Computing the Least Common Multiple of Polynomial Sets

Marilena Mitrouli
Department of Mathematics
National and Kapodistrian University of Athens

Dimitrios Christou
Department of Science and Mathematics
Deree – The American College of Greece

Nicos Karcanias
Systems and Control Centre
City, University of London

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GCD and LCM of polynomials

- **Computation of the Greatest Common Divisor (GCD) of polynomials.**
  - It is linked with the computation of zeros of system representations.
  - Solution of polynomial (matrix) Diophantine equations and applications to control design problems, i.e. computation of stabilizing controllers.
  - Network theory, Communications, Computer Aided Design, Image restoration, Signal processing.

- **Computation of the Least Common Multiple (LCM) of polynomials.**
  - It is linked with the derivation of minimal representations of rational models.
  - Integral part of algebraic synthesis methods in control design.
    - Squaring down problem.
    - Pole assignment by dynamic precompensation.
    - Approximate Pole –Zero cancelations in systems and almost zeros.

GCD and LCM of polynomials

Given a pair of real polynomials $P = \{\ p_1(s), p_2(s)\ \}$ the GCD and LCM problems are naturally interlinked:

$$\boldsymbol{p_1(s) \cdot p_2(s) \;=\; \mathbf{gcd}\{P\} \cdot \mathbf{lcm}\{P\}}$$

Fundamental differences:

1. For an arbitrary set of polynomials the existence of a non-trivial GCD is not always guaranteed (*non-generic property*), but the LCM always exists (*generic property*).

2. The degree of the GCD is always less than (or equal to) the minimum degree of the original polynomials, but the degree of the LCM is always greater than (or equal to) the maximum degree of the original polynomials.

Computation of the ALCM of polynomials

D. Christou, N. Karcanias, and M. Mitrouli, Approximate Least Common Multiple of several polynomials using the ERES Division algorithm, *Linear Algebra and its Applications*, 511, (2016), 141-175

Given a set of several real polynomials in one variable, we aim to:

1. Provide a definition for the *approximate LCM* (*ALCM*) of sets of several polynomials.

2. Develop a method for the computation of the ALCM by avoiding root-finding and GCD computations.

3. Develop an efficient numerical procedure for the computation of the ALCM which allows the derivation of approximate solutions when inaccurate data are given.

Approximate LCM of polynomial sets

We consider sets of many polynomials of the form:

$$\mathcal{P}_{h,n} = \Big\{ p_i(s) \in \mathbb{R}[s],\ i = 1, 2, \ldots, h \text{ with } d_i = \deg\{p_i\} > 0 \text{ and } n = \max_i(d_i) \Big\} \quad (1)$$

$$p_i(s) = \sum_{j=0}^{n} c_j^{(i)}\, s^j \qquad\qquad l(s) = \sum_{j=0}^{d} a_j\, s^j$$

Generic form
d = sum of d_i

Since $d_i \leq n < d$ for all $i = 1..h$, according to the Euclidean identity there exist real polynomials $q_i(s)$ (quotients) and $r_i(s)$ (remainders) such that

$$l(s) = q_i(s)\, p_i(s) + r_i(s)$$

with $\deg\{q_i\} = d - d_i$ and $\deg\{r_i\} \leq d_i - 1$.

Example: LCM matrix representation

$$\begin{cases} p_1(s)=2s^2-14s+20, \\ p_2(s)=2s^2-22s+36, \\ p_3(s)=s-3 \end{cases}$$

d=5, n=2

$$\ell(s) = \alpha_5 s^5 + \alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

$$\left[\begin{array}{cccccccc}
\alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 \\
0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\
0 & 0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\
\hline
\alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 \\
0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\
0 & 0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\
\hline
0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\
0 & 0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0
\end{array}\right]$$

(d_i+1)x $(d+n+1)$

$$\widehat{P} = \left[\begin{array}{cccccccc}
2 & -14 & 20 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -14 & 20 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -14 & 20 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -14 & 20 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -14 & 20 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -14 & 20 \\
\hline
2 & -22 & 36 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -22 & 36 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -22 & 36 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -22 & 36 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -22 & 36 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -22 & 36 \\
\hline
0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3
\end{array}\right]$$

(d+1)x(d+n+1)

$d_i<n$

$$\left[\begin{array}{cccccccccccccccccc}
q_3 & q_2 & q_1 & q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q_3 & q_2 & q_1 & q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_3 & q_2 & q_1 & q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & q'_3 & q'_2 & q'_1 & q'_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q'_3 & q'_2 & q'_1 & q'_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q'_3 & q'_2 & q'_1 & q'_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q''_4 & q''_3 & q''_2 & q''_1 & q''_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q''_4 & q''_3 & q''_2 & q''_1 & q''_0
\end{array}\right]$$

$$\widehat{L} = \widehat{Q} \cdot \widehat{P}$$

Approximate LCM of polynomial sets

Definition 1. Given a set of several polynomials $\mathcal{P}_{h,n}$, the *exact least common multiple* of the polynomials of the set is a polynomial of the smallest possible degree $\ell \leq hn$, such that:

$$\widehat{L} = \widehat{Q} \cdot \widehat{P}$$

Equivalently, using an appropriate matrix norm, denoted by $\|\cdot\|$, the next equality is satisfied:

$$\|\widehat{L} - \widehat{Q} \cdot \widehat{P}\| = 0$$

Definition 2. Given a set of several polynomials $\mathcal{P}_{h,n}$ and a specified small tolerance ε, the *approximate least common multiple* of the set, denoted by ALCM, is a polynomial of the smallest possible degree $\ell \leq hn$, such that the next inequalities are satisfied:

$$\|\widehat{L} - \widehat{Q} \cdot \widehat{P}\| \leq \varepsilon \;\Leftrightarrow\; \|\widehat{R}\| \leq \varepsilon$$

Approximate LCM of polynomial sets

$$P_i \triangleq \mathrm{diag}_{(d+1)}\{\underline{p}_i^t\} = \begin{bmatrix} c_{d_i}^{(i)} & c_{d_i-1}^{(i)} & \dots & c_0^{(i)} & 0 & \dots & 0 \\ 0 & c_{d_i}^{(i)} & c_{d_i-1}^{(i)} & \dots & c_0^{(i)} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & c_{d_i}^{(i)} & c_{d_i-1}^{(i)} & \dots & c_0^{(i)} \end{bmatrix}$$

$$\widehat{P} = \begin{bmatrix} P_1 \\ \vdots \\ P_h \end{bmatrix} \in \mathbb{R}^{(hd+h)\times(d+n+1)}$$

$$Q_i \triangleq \mathrm{diag}_{(d_i+1)}\{\underline{q}_i^t\} \in \mathbb{R}^{(d_i+1)\times(d+1)} \qquad \widehat{Q} = \mathrm{diag}\{Q_1, Q_2, \dots, Q_h\} \in \mathbb{R}^{(d+h)\times(hd+h)}$$

$$R_i \triangleq \left[O_{d+n-d_i} \,|\, \mathrm{diag}_{(d_i+1)}\{\underline{r}_i^t\} \right] \in \mathbb{R}^{(d_i+1)\times(d+n+1)} \qquad \widehat{R} = \begin{bmatrix} R_1 \\ \vdots \\ R_h \end{bmatrix} \in \mathbb{R}^{(d+h)\times(d+n+1)}$$

$$L_i \triangleq \left[O_{n-d_i} \,|\, \mathrm{diag}_{(d_i+1)}\{\underline{a}^t\} \right] \in \mathbb{R}^{(d_i+1)\times(d+n+1)} \qquad \widehat{L} = \begin{bmatrix} L_1 \\ \vdots \\ L_h \end{bmatrix} \in \mathbb{R}^{(d+h)\times(d+n+1)}$$

Computation of the LCM without using the GCD

The Hybrid LCM method

The current approach for the computation of the LCM involves:

A. An appropriate transformation process (symbolic procedure) to formulate a system of linear equations derived from the polynomials of the original set.

B. The formulation of an optimization problem (numerical procedure) to solve the above system of linear equations in the approximate sense (ALCM).

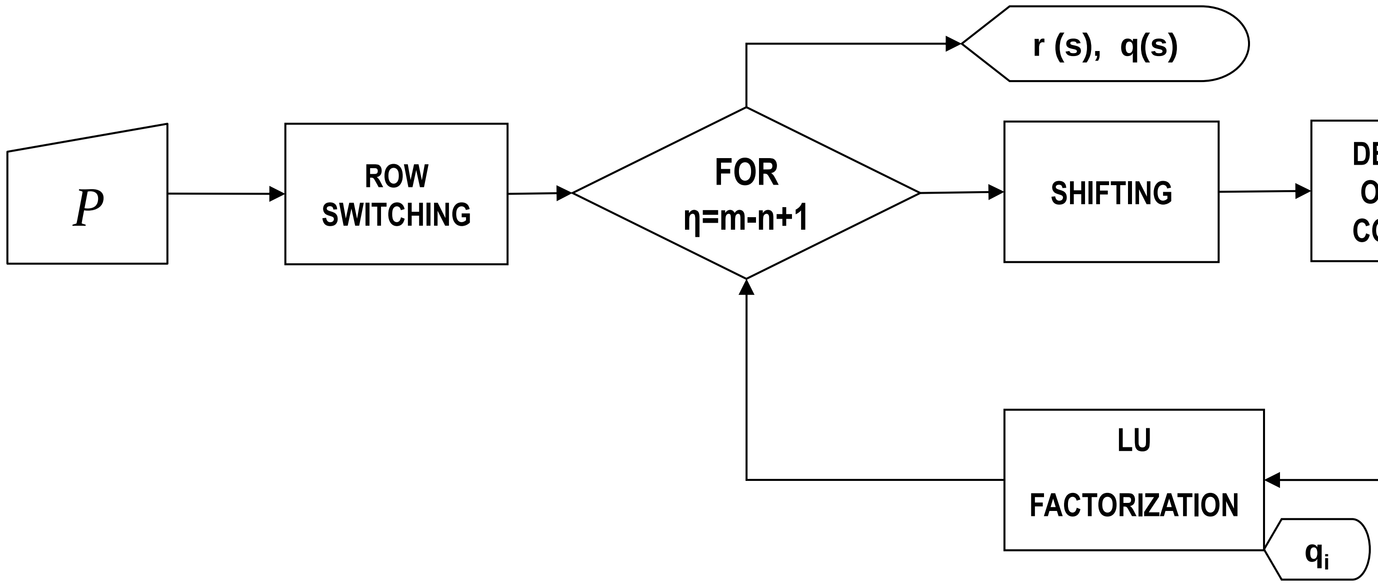
Lemma *Given a set of real polynomials* $\mathcal{P} = \{p_i(s) \in \mathbb{R}[s],\ i = 1, 2, \ldots, h\}$ *the LCM of* $\mathcal{P}$ *is a real polynomial with degree* $\ell \leq \sum_{i=1}^{h} \deg\{p_i(s)\}$ *and every polynomial* $p_i(s)$ *divides evenly into LCM.*

LCM: $l(s) = q_i(s) \cdot p_i(s) + r_i(s)$ *with* $\boxed{r_i(s) = 0}$ ← Remainder

Computation of the ALCM: The ERES Division Algorithm

The ERES Division algorithm* is developed mainly for the symbolic computation of the quotient and the remainder from the division of two polynomials in one variable.

$$\begin{bmatrix} a(s) \\ b(s) \end{bmatrix} \overset{ERES}{\Longrightarrow} \begin{bmatrix} b(s) \\ r(s) \end{bmatrix}$$



* D. Christou, N. Karcanias, and M. Mitrouli, Matrix representation of the shifting operation and numerical properties of the ERES method for computing the greatest common divisor of sets of many polynomials, J. Comput. Appl. Math. 260 (2014) 54-67.

Computation of the ALCM: The ERES Division Algorithm

The *Extended-Row-Equivalence & Shifting (ERES) division algorithm* is an iterative procedure which performs the division of two polynomials by using matrix transformations.

Example: We consider two polynomials $a(s)$ and $b(s)$.

$$\begin{aligned} a(s) &= 2\,s^3 + 3\,s^2 - 7\,s - 32, \qquad & \deg\{a(s)\} = m = 3 \\ b(s) &= s^2 + 4\,s + 5, \qquad & \deg\{b(s)\} = n = 2 \end{aligned}$$

Their coefficients form a matrix P.

$$P = \begin{bmatrix} 2 & 3 & -7 & -32 \\ 0 & 1 & 4 & 5 \end{bmatrix} \in \mathbb{R}^{2\times 4}$$

According to the Euclidean identity, it holds:

$$\frac{a(s)}{b(s)} = \frac{2\,s^3 + 3\,s^2 - 7\,s - 32}{s^2 + 4\,s + 5} = (2\,s - 5) + \frac{3\,s - 7}{s^2 + 4\,s + 5}$$

Computation of the ALCM: The ERES Division Algorithm

After $\eta = m - n + 1$ iterations, the ERES division algorithm provides the 2×2 matrix $\boldsymbol{Q}_\eta$ of the quotient and the $(m+1)\times(n+1)$ matrix $\boldsymbol{S}_\eta$ which corresponds to the iterative application of the shifting operation to the initial matrix P and the deletion of the trailing zero columns.

$$S_\eta = \begin{bmatrix} \frac{13}{2} & \frac{45}{2} & \frac{57}{10} \\ -3 & -5 & \frac{23}{5} \\ 1 & 1 & -\frac{12}{5} \\ 0 & 1 & 2 \end{bmatrix}$$

$$Q_\eta = \begin{bmatrix} 0 & 1 \\ 1 & -\underbrace{(2-5)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

quotient $q(s) = 2\,s - 5$

$$R_\eta = Q_\eta \cdot P \cdot S_\eta = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 3 & -7 \end{bmatrix}$$

Then, the remainder of the division $a(s)/b(s)$ is given by the last row of $\boldsymbol{R}_\eta$

$$r(s) = [0,\ 1] \cdot R_\eta \cdot \begin{bmatrix} s^2 \\ s \\ 1 \end{bmatrix} = 3\,s - 7$$

The Hybrid ALCM algorithm

Given a set of univariate monic polynomials: $\mathcal{P} = \{p_i(s) \in \mathbb{R}[s],\ i = 1, 2, \dots, h\}$

- **Step 1**
  Form the LCM of the polynomial set symbolically for the maximum degree *d* : $l(s) = \sum_{k=0}^{d} a_k\, s^k$ $\begin{cases} d_i = \deg\{p_i(s)\} \\ d = \sum_{i=1}^{h} d_i \end{cases}$

- **Step 2**
  Compute the remainders of the divisions: $\dfrac{l(s)}{p_i(s)}$

$$r_i(s) = r_0^{(i)} + r_1^{(i)}\, s + \ldots + r_{d_i-1}^{(i)}\, s^{d_i-1}$$

by using the ERES Division algorithm: $\begin{bmatrix} l(s) \\ p_i(s) \end{bmatrix} \overset{ERES}{- - - - - >} \begin{bmatrix} r_i(s) \\ p_i(s) \end{bmatrix}$

SYMBOLIC IMPLEMENTATION

The Hybrid ALCM algorithm

ERES Division algorithm $\Longrightarrow$ $\underline{r}_i = \left[r_{d_i-1}^{(i)}, \ldots, r_1^{(i)}, r_0^{(i)} \right] \in \mathbb{R}^{d_i}$

Every coefficient $r_k^{(i)}$ is a linear combination of the coefficients α_k of the LCM.

$$\left.\begin{aligned} \underline{r}_i = \begin{bmatrix} r_{d_i-1}^{(i)} \\ r_{d_i-2}^{(i)} \\ \vdots \\ r_0^{(i)} \end{bmatrix} = \begin{bmatrix} f_{d_i-1,d}^{(i)} a_d + \ldots + f_{d_i-1,1}^{(i)} a_1 + f_{d_i-1,0}^{(i)} a_0 \\ f_{d_i-2,d}^{(i)} a_d + \ldots + f_{d_i-2,1}^{(i)} a_1 + f_{d_i-2,0}^{(i)} a_0 \\ \vdots \\ f_{0,d}^{(i)} a_d + \ldots + f_{0,1}^{(i)} a_1 + f_{0,0}^{(i)} a_0 \end{bmatrix} \\ \Downarrow \\ F_i = \left[\widetilde{F}_i | I_{d_i} \right] = \begin{array}{c} \begin{array}{cccccc} a_d & \ldots & a_{d_i} & a_{d_i-1} & \ldots & a_0 \\ \downarrow & & \downarrow & \downarrow & & \downarrow \end{array} \\ \left[\begin{array}{ccc|ccc} f_{d_i-1,d}^{(i)} & \ldots & f_{d_i-1,d_i}^{(i)} & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{0,d}^{(i)} & \ldots & f_{0,d_i}^{(i)} & 0 & \ldots & 1 \end{array}\right] \end{array} \end{aligned}\right\} \underline{r}_i = F_i \cdot \underline{a}$$

The Hybrid ALCM algorithm

- **Step 3**

  Form a system of linear equations:

$$\left.\begin{array}{l} F_{\mathcal{P}} \cdot \underline{a} = \begin{bmatrix} F_1 \\ \vdots \\ F_h \end{bmatrix} \underline{a} = \begin{bmatrix} \underline{r}_1 \\ \vdots \\ \underline{r}_h \end{bmatrix} \\ \\ r_i(s) = 0, \quad \forall\, i = 1, 2, \dots, h \end{array}\right\} \quad F_{\mathcal{P}} \cdot \underline{a} = \underline{0}$$

$d \times (d+1)$ underdetermined homogeneous linear system

For the matrices $\widehat{R}$ *and* $F_{\mathcal{P}}$ *it holds:* $\quad \|\widehat{R}\|_F \leq \sqrt{n+1}\, \|F_{\mathcal{P}}\, \underline{a}\|_2$

The Hybrid ALCM algorithm

- **Step 4** [NUMERICAL IMPLEMENTATION]

Compute the actual degree and the coefficients of the ALCM.

Let $\rho(F_P)$ and $n(F_P)$ denote the rank and the nullity of F_P , respectively.

The rank of $F_{\mathcal{P}}$ is equal to the degree of the LCM of the set $\mathcal{P}$.

$$\rho(F_{\mathcal{P}}) = \deg\{l(s)\}$$

If $n(F_{\mathcal{P}}) = 1$ only one solution (up to scalar multiples) is obtained and this is actually the generic solution:

$$l(s) = p_1(s) \cdot p_2(s) \cdots p_h(s)$$

If $n(F_{\mathcal{P}}) = \nu > 1$, we can set exactly ν free variables.

The least degree solution will be obtained if we set:

$$a_{d-\nu+1} = 1 \ \text{ and } \ a_{d-\nu+2} = \ldots = a_d = 0$$

Numerical computation of the ALCM

rank(F_p) = ρ

$F_{\mathcal{P}} \cdot \underline{a} = \underline{0}$

d×(d+1) underdetermined homogeneous linear system

$$\Leftrightarrow \check{F}_{\mathcal{P}} \cdot \underline{\check{a}} + \underline{f}_{d-\rho+1} \cdot a_\rho + \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} = \underline{0} \Leftrightarrow$$

first d-ρ columns → $\underline{0}$

last ρ columns → Leading coefficient of the LCM.

$\widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} = -a_\rho \, \underline{f}_{d-\rho+1}$

d×ρ full-rank overdetermined linear system

$$a_\rho = \begin{cases} 1, & \text{if all the polynomials } p_i(s) \text{ are monic, i.e. } c_{d_i}^{(i)} = 1 \\ \operatorname{lcm}\left\{c_{d_i}^{(i)}, i = 1..h\right\}, & \text{if } c_{d_i}^{(i)} \text{ are all integer numbers.} \\ \prod_{i=1}^{h} c_{d_i}^{(i)}, & \text{if } c_{d_i}^{(i)} \text{ are real numbers.} \end{cases}$$

The numerical computation of an approximate LCM can be obtained by solving:

$$\widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} \approx -a_\rho \, \underline{f}_{d-\rho+1} \quad \Leftrightarrow \quad \left\| \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} + a_\rho \, \underline{f}_{d-\rho+1} \right\| = \text{minimum}$$

The above solution can be considered as an approximation in the least-squares sense.

Numerical computation of the ALCM using Least-Squares optimization

Theorem *Let $\mathcal{P}_{h,n}$ a set of real univariate polynomials, as defined by (1) and a small specified tolerance $\varepsilon > 0$. An ALCM of the set $\mathcal{P}_{h,n}$ is given by the solution of the least-squares problem*

$$\mathcal{L} \triangleq \min_{\underline{\hat{a}}} \left\| \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} - \left(- a_\rho \, \underline{f}_{\,d-\rho+1} \right) \right\|_2$$

where ρ is the numerical ε-rank of the matrix $\widehat{F}_{\mathcal{P}}$, a_ρ is the leading coefficient of the ALCM, and $\underline{\hat{a}}$ is the vector of the remaining $\rho - 1$ coefficients of the ALCM.

Example 2

$$\mathcal{P} = \{p_i(s) \in \mathbb{R},\ i = 1, 2, 3\}$$
$$p_1(s) = (s+1)(s+2)^2$$
$$p_2(s) = (s+2)(s+3)(s+4)$$
$$p_3(s) = (s+4)^2(s+5)$$

Maximum degree of LCM : $d = 3+3+3 = 9$

$$\boxed{F_{\mathcal{P}} \cdot \underline{a} = \underline{0}}$$

$$F_{\mathcal{P}} = \begin{bmatrix} -1793 & 769 & -321 & 129 & -49 & 17 & -5 & 1 & 0 & 0 \\ -4868 & 2052 & -836 & 324 & -116 & 36 & -8 & 0 & 1 & 0 \\ -3076 & 1284 & -516 & 196 & -68 & 20 & -4 & 0 & 0 & 1 \\ -111645 & 26335 & -6069 & 1351 & -285 & 55 & -9 & 1 & 0 & 0 \\ -539054 & 125370 & -28286 & 6090 & -1214 & 210 & -26 & 0 & 1 & 0 \\ -632040 & 145656 & -32424 & 6840 & -1320 & 216 & -24 & 0 & 0 & 1 \\ -1101157 & 194017 & -33069 & 5385 & -821 & 113 & -13 & 1 & 0 & 0 \\ -8219432 & 1421064 & -235880 & 36936 & -5288 & 648 & -56 & 0 & 1 & 0 \\ -15521360 & 2645520 & -430800 & 65680 & -9040 & 1040 & -80 & 0 & 0 & 1 \end{bmatrix}$$

(9×10)

ε_t**-rank** $(F_{\mathcal{P}}) = \rho = 7$

Example 2

$$\mathcal{P} = \{p_i(s) \in \mathbb{R},\ i = 1,2,3\}$$
$$p_1(s) = (s+1)(s+2)^2$$
$$p_2(s) = (s+2)(s+3)(s+4)$$
$$p_3(s) = (s+4)^2(s+5)$$

Maximum degree of LCM : $d = 3+3+3 = 9$

$$\check{F}_{\mathcal{P}} \cdot \underline{\check{a}} + \underline{f}_{d-\rho+1} \cdot a_\rho + \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} = \underline{0}$$

$$\boxed{F_{\mathcal{P}} \cdot \underline{a} = \underline{0}}$$

$$F_{\mathcal{P}} = \left[\begin{array}{rr|r|rrrrrrr} -1793 & 769 & \overbrace{-321}^{\underline{f}_{d-\rho+1}} & 129 & -49 & 17 & -5 & 1 & 0 & 0 \\ -4868 & 2052 & -836 & 324 & -116 & 36 & -8 & 0 & 1 & 0 \\ -3076 & 1284 & -516 & 196 & -68 & 20 & -4 & 0 & 0 & 1 \\ -111645 & 26335 & -6069 & 1351 & -285 & 55 & -9 & 1 & 0 & 0 \\ -539054 & 125370 & -28286 & 6090 & -1214 & 210 & -26 & 0 & 1 & 0 \\ -632040 & 145656 & -32424 & 6840 & -1320 & 216 & -24 & 0 & 0 & 1 \\ -1101157 & 194017 & -33069 & 5385 & -821 & 113 & -13 & 1 & 0 & 0 \\ -8219432 & 1421064 & -235880 & 36936 & -5288 & 648 & -56 & 0 & 1 & 0 \\ -15521360 & 2645520 & -430800 & 65680 & -9040 & 1040 & -80 & 0 & 0 & 1 \end{array}\right]$$

Columns 4–10: $\widehat{F}_{\mathcal{P}}$

(9×10)

$$\boxed{\min_{\underline{\hat{a}}} \left\| \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} + \underline{f}_3 \right\|_2}$$

ε_t-rank ($F_{\mathcal{P}}$) = ρ = 7

Example 2

	LS-QR	LS-SVD	LS-PInv
Residual	$6.255761 \cdot 10^{-15}$	$1.533661 \cdot 10^{-11}$	$8.876500 \cdot 10^{-12}$
Relative error	$4.641785 \cdot 10^{-13}$	$1.535942 \cdot 10^{-11}$	$1.405633 \cdot 10^{-11}$

- Residual : $\|u\|_2 = \left\| \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} + \underline{f}_3 \right\|_2$
- Relative error : $\mathrm{Rel} = \dfrac{\|\underline{\hat{a}} - \underline{a}\|_2}{\|\underline{a}\|_2}$
- Exact solution: $l(s) = s^7 + 21\,s^6 + 183\,s^5 + 855\,s^4 + 2304\,s^3 + 3564\,s^2 + 2912\,s + 960$

Example 3

$$\mathcal{P}_3 = \{p_i(s) \in \mathbb{R},\ i = 1,2,3\}$$
$$p_1(s) = (s+1)(s+2+\varepsilon)^2$$
$$p_2(s) = (s+2)(s+3)(s+4+\varepsilon)$$
$$p_3(s) = (s+4)^2(s+5)$$

Added Perturbation $\varepsilon = 10^{-7}$
Coprime polynomials
Degree of exact LCM = 9

Rational LCM method *

Degree of approximate GCD = **2**
(Given by the Hybrid-ERES alg. for $\varepsilon_t = 10^{-4}$)

Degree of approximate LCM = 9-2 = **7**

Remainder norm :

$\|u\|_2 = \|p(s) - g(s)\cdot l(s)\|_2$ = **$1.296138\ 10^{-2}$**

Hybrid LCM method:

No GCD computation.

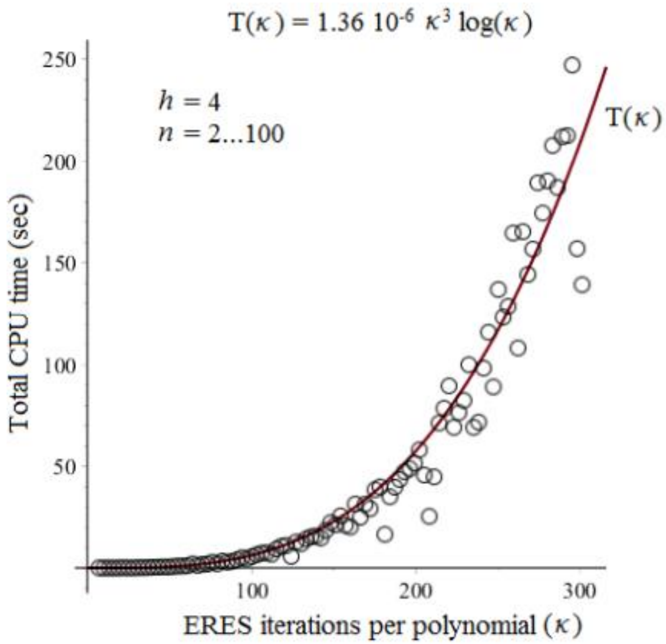
Degree of approximate LCM = **7**
(Given for $\varepsilon_t = 10^{-8}$)

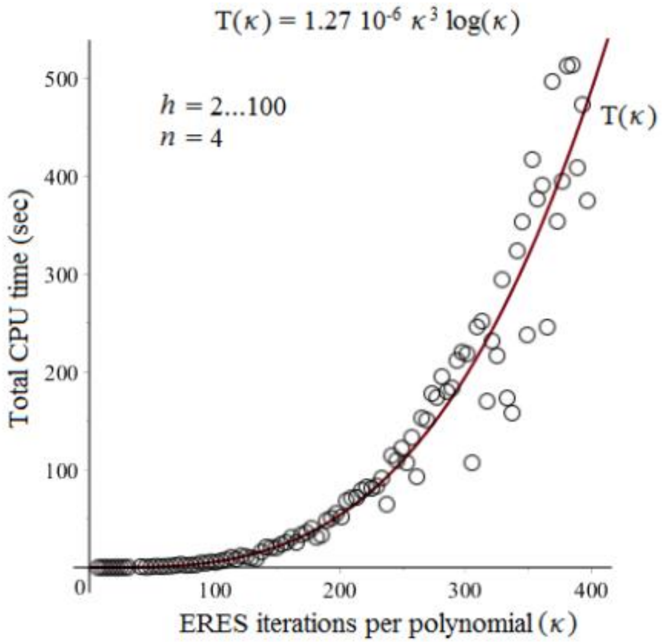
Residual :

$\|u\|_2 = \left\| \widehat{F}_{\mathcal{P}} \cdot \underline{\hat{a}} + \underline{f}_3 \right\|_2$ = **$1.488148\ 10^{-10}$**

* N. Karcanias, M. Mitrouli, Numerical computation of the Least Common Multiple of a set of polynomials, Reliable computing 6 (4) (2000) 439–457.

Computational complexity of the Hybrid-LCM algorithm





Total estimated time for symbolic operations: $T(\kappa) = O\left(\kappa^3 \log(\kappa)\right), \quad \kappa = hn - n + 1$

Total number of floating-point operations: $O\left(\frac{4}{3}(n\,h)^3 + \frac{3}{2}(n\,h)\rho^2 + 5\,\rho^3\right)$

h = number of polynomials, n = the maximum degree of the polynomials,
ρ = the computed degree of the LCM (rank of the system's matrix).

Numerical stability of the Hybrid LCM algorithm

Theorem 4. *The computed ALCM solution* $\underline{a}' = [0,\ldots,0,a_\rho,a_{\rho-1},\ldots,a_0]^t$ *obtained from the H-LCM algorithm, is the exact solution of the nearby problem*

$$(F_{\mathcal{P}} + E)\,\underline{a}' = \underline{0} \tag{70}$$

where

$$\|E\|_F \leq \left\|\Delta\widehat{F}_{\mathcal{P}}\right\|_F + \|\Delta F\|_F + |a_\rho|\left(\left\|\Delta\underline{f}_{\,d-\rho+1}\right\|_2 + \|\Delta f\|_2\right) \tag{71}$$

$$\left\|\Delta\widehat{F}_{\mathcal{P}}\right\|_F \leq (\rho - n + 1)\,\epsilon\,\left\|\widehat{F}_{\mathcal{P}}\right\|_F \quad \text{and} \quad \left\|\Delta\underline{f}_{\,d-\rho+1}\right\|_2 \leq \epsilon\,\left\|\underline{f}_{\,d-\rho+1}\right\|_2$$

$$\begin{aligned}\|\Delta F\|_F &\leq (6h - 3\rho + 41)\,\rho\,\mathbf{u}\,\|\widetilde{F}_{\mathcal{P}}\|_F + O(\mathbf{u}^2)\\ \|\Delta f\|_2 &\leq (6h - 3\rho + 41)\,\rho\,\mathbf{u}\,|a_\rho|\,\left\|\underline{\tilde{f}}_{\,d-\rho+1}\right\|_2 + O(\mathbf{u}^2)\end{aligned}$$

$\mathbf{u} = 2^{-52}$ in 16-digits arithmetic precision

Conclusions

The Hybrid LCM method:

- Combines symbolic and numerical procedures to compute the ALCM of sets of several polynomials.

- Forms an over-determined linear system and computes the ALCM without using any GCD method or root-finding procedures. The numerical rank of this linear system defines the degree of the ALCM.

- The numerical computation of the ALCM is associated with a Least-Squares optimization problem which provides the appropriate approximate solution.


Thank you for
your attention!