On applying the block Arnoldi process to the solution of a particular Sylvester-observer equation

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Consider the LTI (linear time invariant) system

$$\begin{cases} \dot{\hat{x}}(t) & = & A^T \hat{x}(t) + B\, \hat{u}(t), & \hat{x}(0) = \hat{x}_0 \\ \hat{y}(t) & = & C^T \hat{x}(t), & t \geq 0 \end{cases} \quad (1)$$

- The state $\hat{x}(t) \in \mathbb{R}^n$.
- The input $\hat{u}(t) \in \mathbb{R}^p$ and the output $\hat{y}(t) \in \mathbb{R}^q$.
- The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{n\times q}$.

Luenberger Problem :

- We want to approximate $\hat{x}(t)$ by another state $x(t)$.

Luenberger idea :

Introduce the new control system

$$\left\{ \begin{array}{lll} \dot{x}(t) & = & \widehat{H}^T x(t) + G^T \hat{y}(t) + X^T B\, \hat{u}(t), \qquad x(0) = x_0 \\ y(t) & = & C^T x(t), \qquad\qquad\qquad\qquad\qquad\quad t \geq 0, \end{array} \right. \tag{2}$$

where $\widehat{H} \in \mathbb{R}^{q\times q}$, $G \in \mathbb{R}^{q\times q}$ and $X \in \mathbb{R}^{n\times q}$ are to be determined.

Letting

$$e(t) := x(t) - X^T \hat{x}(t),$$

we verify that

$$\dot{e}(t) := \frac{d}{dt}(e(t)) = \widehat{H}^T e(t) - (A\,X - X\,\widehat{H} - C\,G)\,\widehat{x}(t). \tag{3}$$

So

- If A and $\widehat{H}$ have no eigenvalue in common (i.e., $\sigma(A) \cap \sigma(\widehat{H}) = \emptyset$), then the Sylvester equation

$$A\,X - X\,\widehat{H} = C\,G, \tag{4}$$

  has a unique solution X. In this case (3) implies that $\dot{e}(t) = \widehat{H}^T\, e(t)$, and then

$$e(t) = exp(\widehat{H}^T\, t)\, e(0) = exp(\widehat{H}^T\, t)\,(x_0 - X^T\, \hat{x}_0).$$

- Moreover, if $\widehat{H}$ is stable, (i.e., $\Re(\lambda) < 0, \;\; \forall\, \lambda \;\in\; \sigma(\widehat{H})$), then

  $e(t) := x(t) - X^T\, \hat{x}(t)$ converges to zero as t increases.

Proposed approach

To obtain $x(t)$, one can first choose the matrices G and $\widehat{H}$ and then solve the Sylvester equation (4).

Previous works

- small problems : P. Van Dooren : [1984], B. N. Datta and collaborators : Bischof, Purkyastha [1996], Hetti [1997], Sarkissian [2000], Carvalho [2001], ...
- large problems :
  - B. N. Datta and Y. Saad [1991], D. Calvetti, B. Lewis, L. Reichel [2001], ($\mathrm{rank}(C) = 1$, Arnoldi process).
  - B. N. Datta, M. Heyouni and K. Jbilou : [2010]. ($\mathrm{rank}(C) = r$, Global Arnoldi process).

We use the block Arnoldi process and describe another generalization of the Datta-Saad method for solving (4) for a large and sparse matrix A and with $rank(C) = r \geq 1$.

- we choose $G = I_q$ and suppose that $C = \tilde{C}\,\mathbb{E}_m^T \in \mathbb{R}^{n\times mr}$, with

$$\mathbb{E}_m^T = [0_r, \ldots, 0_r, I_r] \in \mathbb{R}^{r\times mr}, \quad rank(\tilde{C}) = r \text{ and } q = mr.$$

- Equation (4) becomes

$$A\,X - X\,\widehat{H} = [0_{n\times r}, \ldots, 0_{n\times r}, \tilde{C}] = \tilde{C}\,\mathbb{E}_m^T; \qquad (5)$$

where $A \in \mathbb{R}^{n\times n}$, $\tilde{C} \in \mathbb{R}^{n\times r}$ are given, while $\widehat{H} \in \mathbb{R}^{mr\times mr}$ and $X \in \mathbb{R}^{n\times mr}$ are to be determined such that

  - $\widehat{H}$ is stable, (i.e. $\Re(\lambda) < 0, \ \ \forall\, \lambda \ \in \ \sigma(\widehat{H})$).
  - $\sigma(\widehat{H}) \cap \sigma(A) = \emptyset$.
  - $(\widehat{H}^T, I)$ is controllable, (i.e., $[I, \widehat{H} - \lambda\, I_q]$ is of maximal rank for every $\lambda \in \mathbb{R}$).

Tools : Let $A \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times r}$

- Matrix valued polynomial : Let $\mathbb{P}_{m,r}$ be the set of $r \times r$ matrix-valued polynomials of degree m, i.e., for $\psi_i \in \mathbb{R}^{r\times r}$ and $i = 1, \ldots, m$

$$\psi = (\psi_i) \in \mathbb{P}_{m,r} \Longleftrightarrow \psi(t) = \sum_{i=0}^{m} t^i \, \psi_i,$$

- The $\circ$ notation : For $\psi = (\psi_i) \in \mathbb{P}_{m,r}$

$$\psi(A) \circ V = \sum_{i=0}^{m} A^i \, V \, \psi_i.$$

- Block Krylov subspace : $\mathbb{K}_m(A, V) = \mathrm{colspan}(\left[V, A\,V, \ldots, A^{m-1}\,V\right])$.
  - $\mathbb{K}_m(A, V)$ is spanned by the $m\,r$ columns of $V, AV, \ldots, A^{m-1}V$.
  - $Z \in \mathbb{K}_m(A, V) \quad \Longleftrightarrow \quad Z = \sum_{i=1}^{m} A^{i-1}\, V\, \Omega_i,$ with $\Omega_i \in \mathbb{R}^{r\times r}$, $i = 1, \ldots, m$.

$$\mathbb{K}_m(A, V) = \{\mathcal{P}(A) \circ V, \ \ \mathcal{P} \in \mathbb{P}_{m-1,r}\}.$$

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Algorithm 1   The block Arnoldi process
1. [V_1, H_{1,0}] = QR(V) ; % QR decomposition of V
2. For j = 1, ..., m do
3.     W = A V_j,
4.     for i = 1, 2, ..., j do
5.         H_{i,j} = V_i^T W ;
6.         W = W - V_i H_{i,j} ;
7.     endfor
8.     [V_{j+1}, H_{j+1,j}] = QR(W) ; % QR decomposition of W
9. EndFor
```

- $\mathbb{V}_m = [V_1, \ldots, V_m] \in \mathbb{R}^{n \times mr}$ is orthonormal, i.e., $\mathbb{V}_m^T \mathbb{V}_m = I_{mr}$.
- $\mathbb{H}_m = [H_{i,j}]$ is a $mr \times mr$ block upper Hessenberg matrix.
- $\mathbb{E}_m = [0_r, \ldots, 0_r, I_r]^T \in \mathbb{R}^{mr \times r}$

$$A\,\mathbb{V}_m = \mathbb{V}_m\,\mathbb{H}_m + V_{m+1}\,H_{m+1,m}\,\mathbb{E}_m^T, \quad (6)$$

$$= \mathbb{V}_m\,\mathbb{H}_m + [0_{n\times r}, \ldots, 0_{n\times r}, V_{m+1}\,H_{m+1,m}]. \quad (7)$$

Observe the similarity between (8) and (9)

$$A\,X - X\,\widehat{H} = \tilde{C}\,\mathbb{E}_m^T = [0_{n\times r}, \ldots, 0_{n\times r}, \tilde{C}]. \tag{8}$$

and

$$A\,\mathbb{V}_m - \mathbb{V}_m\,\mathbb{H}_m = V_{m+1}\,H_{m+1,m}\,\mathbb{E}_m^T = [0_{n\times r}, \ldots, 0_{n\times r}, V_{m+1}\,H_{m+1,m}]. \tag{9}$$

Hence, to solve the Sylvester-Observer equation (8), we propose to

- find a block V_1 such that V_{m+1} is equal to $\widetilde{C}$ (up to a multiplicative $r \times r$ matrix coefficient).
- transform $\mathbb{H}_m$ into a matrix $\widehat{H}$ such that $\sigma(\widehat{H}) = \{\mu_1, \ldots, \mu_{mr}\}$ with $\Re(\mu_j) < 0$.
- take $X = \mathbb{V}_m$ (up to a matrix coefficient).

To find V_1, we use

Proposition

The orthonormal matrices $V_i \in \mathbb{R}^{n\times r}$ generated by the block Arnoldi process are such that

$$V_{i+1} = \mathcal{P}_i(A) \circ V_1, \ \text{for } i = 0, \ldots, m. \tag{10}$$

where $\mathcal{P}_i$ is an $r \times r$ matrix-valued polynomial of degree i.

Proposition

Let $\mathcal{P}_i$, $(i = 1, \ldots, m)$, be the $r \times r$ matrix-valued polynomial of degree i given by (10). Then, up to a multiplicative scalar $\rho_i \in \mathbb{R}$, the determinant of the matrix-valued polynomial $\mathcal{P}_i(t)$ is the characteristic polynomial of the block upper Hessenberg matrix $\mathbb{H}_i$, i.e.,

$$\det(\mathcal{P}_i(t)) = \rho_i \det(\mathbb{H}_i - t\, I_{i\,r}).$$

For a similar result see [Simoncini and Gallopulous, LAA (1996)]

- Hence, since $\mathbb{H}_m$ must be transformed by an eigenvalue assignment algorithm into $\widehat{H}$, in order to have the pre-assigned spectrum $\{\mu_1, \ldots, \mu_{mr}\}$, we propose to look for a polynomial $\mathcal{P}_m$ such that

$$\mathcal{P}_m(A) \circ Y = \widetilde{C}, \tag{11}$$

with

$$\det(\mathcal{P}_m(t)) = \rho \prod_{j=1}^{mr} (t - \mu_j). \tag{12}$$

- Once the block Y is computed, we apply the block Arnoldi process to the pair (A, Y) to get $\mathbb{V}_m = [V_1, \ldots, V_m]$.

To solve the block linear system (11) satisfiying (12) :

- Let $\widetilde{C} = [\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_r]$, with $\widetilde{c}_i \in \mathbb{R}^n$ for $i = 1, \ldots, r$.
- Define $\Gamma = \bigcup_{i=1}^{r} \Gamma_i$ with $\Gamma_i = \{\mu_{i+jr}\}_{j=0,1,\ldots,m-1}$ and $\mu_{i+jr} \neq \mu_{i+kr}$ for $j \neq k$.
- For $i = 1, \ldots, r$, we denote by $p_m^{(i)}$ the polynomial of degree m defined by

$$p_m^{(i)}(t) = \prod_{\mu \in \Gamma_i} (t - \mu) = \prod_{j=0}^{m-1} (t - \mu_{i+jr}). \tag{13}$$

- Take $\mathcal{P}_m(t) = \mathrm{diag}\left(p_m^{(1)}(t), \ldots, p_m^{(r)}(t)\right)$.

We verify that

$$\mathcal{P}_m(A) \circ Y = \widetilde{C} \iff \left[p_m^{(1)}(A)\, y_1, \ldots, p_m^{(r)}(A)\, y_r\right] = [\widetilde{c}_1, \ldots, \widetilde{c}_r]$$

- For $i = 1, \ldots, r$, let $y_i \in \mathbb{R}^n$ be the solution of the following linear system

$$p_m^{(i)}(A)\, y_i = \widetilde{c}_i, \tag{14}$$

To solve the above systems, we proceed as in the Datta-Saad method

- Let $y_i = f^{(i)}(A)\, \widetilde{c}_i$ where $f^{(i)}(t) = \dfrac{1}{p_m^{(i)}(t)} = \displaystyle\prod_{j=0}^{m-1} \dfrac{1}{(t - \mu_{i+jr})}$.
- Denoting by $[p_m^{(i)}]'(t)$ the derivative of $p_m^{(i)}(t)$, we show that

$$y_i = \sum_{j=0}^{m-1} \frac{1}{[p_m^{(i)}]'(\mu_{i+j\,r})}\, z_j^{(i)}, \text{ with } [p_m^{(i)}]'(\mu_{i+j\,r}) = \prod_{\substack{k=0\\k\neq j}}^{m-1} (\mu_{i+j\,r} - \mu_{i+kr}), \tag{15}$$

where $z_j^{(i)}$ for $j = 0, \ldots, m-1$ are solutions of the shifted linear systems

$$(A - \mu_{i+j\,r}\, I)\, z_j^{(i)} = \widetilde{c}_i, \quad \text{for } i = 1, \ldots, r. \tag{16}$$

Now, we have

$$A\,\mathbb{V}_m - \mathbb{V}_m\,\mathbb{H}_m = V_{m+1}\,H_{m+1,m}\,\mathbb{E}_m^T = [0_{n\times r}, \ldots, 0_{n\times r}, V_{m+1}\,H_{m+1,m}].$$

- The eigenvalues of $\mathbb{H}_m$ do not necessarily coincide with the chosen scalars $\{\mu_k\}_{k=1,\ldots,mr}$.
- **Idea :** Transform $\mathbb{H}_m$ into (a stable matrix) $\widehat{H}$ so that $\sigma(\widehat{H}) = \{\mu_1, \ldots, \mu_{mr}\}$.
  - Define : $L_1 = \mathbb{E}_1\,H_{1,0} = (H_{1,0}^T, 0_r, \ldots, 0_r)$, $\quad L_{i+1} = \mathbb{H}_m\,L_i - L_i\,\widehat{\Lambda}_i$, $(i = 1, \ldots, m+1)$ with $\widehat{\Lambda}_i = \mathrm{diag}(\mu_{1+(i-1)\,r}, \mu_{2+(i-1)\,r}, \ldots, \mu_{r+(i-1)\,r})$
  - Let $S = L_{m+1}$, $\alpha = \prod_{i=1}^{m-1} H_{i+1,i}^{-1}$.
  - Let

$$\widehat{H}_m = \mathbb{H}_m - S\,H_{1,0}^{-1}\,\alpha\,\mathbb{E}_m^T. \tag{17}$$

The eigenvalues of $\widehat{H}_m$ are $\mu_1, \ldots, \mu_{mr}$.

Proposition

Let $\mathbb{V}_{m+1} = [\mathbb{V}_m, V_{m+1}]$, $\mathbb{H}_m$ *be respectively the Krylov and the upper block Hessenberg matrices constructed by the block Arnoldi process. Set also*

$$\beta_m = (V_{m+1}^T \widetilde{C})^{-1} H_{m+1,m}, \tag{18}$$

Then the matrix $\widehat{H}_m$ *can be expressed as*

$$\widehat{H}_m = \mathbb{H}_m - F\,\mathbb{E}_m^T, \tag{19}$$

where $F := \mathbb{V}_m^T \widetilde{C}\,\beta_m$*. Moreover, the matrix* $\widehat{H}_m$ *satisfies the Arnoldi-like relation*

$$A\,\mathbb{V}_m - \mathbb{V}_m\,\widehat{H}_m = \widetilde{C}\,\beta_m\,\mathbb{E}_m^T. \tag{20}$$

Remarks

- From a computational viewpoint, (19) is more convenient than (17)
- Another expression for β_m : $\beta_m = \prod_{i=0}^{m-1} H_{i+1,i}^{-1}$.

To recover the Sylvester-observer form

$$A\,X - X\,\widehat{H} = \left[0_{n\times r}, \ldots, 0_{n\times r}, \widetilde{C}\right] = \widetilde{C}\,\mathbb{E}_m^T,$$

- Define D as the last column block of the matrix $A\,\mathbb{V}_m - \mathbb{V}_m\,\widehat{H}_m$, i.e., $D = \widetilde{C}\,\beta_m$.
- Define the diagonal matrix

$$\Theta = \begin{pmatrix} I_r & 0_r & \ldots & 0_r \\ 0_r & \ddots & \ddots & \vdots \\ \vdots & & I_r & 0_r \\ 0_r & \ldots & 0_r & \beta_m^{-1} \end{pmatrix}. \qquad (21)$$

Then,

$$A\,\mathbb{V}_m\,\Theta - \mathbb{V}_m\,\Theta\,\Theta^{-1}\,\widehat{H}_m\,\Theta = \left[0_{n\times r}, \ldots, 0_{n\times r}, \widetilde{C}\right].$$

- Take

$$X = \mathbb{V}_m\,\Theta \quad \text{and} \quad \widehat{H} = \Theta^{-1}\,\widehat{H}_m\,\Theta. \qquad (22)$$

Algorithm 2. The block Arnoldi alg. for multiple-output Sylvester-Observer equation

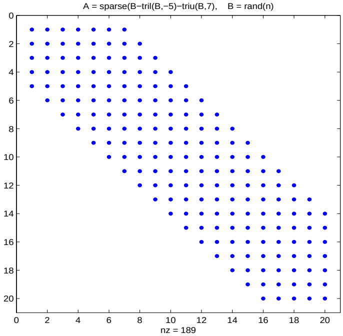
- *Inputs* : $A \in \mathbb{R}^{n\times n}$, $\widetilde{C} = (\widetilde{c}_1, \ldots, \widetilde{c}_r) \in \mathbb{R}^{n\times r}$ and $\Gamma = \{\mu_1, \mu_2, \ldots, \mu_{mr}\}$.
- *Output* : $X, \widehat{H}$ solution of the Sylvester-Observer equation.
- *Step 1.* Solve the linear problem $\mathcal{P}_m(A) \circ Y = \widetilde{C}$, i.e.,
  - *Step 1.1.* Solve $(A - \mu_{i+j\,r}\, I_{mr})\, z_j^{(i)} = \widetilde{c}_i$, for $i = 1, \ldots, r$ and $j = 0, \ldots, m-1$.
  - *Step 1.2.* Compute $y_i = \sum_{j=0}^{m-1} \gamma_j\, z_j^{(i)}$; $i = 1, \ldots, r$, where $\gamma_j = \prod_{\substack{k=0 \\ k\neq j}}^{m-1} \frac{1}{(\mu_{i+j\,r} - \mu_{i+k\,r})}$.
- *Step 2.* Define $Y = [y_1, \ldots, y_r]$; apply block Arnoldi to (A, Y) to get $\mathbb{H}_m = [H_{i,j}]$ and $\mathbb{V}_{m+1} = [V_1, \ldots, V_m, V_{m+1}]$;
- *Step 3.* Modify $\mathbb{H}_m$ to get $\widehat{H}$ such that $\sigma(\widehat{H}) = \{\mu_1, \mu_2, \ldots, \mu_{mr}\}$, i.e.,
  - *Step 3.1.* Compute $\beta_m = \left(V_{m+1}^T\, \widetilde{C}\right)^{-1} H_{m+1,m} = \prod_{i=0}^{m-1} H_{i+1,i}^{-1}$ and $F = \mathbb{V}_m^T\, \widetilde{C}\, \beta_m$;
  - *Step 3.2.* Define $\widehat{H}_m = \mathbb{H}_m - F\, \mathbb{E}_m^T$.
  - *Step 3.3.* Determine D the last block column of $A\, \mathbb{V}_m - \mathbb{V}_m\, \widehat{H}_m$
  - *Step 3.4.* Construct $\Theta = \mathrm{diag}(I_r, \ldots, I_r, \beta_m^{-1})$.
- *Step 4.* Take $X = \mathbb{V}_m\, \Theta$, and $\widehat{H} = \Theta^{-1}\, \widehat{H}_m\, \Theta$.

Numerical tests

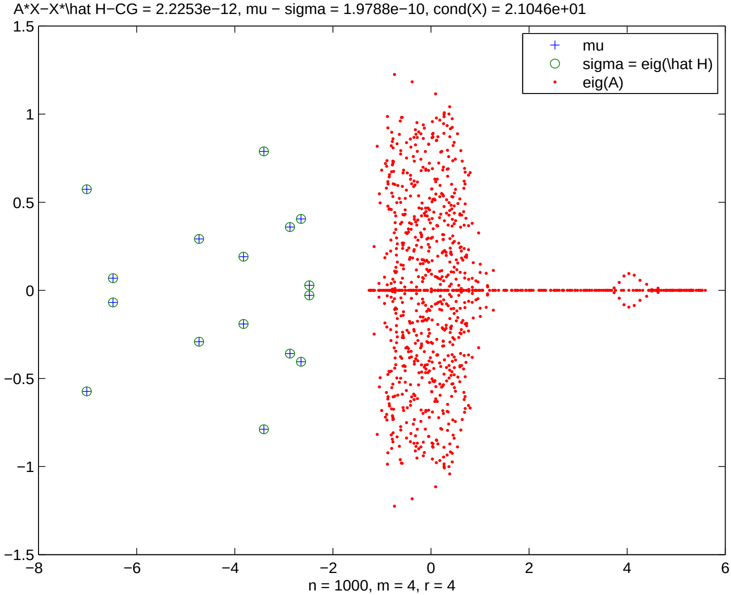
- Experiments were performed on a laptop CORE i5 at 1.70GHz and 6.00Go of RAM.
- The algorithms were coded in Matlab R2014.a.
- The entries of the $n \times r$ matrix $\widetilde{C}$ were random values uniformly distributed on $[0, 1]$.
- To solve mr linear systems , in *Step 1.1* of Algorithm 2, we can use
  - a (preconditioned) Krylov method for shifted linear systems.
  - the Gaussian elimination method.

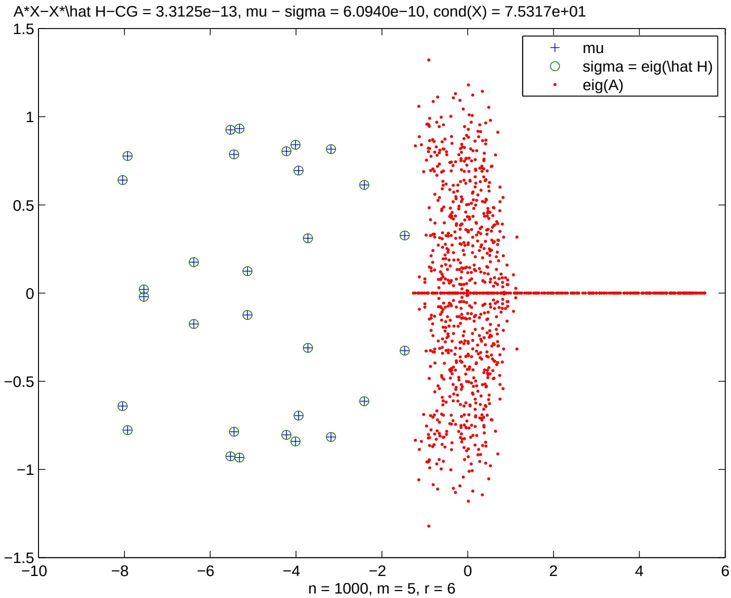
Experiment 1 :

- $A = \texttt{sparse}(\texttt{B} - \texttt{tril}(\texttt{B}, -5) - \texttt{triu}(\texttt{B}, 7))$, $n = 1000$, $B = \texttt{rand}(\texttt{n})$



- $\Gamma = \{z_k, \bar{z}_k\}$, with $\Re(z_k) = -7 * rand + min(real(eig(A)))$, $\Im(z_k) = rand$
- $\widetilde{C} \in \mathbb{R}^{n \times r}$ is generated randomly.
- Gaussian elimination method is used to solve the $m\,r$ linear systems.


A*X−X*\hat H−CG = 2.2253e−12, mu − sigma = 1.9788e−10, cond(X) = 2.1046e+01
mu
sigma = eig(\hat H)
eig(A)
1.5
1
0.5
0
−0.5
−1
−1.5
−8
−6
−4
−2
0
2
4
6
n = 1000, m = 4, r = 4



A*X−X*\hat H−CG = 3.3125e−13, mu − sigma = 6.0940e−10, cond(X) = 7.5317e+01
mu
sigma = eig(\hat H)
eig(A)
1.5
1
0.5
0
−0.5
−1
−1.5
−10
−8
−6
−4
−2
0
2
4
6
n = 1000, m = 5, r = 6

Experiment 2 :

- A=`gallery('wathen',70,100)`, $n = 21341$, $nnz(A_4) = 330361$
- To test the influence of the pre-scripted set of eigenvalues Γ, we consider a set $\Gamma = \{\mu_1, \dots, \mu_r\}$ of negative real values, $\Gamma = \Gamma^c = -c * \mathrm{rand}(mr, 1)$, where c is a positive integer.
- $\widetilde{C}$ is generated randomly.
- Restarted Shifted FOM(50) is used to solve the linear systems. (Initial guess : $(Y_i)_0 = 0_{n\times r}$ Relative tolerance : $\varepsilon = 10^{-10}$).

m	r	c	$SylvErr$	$EigErr$	$\kappa(X)$
2	5	10	$1.04\,10^{-10}$	$8.24\,10^{-09}$	$4.59\,10^{+00}$
2	5	30	$1.55\,10^{-13}$	$6.65\,10^{-12}$	$4.70\,10^{+00}$
3	10	10	$4.25\,10^{-09}$	$2.39\,10^{-08}$	$5.17\,10^{+00}$
3	10	30	$3.24\,10^{-10}$	$3.22\,10^{-09}$	$1.10\,10^{+01}$
4	5	10	$3.85\,10^{-06}$	$6.41\,10^{-05}$	$5.54\,10^{+00}$
4	5	30	$2.51\,10^{-09}$	$7.55\,10^{-06}$	$1.73\,10^{+01}$

Experiment 2 bis :

- A=`gallery('wathen',70,100)`, $n = 21341$, $nnz(A_4) = 330361$
- To test the influence of the pre-scripted set of eigenvalues Γ, we consider a set $\Gamma = \{\mu_1, \ldots, \mu_r\}$ of negative real values, $\Gamma = \Gamma^c = -c * \mathrm{rand}(mr, 1)$, where c is a positive integer.
- $\widetilde{C}$ is generated randomly.
- Gaussian elimination is used to solve the linear systems.

m	r	c	$SylvErr$	$EigErr$	$\kappa(X)$
2	5	10	$1.22\,10^{-13}$	$2.97\,10^{-13}$	$4.59\,10^{+00}$
2	5	30	$3.25\,10^{-14}$	$1.91\,10^{-13}$	$4.70\,10^{+00}$
3	10	10	$1.33\,10^{-11}$	$2.39\,10^{-09}$	$5.17\,10^{+00}$
3	10	30	$1.62\,10^{-12}$	$2.57\,10^{-10}$	$1.10\,10^{+01}$
4	5	10	$8.92\,10^{-09}$	$1.00\,10^{-05}$	$5.54\,10^{+00}$
4	5	30	$2.82\,10^{-10}$	$1.56\,10^{-07}$	$1.73\,10^{+01}$

Experiment 3 :

- A is of size $n = 20000$ ($p = n/2 = 10000$)

$$A = \begin{pmatrix} 0_p & I_p \\ L & D \end{pmatrix}, \text{ where } L = \begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_p \end{pmatrix} \text{ and } D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{pmatrix}.$$

- For $d_k = 2\,\alpha_k$, $l_k = -(\alpha_k^2 + \beta_k^2)$ then : $\sigma(A) = \{\lambda_k, \bar{\lambda}_k\}_{k=1,\ldots,p}$, where $\lambda_k = \alpha_k + \imath\,\beta_k$. ($\alpha_k,\ \beta_k$ were random values uniformly distributed in [-1, 1].)
- The μ_k are the zeros of the Chebyshev polynomial of 1st kind of degree *mr* for $[a + \imath\, b,\ a - \imath\, b]$, where $a = -1 + \min_{d \in \sigma(A)} \mathrm{Re}(d)$ and $b = \max_{d \in \sigma(A)} \mathrm{Im}(d)$.
- Gaussian elimination is used to solve the linear systems.

m	r	*SylvErr*	*EigErr*	$\kappa(X)$
3	10	$6.91\,10^{-14}$	$3.23\,10^{-14}$	$2.80\,10^{+01}$
5	4	$1.00\,10^{-12}$	$2.12\,10^{-12}$	$4.82\,10^{+00}$
5	6	$2.96\,10^{-13}$	$2.26\,10^{-12}$	$5.42\,10^{+00}$
6	10	$9.95\,10^{-13}$	$5.11\,10^{-11}$	$3.97\,10^{+00}$
8	6	$8.81\,10^{-11}$	$8.64\,10^{-08}$	$4.44\,10^{+00}$
8	10	$8.96\,10^{-12}$	$1.08\,10^{-07}$	$4.61\,10^{+00}$

Conclusion

- We used the block Arnoldi for solving the multi-output Sylvester-Observer equation arising in state-estimation in a linear time-invariant control system.
- The proposed method is suitable for large and sparse computing.
- The method can be considered as a generalization of the Arnoldi-method proposed earlier by Datta and Saad in the single-output case.

Thanks for your attention

Comparing to the global Arnoldi process

Noticing the similarity between the particular Sylv. obs. eqt. (23)

$$A\,X - X\,\widehat{H} = \tilde{C}\,\mathbb{E}_m^T = [0_{n\times r}, \ldots, 0_{n\times r}, \tilde{C}]. \tag{23}$$

and the global Arnoldi iteration (24)

$$A\,\mathcal{V}_m - \mathcal{V}_m\,(H_m \otimes I_r) = h_{m+1,m}\,V_{m+1}\,(e_m \otimes I_r)^T = [0_{n\times r}, \ldots, 0_{n\times r}, h_{m+1,m}\,V_{m+1}]. \tag{24}$$

To obtain a solution to the Sylvester-Observer equation (23), we applied the Datta-Saad approach the $m \times m$ upper Hessenberg matrix H_m, i.e.,

- find $V_1 \in \mathbb{R}^{n\times r}$ such that $V_{m+1} = \widetilde{C}$ (a part from a multiplicative scalar).
- transform H_m to $\widehat{H}_m$ such that $\sigma(\widehat{H}_m) = \{\mu_1, \ldots, \mu_m\}$ with $\Re(\mu_j) < 0$.
- take $\widehat{H} = (\widehat{H}_m \otimes I_r)$ and observe that $\sigma(\widehat{H}) = \{\mu_1, \ldots, \mu_m\}$.
  $\text{multiplicity}(\mu_k) = r$.
- take $X = \mathcal{V}_m$ (a part from a multiplicative scalar).