Identification of hydraulic conductivity in the unsteady case

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25 octobre 2016

Plan

Introduction:

- saltwater intrusion problem
- control problem
- Definition of the set of admissible parameters

Main results:

- Existence of the optimal control solution
- Existence and uniqueness of the adjoint problem solution
- Existence of the optimality system solution

Numerical results

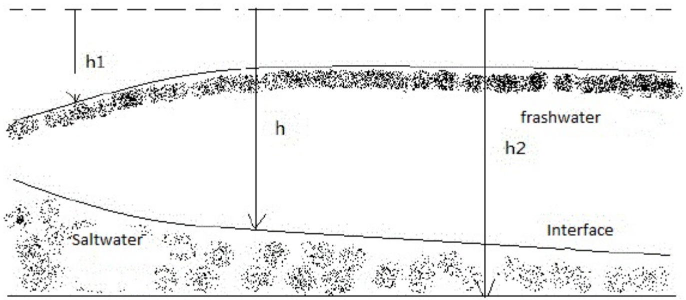
- Algorithm
- Experiments

Saltwater intrusion problem - case of unconfined aquifer with diffuse interface

The saltwater intrusion problem is defined by:

$$\phi\partial_t h - \nabla\cdot\left(\alpha K T_s(h)\nabla h\right) - \nabla\cdot\left(\delta\nabla h\right) - \nabla\cdot\left(K T_s(h)\mathcal{X}_0(h_1)\nabla h_1\right) = Q_s, \quad (1)$$

$$\phi\partial_t h_1 - \nabla\cdot\left(K\left(T_f(h-h_1) + T_s(h)\right)\right)\mathcal{X}_0(h_1)\nabla h_1\right) - \nabla\cdot\left(\delta\nabla h_1\right) - \nabla\cdot\left(\alpha K T_s(h)\nabla h\right)$$
$$= Q_f + Q_s, \quad (2)$$



Saltwater intrusion problem - case of unconfined aquifer with diffuse interface

Boundary and initial conditions

with the following initial and boundary conditions :

$$\begin{cases} h = h_D \quad , \quad h_1 = h_{1,D} \quad \text{on} \quad \Gamma \times (0,T), \\ h(0,x) = h_0(x) \quad , \quad h_1(0,x) = h_{1,0}(x) \quad \text{in} \quad \Omega, \end{cases} \tag{3}$$

and

$$\begin{array}{ll} 0 \leq h_{1,D} \leq h_D \leq h_2 \quad , & \text{a.e. in} \ \ \Gamma \times (0,T), \\ 0 \leq h_{1,0} \leq h_0 \leq h_2 \quad , & \text{a.e. in} \ \Omega. \end{array} \tag{4}$$

such that $(Q_s, Q_f) \in \ (L^2(0,T;H))^2$, and
$(h_D, h_{1,D}) \in (L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')^2$, while $(h_0, h_{1,0}) \in (H^1(\Omega))^2$.

The control problem

The control problem is defined by:

$$(\mathcal{O}_c)\begin{cases}\text{find } K^* \in U_{adm} \text{ such that} \\ \mathcal{J}(K^*) = \inf_{K\in U_{adm}} \mathcal{J}(K),\end{cases}$$

with

$$\mathcal{J}(K) = \frac{1}{2}||h_1(K) - h_{1,obs}||^2_{L^2(\Omega_T)} + \frac{1}{2}||h(K) - h_{obs}||^2_{L^2(\Omega_T)},$$

and

$$U_{adm} = \{K \in BV(\Omega) \cap L^\infty(\Omega), K_m \leq K \leq K_M \text{ et } TV(K) \leq c\},$$

where K_m and K_M are strictly positive real constants. $(BV(\Omega); ||.||_{BV(\Omega)})$ is the space of functions with bounded variation on Ω.

Existence of optimal control

Theorem 1:

There exists at least one optimal control for the problem $(\mathcal{O}_c)$.

Existence of optimal control

Proof

- Let $(K_n)_{n\in\mathbb{N}} \subset U_{adm}$ be a minimizing sequence such that

$$\mathcal{J}(K_n) \longrightarrow \mathcal{J}^* = \inf_{K\in U_{adm}} \mathcal{J}(K).$$

- U_{adm} is a compact subset of $L^2(\Omega)$, so

$$K_n \longrightarrow K^* \text{ strongly in } L^2(\Omega).$$

- From the existence theorem of the exact solution, $(h_1^n, h^n) = (h_1(K_n), h(K_n))$, satisfies:

$$||h_1^n||_{L^2(0,T,H^1(\Omega))} + ||h^n||_{L^2(0,T,H^1(\Omega))} \leq C,$$

$$||\partial_t h^n||_{L^2(0,T,V')} \leq C,$$

$$||\partial_t h_1^n||_{L^2(0,T,V')} \leq C,$$

where C is a constant independent of n.

Existence of optimal control
Proof

- We deduce from Aubin compactness that there exist $(h_1^*, h^*) \in W(0,T)^2$ such that :

$$h^n \longrightarrow h^* \text{ in } L^2(0,T;H) \text{ and a.e. in } [0,T]\times\Omega,$$
$$\partial_t h^n \longrightarrow \partial_t h^* \text{ weakly in } L^2(0,T;V'),$$
$$h_1^n \longrightarrow h_1^* \text{ in } L^2(0,T;H) \text{ and a.e. in } [0,T]\times\Omega,$$
$$\partial_t h_1^n \longrightarrow \partial_t h_1^* \text{ weakly in } L^2(0,T;V'),$$

  With

$$W(0,T) := \left\{\omega \in L^2(0,T;V),\ \partial_t\omega \in L^2(0,T;V')\right\}$$

  and

$$V = H_0^1(\Omega),$$

- From the passage to limit in the variational formulation of the exact problem and from the uniqueness of the exact solution, we obtain

$$(h_1^*, h^*) = (h_1(K^*), h(K^*)) \text{ and } \mathcal{J}(K^*) = \mathcal{J}^*.$$

optimality conditions

Introduce the Lagrangian $\mathcal{L}$

We introduce the Lagrangian $\mathcal{L}$ defined as follows:

$$
\begin{aligned}
\mathcal{L} \quad & (h_1, h, \lambda_f, \lambda_i, K) = \mathcal{J}(K) + \int_{t_0}^{t_f}\int_\Omega \phi \frac{\partial h}{\partial t}\lambda_i \, dxdt \\
+ \quad & \int_{t_0}^{t_f}\int_\Omega (\delta + \alpha K(x) T_s(h))\nabla h \cdot \nabla \lambda_i \, dxdt + \int_{t_0}^{t_f}\int_\Omega K(x)\, T_s(h)\, \mathcal{X}_0(h_1)\nabla h_1 \cdot \nabla \lambda_i \, dxdt \\
+ \quad & \int_{t_0}^{t_f}\int_\Omega \phi \frac{\partial h_1}{\partial t}\lambda_f \, dxdt + \int_{t_0}^{t_f}\int_\Omega [K(x)\,(T_f(h-h_1) + T_s(h)) + \delta]\mathcal{X}_0(h_1)\nabla h_1 \cdot \nabla \lambda_f \, dxdt \\
+ \quad & \int_{t_0}^{t_f}\int_\Omega \alpha K(x)\, T_s(h)\nabla h \cdot \nabla \lambda_f \, dxdt - \int_{t_0}^{t_f}\int_\Omega Q_s\, \lambda_i \, dxdt \\
- \quad & \int_{t_0}^{t_f}\int_\Omega (Q_s + Q_f)\lambda_f \, dxdt.
\end{aligned}
\tag{5}
$$

Existence and uniqueness of the adjoint problem

Introduce the adjoint system

The state system is given by:

$$\begin{cases} -\phi\dfrac{\partial h}{\partial t} + div\left((\delta + \alpha K T_s(h))\nabla h\right) + div\left(K(x)(h_2 - h)\mathcal{X}_0(h_1)\nabla h_1\right) = Q_s \\ -\phi\dfrac{\partial h_1}{\partial t} + div\left((K(x)(T_f(h - h_1) + T_s(h)) + \delta)\mathcal{X}_0(h_1)\nabla h_1\right) + div\left(\alpha K(x) T_s(h)\nabla h\right) \\ = Q_f + Q_s \end{cases}$$

The associated adjoint state system is given by the following retrograde system:

$$\begin{cases} -\phi\dfrac{\partial \lambda_i}{\partial t} - div((\delta + \alpha K T_s(h))\nabla\lambda_i) - \alpha K(x)\nabla h\cdot\nabla\lambda_i + K(x)\nabla h_1\cdot\nabla\lambda_i, \\ -div(\alpha K(x) T_s(h)\nabla\lambda_f) - \alpha K(x)\nabla h\cdot\nabla\lambda_f = h_{obs} - h, \\ -\phi\dfrac{\partial \lambda_f}{\partial t} - div(K(x)(h_2 - h)\nabla\lambda_i) - div((K(x)(T_f(h - h_1) + T_s(h)) + \delta)\nabla\lambda_f) \\ -K(x)\,\nabla h_1\cdot\nabla\lambda_f = h_{1,obs} - h_1, \end{cases}$$

$$\lambda_i = 0,\ \lambda_f = 0 \ \text{ on } \Gamma_D,\ \ \lambda_i(t_f, x) = 0,\ \ \lambda_f(t_f, x) = 0,\ \ \forall x \in \mathbb{R}.$$

Existence and uniqueness of the adjoint problem

Theorem

Let $(h_1, h) = (h_1(K), h(K))$ the exact solution associated with the hydraulic conductivity $K \in U_{adm}$, the adjoint problem defined by:

$$\left\{\begin{array}{l} Find \quad (\lambda_i, \lambda_f) \in W(0,T)^2 \text{ such that } \forall (u_f, u_i) \in H_0^1(\Omega)^2 : \\ \int_{\Omega_T} [-\phi \frac{\partial \lambda_i}{\partial t} u_i + (\delta + \alpha K(x) T_s(h)) \nabla \lambda_i \cdot \nabla u_i - \alpha K(x) \nabla h \cdot \nabla \lambda_i \cdot u_i] dxdt \\ + \int_{\Omega_T} [\alpha K(x)\, T_s(h) \nabla \lambda_f \cdot \nabla u_i - K(x) \nabla h_1 \cdot \nabla \lambda_i \cdot u_i - \alpha K(x) \nabla h \cdot \nabla \lambda_f \cdot u_i dxdt \\ = \int_{\Omega_T} (h_{obs} - h) u_i dxdt, \\ \int_{\Omega_T} [-\phi \frac{\partial \lambda_f}{\partial t} u_f + K(x)(h_2 - h) \nabla \lambda_i \cdot \nabla u_f - \int_{\Omega_T} K(x) \nabla h_1 \cdot \nabla \lambda_f \cdot u_f dxdt \\ + \int_{\Omega_T} (K(x)(T_f(h - h_1) + T_s(h)) + \delta) \nabla \lambda_f \cdot \lambda u_f] dxdt \\ = \int_{\Omega_T} (h_{1,obs} - h_1) u_f \, dxdt, \end{array}\right. \tag{6}$$

has a unique solution.

Existence of the solution of the optimality system

Theorem

Let K^* be a solution of problem $(\mathcal{O}_c)$, there exists a couple $(h^* - h_D, h_1^* - h_{1,D}) \in W(0,T)^2$ and a couple $\lambda^* = (\lambda_i^*, \lambda_f^*) \in W(0,T)^2$ satisfying the optimality system determined by the direct problem, the adjoint problem (6) and, for all $K \in U_{adm}$

$$D_K \mathcal{J}(K) \cdot (K(x) - K^*(x)) \geq 0.$$

Where the gradient of the cost function is given by:

$$D_K \mathcal{J}(K^*)\, \delta_K = \int_{t_0}^{t_f} \int_\Omega \delta_K\, T_s(h^*)\, (\alpha \nabla h^* + \nabla h_1^*) \cdot \nabla \lambda_i^* \, dxdt$$

$$+ \int_{t_0}^{t_f} \int_\Omega \delta_K \left((h_2 - h_1^*) \nabla h_1^* + \alpha\, T_s(h^*) \nabla h^* \right) \cdot \nabla \lambda_f^* \, dxdt, \quad \text{with} \quad \delta_K \in U_{adm}.$$

Existence of the solution of the optimality system

Proof

- We introduce the application $\mathcal{Q}: \ K \longrightarrow (h(K), h_1(K))$ implicitly defined by the direct problem,
- we thus define the mapping

$$\begin{array}{rcl} \mathcal{R}: \ Z(0,T)^2 \times Int(U) & \longrightarrow & L^2(0,T;H^{-1}(\Omega)) \\ (\bar{h_1}, \bar{h}, K) & \longrightarrow & \mathcal{R}(\bar{h_1}, \bar{h}, K) \end{array}$$

where

$$Z(0,T) = W(0,T) \cap L^\infty(0,T;L^2(\Omega))$$

and

$$U = \{K \in BV(\Omega) \cap L^\infty(\Omega), K_m \leq K \leq K_M \text{ and } TV(K) \leq C\}, \text{ with } c < C,$$

Where the constant c is the constant defining U_{adm} and $(\bar{h_1}, \bar{h}) = (h_1 - h_{1,D}, h - h_D)$.

Existence of the solution of the optimality system
Proof

such that $\forall (\varphi_i, \varphi_f) \in L^2(0, T; H_0^1(\Omega))^2$, we have

$$
\begin{aligned}
& < \mathcal{R}(\bar{h}_1, \bar{h}, K), (\varphi_i, \varphi_f) > \\
= \;& \int_{t_0}^{T} \int_\Omega \phi \frac{\partial h}{\partial t} \varphi_i \, dxdt + \int_{t_0}^{T} \int_\Omega \phi \frac{\partial h_1}{\partial t} \varphi_f \, dxdt \\
+ \;& \int_{t_0}^{T} \int_\Omega (\delta + \alpha K(x) T_s(h)) \nabla h \cdot \nabla \varphi_i \, dxdt + \int_{t_0}^{T} \int_\Omega K(x)\, T_s(h)\, \mathcal{X}_0(h_1)\, \nabla h_1 \cdot \nabla \varphi_i \, dxdt \\
+ \;& \int_{t_0}^{T} \int_\Omega (\delta + K(x)(T_s(h) + T_f(h - h_1))\mathcal{X}_0(h_1)\,)\, \nabla h_1 \cdot \nabla \varphi_f \, dxdt \\
+ \;& \int_{t_0}^{T} \int_\Omega \alpha\, K(x)\, T_s(h)\, \nabla h \cdot \nabla \varphi_f \, dxdt - \int_{t_0}^{T} \int_\Omega Q_s \varphi_i \, dxdt + \int_{t_0}^{T} \int_\Omega (Q_s + Q_f) \varphi_f \, dxdt.
\end{aligned}
\tag{7}
$$

Existence of the solution of the optimality system
Proof

- T_s and T_f belong to $L^\infty(\Omega_T)$, so the continuity of $\mathcal{R}$ and $D_K\mathcal{R}(\bar{h_1}, \bar{h}, K)$ is clear,
- We use the regularity of exact solution to demonstrate the continuity of $D_{(\bar{h_1},\bar{h})}\mathcal{R}(\bar{h_1}, \bar{h}, K)$,
- $D_{(\bar{h_1},\bar{h})}\mathcal{R}(\bar{h_1}, \bar{h}, K)$ is an isomorphism,
- Applying the theorem of implicit function, we state that the application $\mathcal{Q}$ is continuous and differentiable from U_{adm} to $Z(0, T)$.

Existence of the solution of the optimality system
Proof

- The application $K \longrightarrow \mathcal{J}(K)$ is differentiable and

$$D_K\mathcal{J}(K^*) = \partial_K \mathcal{L}(K^*, h(K^*), h_1(K^*), \lambda_i^*, \lambda_f^*),$$

with

$$D_K\mathcal{J}(K^*)(\delta_K) = \int_{t_0}^{t_f}\int_\Omega \delta_K T_s(h(K^*))\,(\alpha\nabla h(K^*) - \nabla h_1(K^*) \cdot \nabla\lambda_i^*\, dxdt$$
$$+ \int_{t_0}^{t_f}\int_\Omega \delta_K\,((h_2 - h_1(K^*))\nabla h_1(K^*) - \alpha\, T_s(h(K^*))\nabla h(K^*)) \cdot \nabla\lambda_f^*\, dxdt, \quad \forall\delta_K \in U_{adm}$$

- Furthermore, if K^* is a minimum of $\mathcal{J}$, we have

$$\partial_K \mathcal{L}(K^*, h^*, h_1^*, \lambda_i^*, \lambda_f^*)(K - K^*) \geq 0, \forall K \in U_{adm},$$

where

$$h^* = h(K^*),\ h_1^* = h_1(K^*),\ \lambda_i^* = \lambda_i(K^*, h^*, h_1^*),\ \lambda_f^* = \lambda_f(K^*, h^*, h_1^*).$$

Algorithm

Data:

- K_0 initial shooting;
- $H_0 = I$ a first approximation of the inverse of the Hessian matrix,

Exit: a parameter K verifies $||\nabla \mathcal{J}(K)|| \leq \epsilon$.
while $||\nabla \mathcal{J}(K_i)|| > \epsilon$, do:

- $d_1 = H_i * \nabla \mathcal{J}(K_i)$,
- if $(-d_1, \nabla \mathcal{J}(K_i)) \geq 0$, put $d_i = d_1$, else $d_i = -\nabla \mathcal{J}(K_i)$,
- calculate s_i (the step length) verifies the Wolf conditions using the line search algorithm,

Algorithm

- $K_{i+1} = K_i + s_i * d_i$,
- calculate $\mathcal{J}(K_{i+1})$ and $\nabla\mathcal{J}(K_{i+1})$,
- calculate y_i and c_i:
$$y_i = \nabla\mathcal{J}(K_{i+1}) - \nabla\mathcal{J}(K_i);$$
and
$$c_i = K_{i+1} - K_i;$$
- $$H_{i+1} = (I - \frac{c_i y_i^T}{y_i^T c_i})\ H_i\ (I - \frac{y_i c_i^T}{y_i^T c_i}) + \frac{c_i c_i^T}{y_i^T c_i}.$$

The line search algorithm

Data: s_0 the initial step length,
Exit: s_i a step length satisfy the Wolf conditions.
The Wolf conditions is:

- The function $\mathcal{J}$ must increase significantly:

$$\mathcal{J}(K_i + s_i d_i) \leq \mathcal{J}(K_i) + W_1 s_i \cdot \nabla\mathcal{J}(K_i)d_i \qquad (A)$$

- The step length s_i must be large enough:

$$(\nabla\mathcal{J}(K_i + s_i d_i))^T \cdot d_i \geq W_2 \nabla\mathcal{J}(K_i)d_i \qquad (B)$$

where $W_1 = 10^{-4}$ and $W_2 = 0.99$.

The line search algorithm

- Set $s_1 = 0$ and $s_2 = \infty$,
- we take

$$s_0 = \frac{-2\Delta^{(i)}}{\nabla \mathcal{J}(K_i) \cdot d_i} \qquad (\textit{fletcher} \quad \textit{step}),$$

with $\Delta^{(i)} = \gamma\left(\mathcal{J}(K_i) - \mathcal{J}_{min}\right)$ and γ of the order 10^{-2} or 10^{-1}.
For $k = 0, 1, ...$

a. if s_i does not satisfy the Wolf condition (A) :

- decreases the upper bound : $s_2 = s_i$;
- Choosing a new step length: $s_i = \frac{1}{2}(s_1 + s_2)$.

b. if s_i satisfy the Wolf condition (A) and does not satisfy (B):

- increasing the lower bound: $s_1 = s_i$,
- Choosing a new step length: $s_i = \frac{1}{2}(s_1 + s_2)$,

Experience 1

Table: experience 1

Case	Number of wells	exact values	initial values	identified values
i	1	K= 50 m/d	K0= 70 m/d	K1= 49.6266 m/d
ii	2	K= 50 m/d	K0= 70 m/d	K2= 50.3181 m/d
iii	3	K= 50 m/d	K0= 70 m/d	K3= 50.0101 m/d

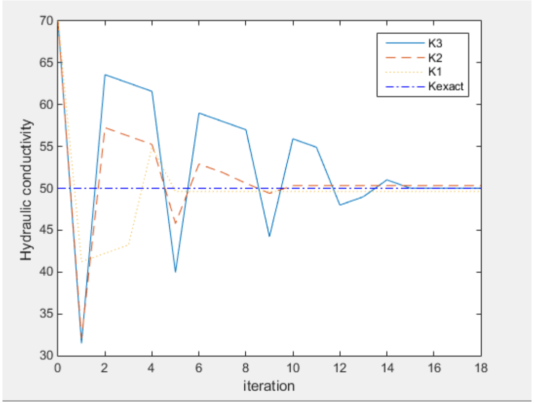


Figure: Graph representing the convergence of the hydraulic conductivity in the experience 1.

Experience 2

Table: experience 2

Case	Number of wells	exact values	initial values	identified values
i	2	K1= 50 m/d K2= 90 m/d	K1= 60 m/d K2= 100 m/d	K1= 49.490 m/d K2= 91.115 m/d
ii	4	K1= 50 m/d K2= 90 m/d	K1= 60 m/d K2= 100 m/d	K1= 49.963 m/d K2= 90.090 m/d

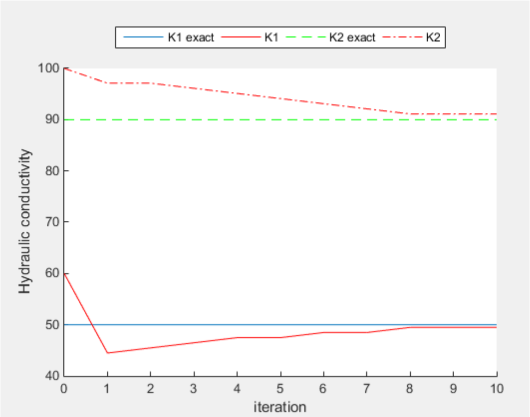


Figure: Graph representing the convergence of hydraulic conductivity in Experience 2 for case (i).

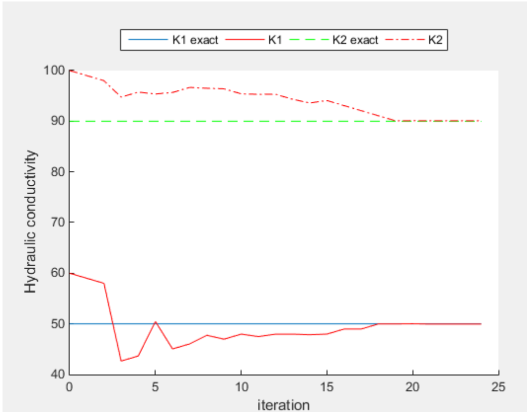


Figure: Graph representing the convergence of hydraulic conductivity in Experience 2 for case (ii).

Conclusion:
We solved parameter identification problem by the adjoint method. We are interested in the identification of the hydraulic conductivity K. We estimated that parameter in terms of the observations or the measures on the ground, made on the depth of the interface between the saturated zone and the dry area, and on the depth of the interface freshwater/saltwater.
Perspectives:

- Studying the saltwater intrusion problem considering that parameters such as hydraulic conductivity and porosity are stochastic,
- compare the results for problem of saltwater intrusion as the parameters are deterministic or stochastic.

Thank you