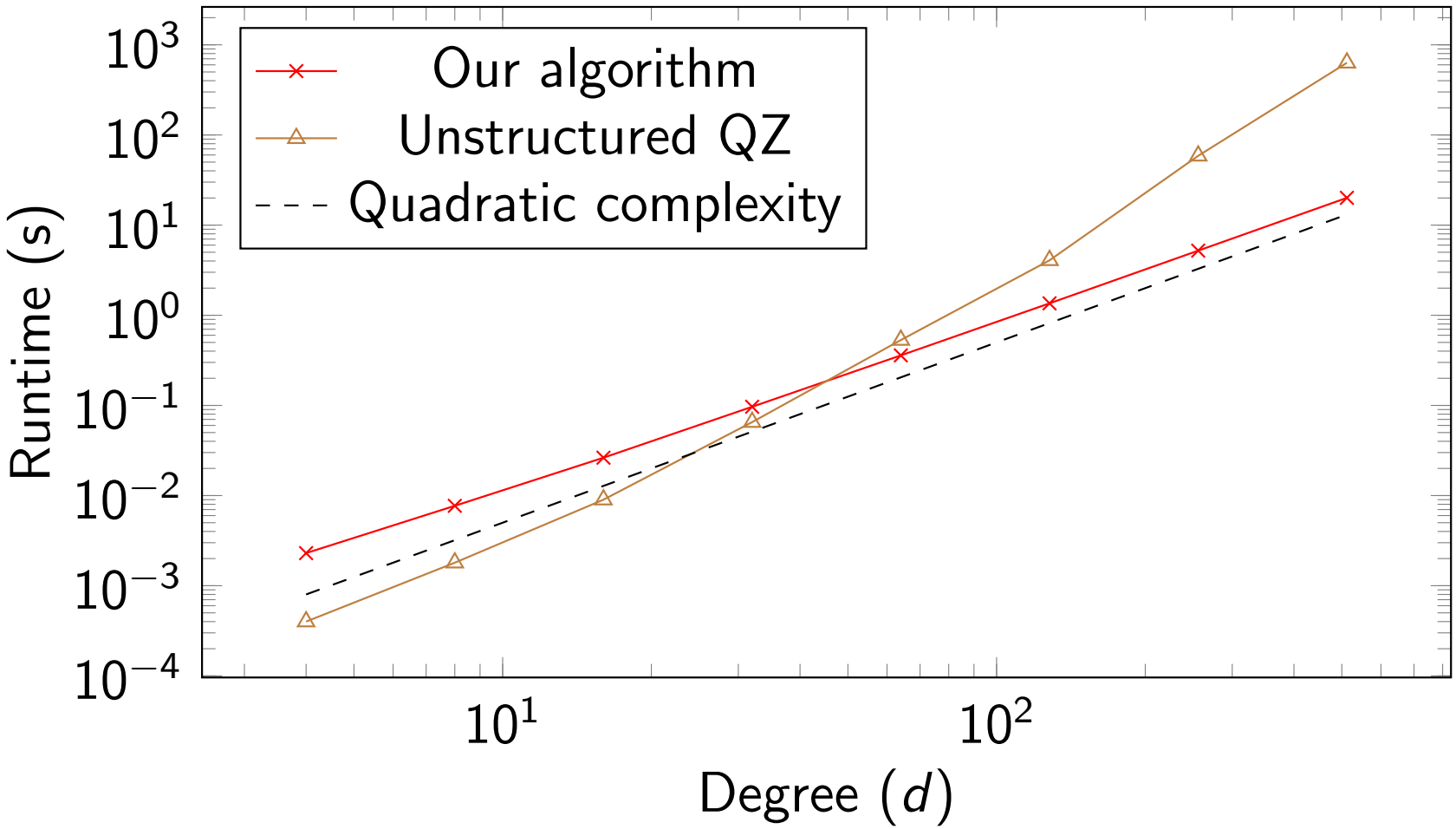
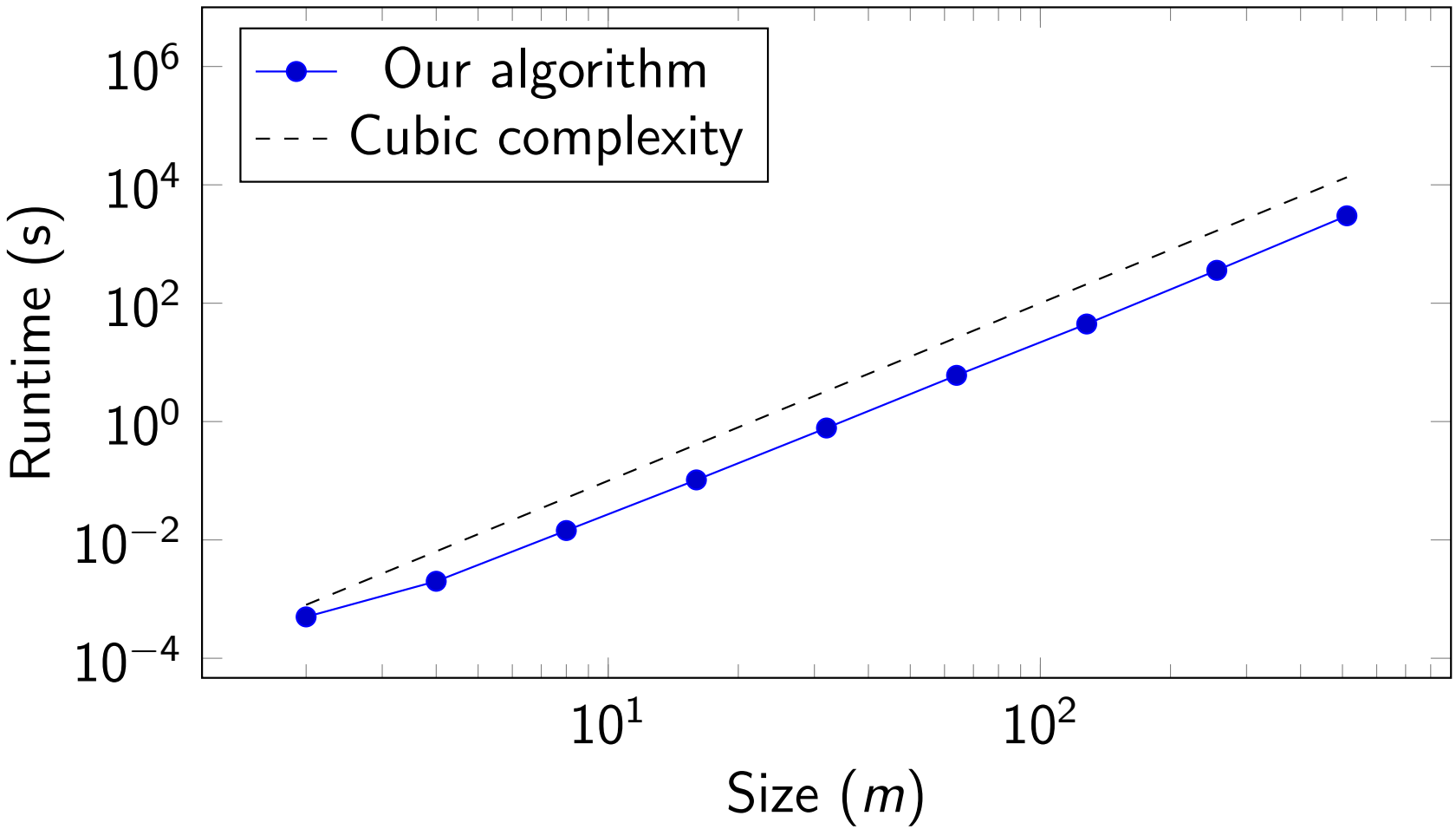
Assuming $\mathcal{O}(md)$ steps are required to reach convergence, the cost is $\mathcal{O}(m^3 d^2)$.

Complexity in d (4 x 4 matrix polynomial)



Complexity in m ($d = 10$)



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- We can prove that the rank 1 part (of each factor) can be recovered from the unitary one.
- $\mathcal{O}(u)$ backward error on the unitary part introduces a $\mathcal{O}(\|y_j\|^2 u)$ backward error on the rank 1 part.
- Total backward error of the Schur form (on the original matrix pencil): $\mathcal{O}(\|V\|^2 u)$ and $\mathcal{O}(\|W\|)^2 u)$.

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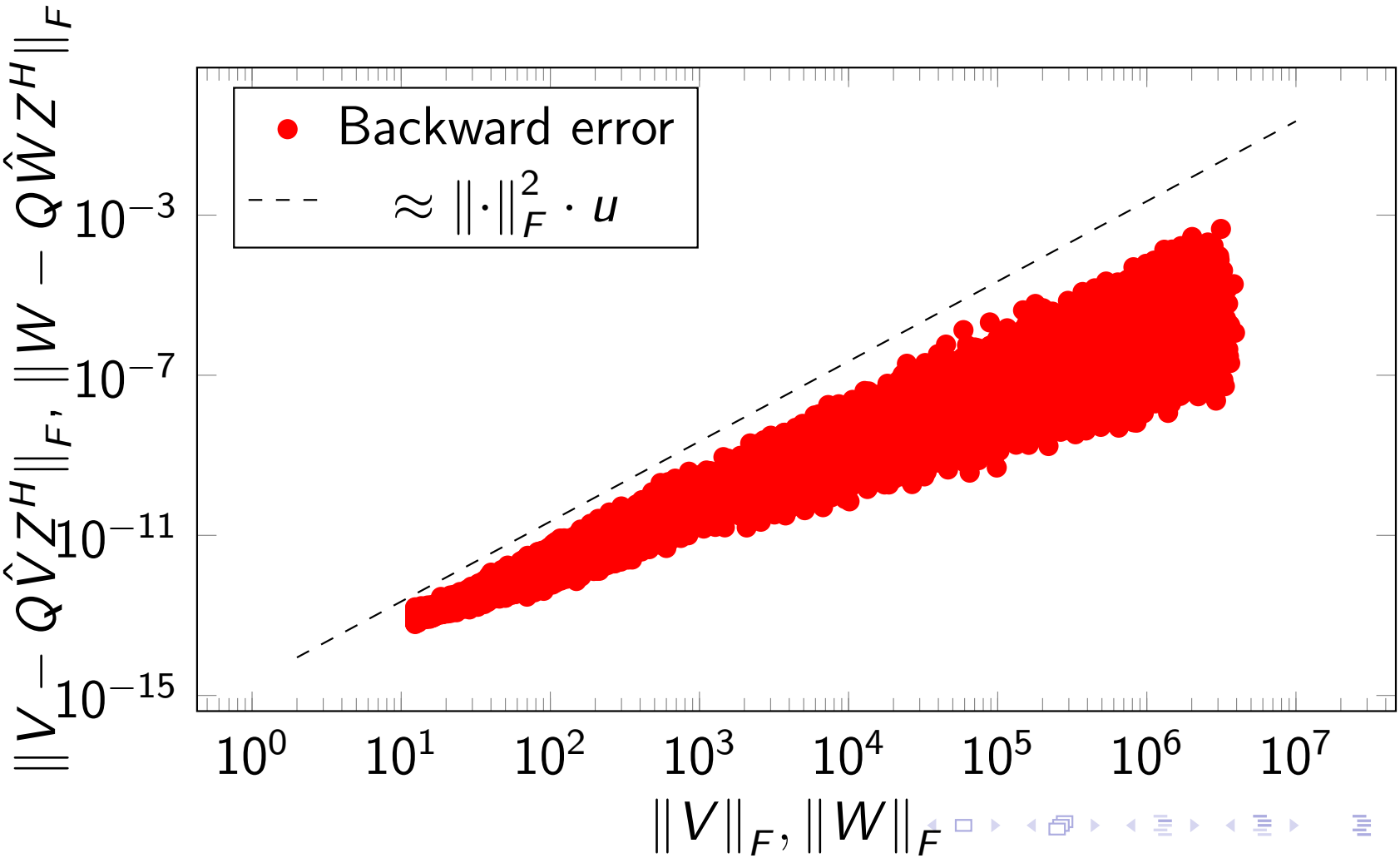

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Thank you for your attention!