Gauss quadrature for quasi-definite linear functionals

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- S. Pozza, M.P. and Z. Strakos: Gauss quadrature for quasi-definite linear functionals, IMA J. Numer. Anal. (2016)

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- $\mathcal{L}$ is said to be quasi-definite on $\mathcal{P}_k$ if $\Delta_j \neq 0$, for $j = 0, 1, \ldots, k$.

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- G3: The Gauss quadrature of a function f can be written in the form

$$m_0 \mathbf{e}_1^T f(J_n) \mathbf{e}_1.$$

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- Unlike in the positive-definite case, their coefficients are not necessarily real, the coefficients in the three-term recurrence relation are, in general, complex, and zeros of OP can be complex and multiple.

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- J_n - complex Jacobi matrix: three-diagonal, symmetric, no zeros on sub-diagonal

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- Should we call it Gauss quadrature? (G1, G2 and G3)

Theorem

Quasi-definitness of the linear functional $\mathcal{L}$ is the necessary and sufficient condition for the quadrature

$$\mathcal{L}(f) = \sum_{i=1}^{\ell} \sum_{j=0}^{s_i-1} \omega_{i,j}\, f^{(j)}(\lambda_i) + R_n(f)\,.$$

to have all three properties G1, G2 and G3. For non-definite linear functionals all three properties cannot hold.

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- The zeros of π_2 are $x_1 = x_2 = 2$, which means that the Gauss quadrature in the standard form does not exist. In other words, the nonlinear system

$$A_1\, z_1^k + A_2\, z_2^k = m_k, \quad k = 0, 1, 2, 3,$$

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- $J_3 = \begin{bmatrix} 3 & i & 0 \\ i & 1 & 2i \\ 0 & 2i & 3 \end{bmatrix}$
- J_3 is diagonalizable, J_2 is not diagonalizable

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- For $n = 2$ the Gauss quadrature is of the form $A_1 f(2) + A_2 f'(2)$. It is easy to check that the nonlinear system

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- The functional $\mathcal{L}_1$ whose first five moments are

$$m_0 = 1,\ m_1 = 3,\ m_2 = 8,\ m_3 = 20,\ m_4 = 48,$$

  is not quasi-definite on $\mathcal{P}_2$. If $m_5 = 2^5 + 5 \cdot 2^4 = 112$ then the quadrature $f(2) + f'(2)$ would have degree of exactness at least 5.

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- $$\mathbf{v}_t = \sum_{k=j}^{s_i - 1} k!\, \omega_{i,k}\, \mathbf{w}^{i,k-j}$$
- i and j are unique integers such that

$$t = s_0 + s_1 + \cdots + s_{i-1} + j + 1$$

with $s_0 = 0$.

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- $$\tilde{\ell} = \ell,\;\; \tilde{s}_i = s_i,\;\; \frac{A_{i,j}}{j!} = \omega_{i,j},\;\; \alpha_i = \lambda_i \;\; i = 1, \ldots, \ell,\;\; j = 0, \ldots, s_i - 1$$

THANK YOU FOR YOUR ATTENTION!