Fast Nonnegative Least Squares through flexible Krylov subspaces

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What is this talk about?

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Solution of

$$\min_{x\in\mathbb{R}^N} \|b - Ax\|_2, \quad A \in \mathbb{R}^{N\times N}, \quad b \in \mathbb{R}^N,$$

coming from suitable discretization of

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- the process k, the output g ($g = g^{ex} + \varepsilon$) are known;
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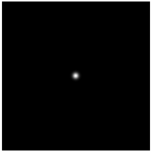

PSF

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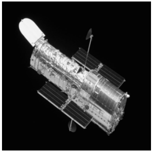

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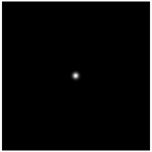
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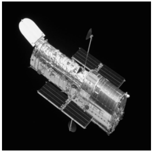
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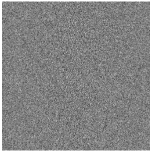
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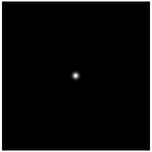

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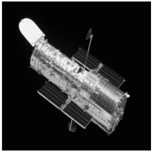

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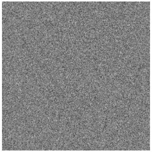

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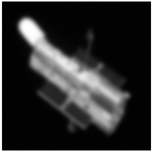

available

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Regularization...

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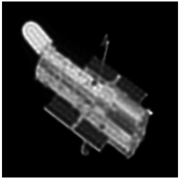
Nonnegative constraints!

$$\min_{x \geq 0} \|b - Ax\|_2$$



Outline

1 Nonnegative LS problems
- Brief survey of the available methods
- Krylov methods for unconstrained problems

2 Flexible Krylov subspaces
- The need of introducing "flexibility"
- Deriving FCGLS
- MFCGLS for nonnegative LS problems

3 Numerical Experiments
- Restoration of Astronomical Images
- CT Reconstruction

4 Looking at Poisson noise
- Modeling Poisson Noise
- CP-MFCGLS

5 Final Remarks

Popular approaches

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- Projected gradient

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$$x_{m+1} = P_+(x_m + \alpha_m A^T(b - Ax_m)) \, .$$

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  [Beck and Teboulle. FISTA, *SIIMS*, 2009]

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Fixed point iterations:

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- Active set methods
  [Morigi, Plemmons, Reichel, Sgallari. Hybrid multilevel-active set method for box-constr. Calcolo, 2011]

Krylov methods for unconstrained problems

Unconstrained LS problem:

$$\min_{x \in \mathbb{R}^N} \|b - Ax\|_2^2 \,.$$

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Iterative methods such that:

- $x_m \in \mathcal{K}_m(C,d)$, where

$$\mathcal{K}_m(C,d) = \text{span}\{d, Cd, \dots, C^{m-1}d\}\,,$$

  and $C = A,\ A^TA,\ AA^T,\ d = b,\ A^Tb,\ A^\ell b\,(\ell \geq 1)$.

- $r_m := b - Ax_m$ satisfies some conditions, e.g.,

$$\min_{\hat{x}\in\mathcal{K}_m(C,d)} \|b - A\hat{x}\|_2^2 \,.$$

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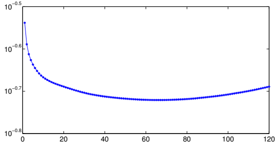
Extremely efficient!

Reichel & CO., Hansen & CO., Nagy & CO., Strakos & CO., Novati & CO [...]

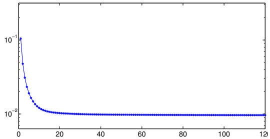
Lewis, Sgallari, Morigi, Lanza, Calvetti, Hanke, Donatelli, Chung, Elden, Simoncini, Jensen, Rodriguez, Russo, O'Leary, Plemmons, Jorgensen, Kilmer, Hnetynkova, Hochstenbach, Noschese, Dykes, Hayami, Ye, Saunders, Palmer, Huang, Jia [...]

Krylov methods (CGLS) in action

relative errors



relative residuals



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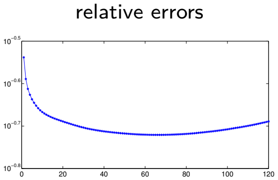
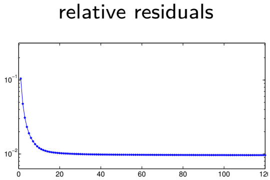
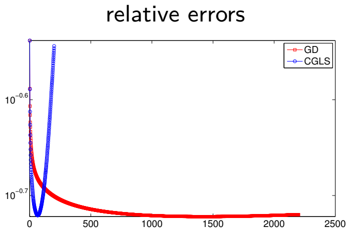




Gradient Descent approach VS. Krylov Subspaces approach



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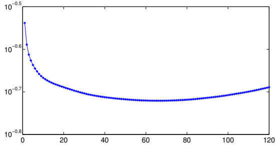

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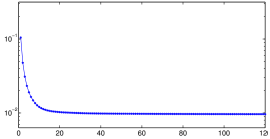

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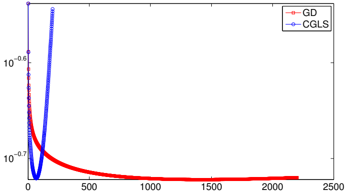

They work efficiently as:

- they mimic the TSVD;
- $\mathcal{K}_m(C, d) \simeq \mathcal{K}_{m+1}(C, d)$ for small m.

Defining Krylov methods

[Axelsson. Iterative Solution Methods, *Cambridge*, 1994]

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Explicitly computing the solution along descent directions:

$$x_{m+1} = x_m + \sum_{j=0}^{m} \alpha_j^{(m)} d_j$$

$$d_{m+1} = \bar{z}_{m+1} + \sum_{j=0}^{m} \beta_j^{(m)} d_j \,.$$

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Requiring Ad_j orthogonal and minimal residual:

- $x_{m+1} = x_m + \alpha_m d_m$, where $\alpha_m = \frac{(r_m, Ad_m)}{(Ad_m, Ad_m)}$;
- $\beta_j^{(m)} = -\frac{(A\bar{z}_{m+1}, Ad_j)}{(Ad_j, Ad_j)}\,, \quad j = 0, \ldots, m\,.$

Special case: CGLS.

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$$XA^T(b - Ax_m) = 0\,, \quad x_m \geq 0$$

- make sure that $x_m \geq 0$ for each m.

The need of introducing "flexibility"

Back the KKT conditions, most general case:

$$X^{(m)}A^T(b - Ax_m) = 0\,, \quad x_m \geq 0$$

where, at the mth step, $X^{(m)} = \mathrm{diag}(x_{m-1})$.

- variable "preconditioners";
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We need a Krylov method (CGLS) that handles:

- variable "preconditioners";
- make sure that $x_m \geq 0$ for each m.

[Simoncini and Szyld. Recent computational developments in Krylov meth, *NLAA*, 2007]

Flexible Krylov methods for regularization

"Preconditioners" that enforce "regularity"

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- $\min_{x\in\mathbb{R}^N} \|b - Ax\|_2^2 + \lambda\|x\|_1 \quad \equiv \quad \min_{x\in\mathbb{R}^N} \|b - AX^{(m)}x\|_2^2 + \lambda\|x\|_2^2$

  [G. and Nagy. GAT for sparse reconstruction. SISC, 2014]

- $\min_{x\in\mathbb{R}^N} \|b - Ax\|_2^2 \quad \longrightarrow \quad x_m = Z_m y_m$

  [Morikuni, Reichel, Hayami. FGMRES for linear ill-posed pb. Appl.Numer.Math., 2014]

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  [Reichel, Yu. Tikhonov regularization via flexible Arnoldi. BIT, 2015]

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now... deriving FCGLS

If no preconditioners: CGLS

Input: A, b, x_0.
Initialize: $r_0 = b - Ax_0$, $\bar{z}_0 = A^T r_0$.

Take $d_0 = \bar{z}_0$.
For $m = 0, \dots,$ till a stopping criterion is satisfied

- Set $\alpha_m = \frac{(r_m, Ad_m)}{(Ad_m, Ad_m)}$.
- Update $x_{m+1} = x_m + \alpha_m d_m$.
- Update $r_{m+1} = r_m - \alpha_m Ad_m$.
- Compute $\bar{z}_{m+1} = A^T r_{m+1}$.

- Set $\beta_m = -\frac{(A\bar{z}_{m+1}, Ad_m)}{(Ad_m, Ad_m)}$.
- Update $d_{m+1} = \bar{z}_{m+1} + \beta_m d_m$.

If the preconditioner L is fixed: PCGLS

Input: A, L, b, x_0.
Initialize: $r_0 = b - Ax_0$, $z_0 = A^T r_0$.
Compute $\bar{z}_0 = Lz_0$.
Take $d_0 = \bar{z}_0$.
For $m = 0, \dots,$ till a stopping criterion is satisfied

- Set $\alpha_m = \frac{(r_m, Ad_m)}{(Ad_m, Ad_m)}$.
- Update $x_{m+1} = x_m + \alpha_m d_m$.
- Update $r_{m+1} = r_m - \alpha_m Ad_m$.
- Compute $z_{m+1} = A^T r_{m+1}$.

- Compute $\bar{z}_{m+1} = Lz_{m+1}$.
- Set $\beta_m = -\frac{(A\bar{z}_{m+1}, Ad_m)}{(Ad_m, Ad_m)}$.
- Update $d_{m+1} = \bar{z}_{m+1} + \beta_m d_m$.

If the "preconditioners" $L^{(m)}$ are variable: FCGLS

Input: A, $L^{(0)}$, b, x_0.
Initialize: $r_0 = b - Ax_0$, $z_0 = A^T r_0$.
Compute $\bar{z}_0 = L^{(0)} z_0$.
Take $d_0 = \bar{z}_0$.
For $m = 0, \ldots,$ till a stopping criterion is satisfied

- Set $\alpha_m = \frac{(r_m, Ad_m)}{(Ad_m, Ad_m)}$.
- Update $x_{m+1} = x_m + \alpha_m d_m$.
- Update $r_{m+1} = r_m - \alpha_m A d_m$.
- Compute $z_{m+1} = A^T r_{m+1}$.
- Compute $L^{(m+1)}$.
- Compute $\bar{z}_{m+1} = L^{(m+1)} z_{m+1}$.
- Set $\beta_j^{(m)} = -\frac{(A\bar{z}_{m+1}, Ad_j)}{(Ad_j, Ad_j)}$, $j = 0, \ldots, m$.
- Update $d_{m+1} = \bar{z}_{m+1} + \sum_{j=0}^{m} \beta_j^{(m)} d_j$.

[Notay. Flexible CG, *SISC*, 2000]

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Take $d_0 = \bar{z}_0$.
For $m = 0, \ldots,$ till a stopping criterion is satisfied

- Set $\alpha_m = \frac{(r_m, Ad_m)}{(Ad_m, Ad_m)}$.
- Update $x_{m+1} = x_m + \alpha_m d_m$.
- Update $r_{m+1} = r_m - \alpha_m A d_m$.
- Compute $z_{m+1} = A^T r_{m+1}$.
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- Compute $\bar{z}_{m+1} = L^{(m+1)} z_{m+1}$.
- Set $\beta_j^{(m)} = -\frac{(A\bar{z}_{m+1}, Ad_j)}{(Ad_j, Ad_j)}$, $j = \hat{m}, \ldots, m$.
- Update $d_{m+1} = \bar{z}_{m+1} + \sum_{j=\hat{m}}^{m} \beta_j^{(m)} d_j$.

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Solving:

$$X^{(m)}A^T(b - Ax) = 0\,, \quad x \geq 0\,, \quad \text{with} \quad X^{(m)} = \text{diag}(x_m)\,,$$

by FCGLS:

$$x_{m+1} = x_m + \alpha_m d_m\,.$$

MFCGLS for nonnegative LS problems

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by FCGLS:

$$x_{m+1} = x_m + \alpha_m d_m\,.$$

To guarantee nonnegativity at each step:

$$\alpha_m = \min\left(\alpha_m, \min\left(-\frac{x_m(d_m < 0)}{d_m(d_m < 0)}\right)\right).$$

MFCGLS for nonnegative LS problems

Solving:

$$X^{(m)}A^T(b - Ax) = 0\,, \quad x \geq 0\,, \quad \text{with} \quad X^{(m)} = \text{diag}(x_m)\,,$$

by FCGLS:

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Prone to stagnation: $\alpha_m = 0$!!!

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Prone to stagnation: $\alpha_m = 0$!!!

MFCGLS (Modified FCGLS)

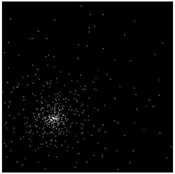
Input: A, b, $x_0^0 \geq 0$.

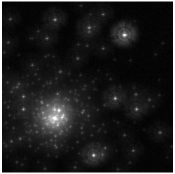
- For $k = 0, \dots,$ till a stopping criterion is satisfied
  - For $m = 0, \dots,$ till $m_{\max}$ or $\alpha_m = 0$
    - Run a FCGLS with initial guess x_0^k, preconditioner $X^{(m)}$, and NN:
      $x_{m+1} = x_m + \alpha_m d_m$;
      $d_{m+1} = \bar{z}_{m+1} + \sum_{j=\hat{m}}^{m} \beta_j^{(m)} d_j$.
    - Restart with the last approximation.

Numerical experiments

Numerical experiments

- star_cluster test problem [Nagy et al. *Restore Tools*, 2012]





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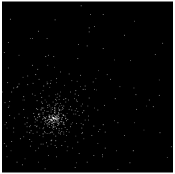
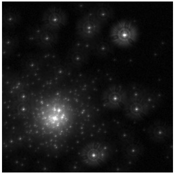
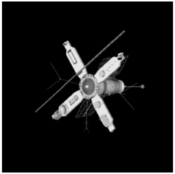
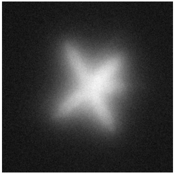



- satellite test problem [Nagy et al. *Restore Tools*, 2012]

Numerical experiments

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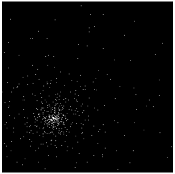

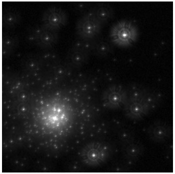

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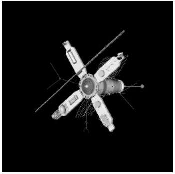
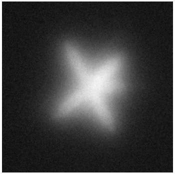




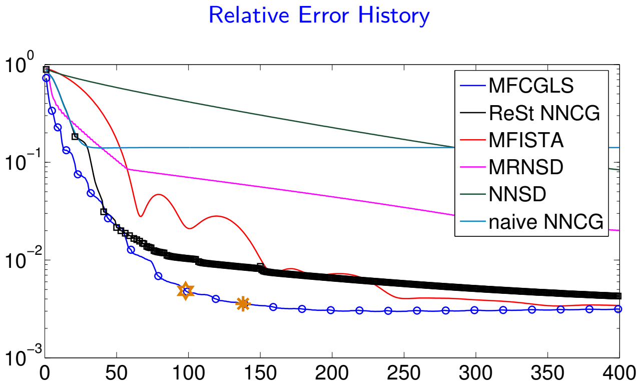
- paralleltomo test problem [Hansen et al. *AIR Tools*, 2012]

star_cluster test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-2}$

star_cluster test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-2}$



ReSt NNCG by

[Calvetti et al. Non-neg. and iterative methods for ill-posed pb, *Inv. Problems*, 2004]

star_cluster test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-2}$

	rel.error	iterations	tot.time	av.time
MFCGLS	2.8132e-03	248.67	62.56	0.25
ReSt NNCG	5.3699e-03	261.00	113.51	0.43
FISTA	9.1283e-02	72.00	42.06	0.58
MFISTA	3.2803e-03	400.00	216.11	0.54
MFISTA(0.2)	3.2445e-03	400.00	194.78	0.49
MFISTA(5)	4.2834e-03	400.00	185.22	0.46
MRNSD	1.9889e-02	400.00	91.11	0.23
NNSD	8.3206e-02	400.00	91.59	0.23
naive NNCG	1.4028e-01	400.00	105.02	0.26

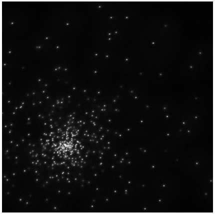
star_cluster test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-2}$, 200th iteration

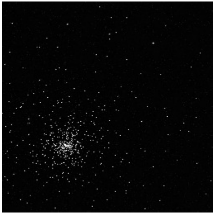
star_cluster test problem

$A \in \mathbb{R}^{65536\times 65536}$, $\widetilde{\varepsilon} = 10^{-2}$, 200th iteration

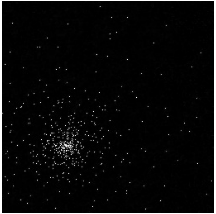
MRNSD



ReSt NNCG



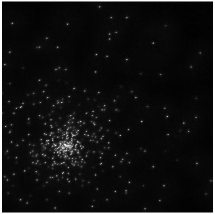
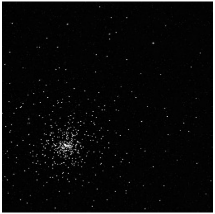
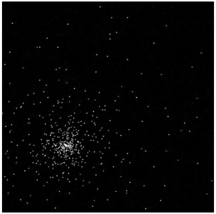
MFCGLS

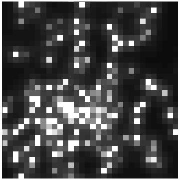
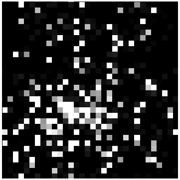
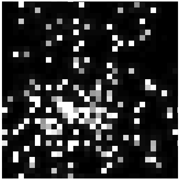


star_cluster test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-2}$, 200th iteration

MRNSD **ReSt NNCG** **MFCGLS**

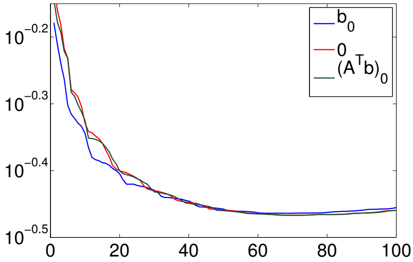




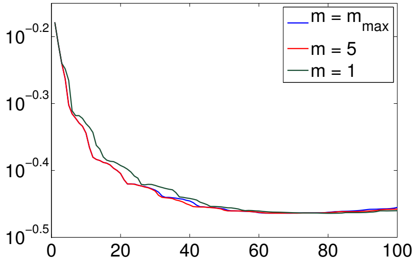
satellite test problem

satellite test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-1}$

Relative Error Histories, varying some "side" parameters





satellite test problem

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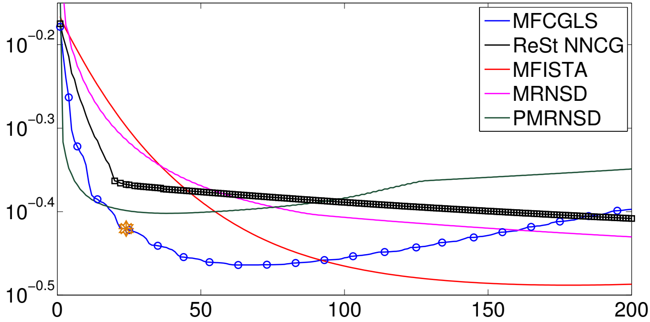
Relative Error History



ReSt NNCG by

[Hanke, Nagy, and Plemmons. Preconditioned iterative regularization, *Num.Lin.Alg.*, 1993]

satellite test problem

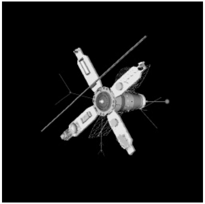
$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-1}$

	rel.error	iterations	tot.time	av.time
MFCGLS	3.5098e-01	70.33	5.49	0.08
ReSt NNCG	4.0957e-01	106.67	9.38	0.08
FISTA	3.2969e-01	164.33	21.22	0.12
MFISTA	3.2583e-01	177.00	23.10	0.13
MFISTA(0.2)	3.3318e-01	137.00	20.58	0.15
MFISTA(5)	3.3397e-01	200.00	26.86	0.13
MRNSD	3.7720e-01	200.00	12.55	0.06
PMRNSD	4.0032e-01	37.33	2.62	0.07
NNSD	4.3095e-01	200.00	13.82	0.07

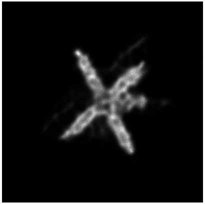
satellite test problem

$A \in \mathbb{R}^{65536 \times 65536}$, $\widetilde{\varepsilon} = 10^{-1}$

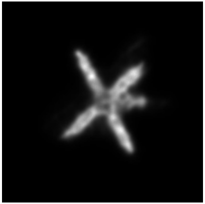
exact



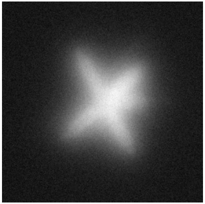
MFCGLS (# 84)



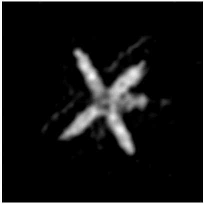
MRNSD (# 200)



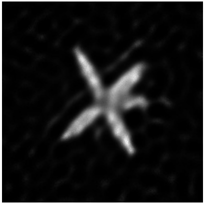
blurred & noisy



MFISTA (# 200)



PMRNSD (# 30)



`paralleltomo` test problem

paralleltomo test problem

overdetermined (81088×65536 coefficient matrix, $\widetilde{\varepsilon} = 5 \cdot 10^{-2}$)

	rel.error	iterations	tot.time	av.time
MFCGLS	1.8268e-01	17.33	0.92	0.09
ReSt NNCG	2.0133e-01	56.00	16.31	0.11
MFISTA	2.0029e-01	37.00	53.13	1.44
MRNSD	1.8506e-01	45.00	4.10	0.09
Cimmino	1.9982e-01	100.00	33.47	0.33

paralleltomo test problem

overdetermined (81088×65536 coefficient matrix, $\widetilde{\varepsilon} = 5 \cdot 10^{-2}$)

	rel.error	iterations	tot.time	av.time
MFCGLS	1.8268e-01	17.33	0.92	0.09
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MRNSD	1.8506e-01	45.00	4.10	0.09
Cimmino	1.9982e-01	100.00	33.47	0.33

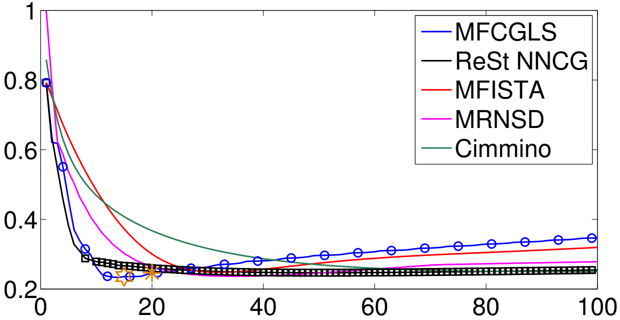
underdetermined (32580×65536 coefficient matrix, $\widetilde{\varepsilon} = 5 \cdot 10^{-2}$)

	rel.error	iterations	tot.time	av.time
MFCGLS	2.3145e-01	13.00	0.15	0.07
ReSt NNCG	2.4572e-01	51.00	0.59	0.05
MFISTA	2.4634e-01	32.00	12.41	0.39
MRNSD	2.3485e-01	35.00	3.43	0.09
Cimmino	2.4715e-01	94.33	8.84	0.09

underdetermined `paralleltomo` test problem

$A \in \mathbb{R}^{65160\times 65536}$, $\widetilde{\varepsilon} = 5 \cdot 10^{-2}$.

Relative Error History

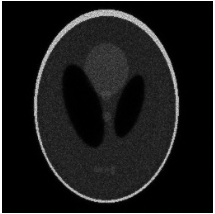


underdetermined `paralleltomo` test problem

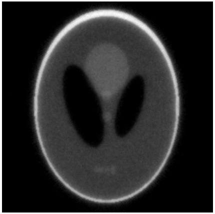
$A \in \mathbb{R}^{65160 \times 65536}$, $\widetilde{\varepsilon} = 5 \cdot 10^{-2}$, 17th iteration

exact

MFCGLS



MFISTA



Incorporating Poisson noise

$$b = \text{Poisson}(Ax^{ex}) + \text{Poisson}(\beta \mathbf{1}) + \text{Normal}(0, \sigma^2 I)$$

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$$b = \text{Poisson}(Ax^{ex}) + \text{Poisson}(\beta \mathbf{1}) + \text{Normal}(0, \sigma^2 I)$$

[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]

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[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]

$$\underbrace{b - \beta\mathbf{1}}_{=:\, b_\beta} = Ax^{ex} + \text{Normal}(0, \underbrace{\text{diag}(Ax^{ex} + \beta\mathbf{1} + \sigma^2\mathbf{1})}_{=:\, C_\eta}) \,.$$

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[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]
[Bardsley and Nagy. Covariance-prec. meth. for NN, *SIMAX*, 2006]

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$$\underbrace{b - \beta\mathbf{1}}_{=: b_\beta} = Ax^{ex} + \text{Normal}(0, \underbrace{\text{diag}(Ax^{ex} + \beta\mathbf{1} + \sigma^2\mathbf{1})}_{=: C_\eta}).$$

Problem to solve:

$$\min_{x \geq 0} \| C_\eta^{-1/2}(b_\beta - Ax)\|_2^2,$$

by the class of MRNSD methods:

Incorporating Poisson noise

$$b = \text{Poisson}(Ax^{ex}) + \text{Poisson}(\beta\mathbf{1}) + \text{Normal}(0, \sigma^2 I)$$

[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]
[Bardsley and Nagy. Covariance-prec. meth. for NN, *SIMAX*, 2006]

$$\underbrace{b - \beta\mathbf{1}}_{=:\, b_\beta} = Ax^{ex} + \text{Normal}(0, \underbrace{\text{diag}(Ax^{ex} + \beta\mathbf{1} + \sigma^2\mathbf{1})}_{=:\, C_\eta}) \,.$$

Problem to solve:

$$\min_{x\geq 0} \| C_\eta^{-1/2}(b_\beta - Ax)\|_2^2,$$

by the class of MRNSD methods:

- $$C_\eta = \text{diag}(b + \sigma^2\mathbf{1});$$

Incorporating Poisson noise

$$b = \text{Poisson}(Ax^{ex}) + \text{Poisson}(\beta\mathbf{1}) + \text{Normal}(0, \sigma^2 I)$$

[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]
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$$\underbrace{b - \beta\mathbf{1}}_{=:\, b_\beta} = Ax^{ex} + \text{Normal}(0, \underbrace{\text{diag}(Ax^{ex} + \beta\mathbf{1} + \sigma^2\mathbf{1})}_{=:\, C_\eta})\,.$$

Problem to solve:

$$\min_{x \geq 0} \|C_\eta^{-1/2}(b_\beta - Ax)\|_2^2,$$

by the class of MRNSD methods:

- WMRNSD : $\quad C_\eta = \text{diag}(b + \sigma^2\mathbf{1});$

Incorporating Poisson noise

$$b = \text{Poisson}(Ax^{ex}) + \text{Poisson}(\beta \mathbf{1}) + \text{Normal}(0, \sigma^2 I)$$

[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]
[Bardsley and Nagy. Covariance-prec. meth. for NN, *SIMAX*, 2006]

$$\underbrace{b - \beta \mathbf{1}}_{=: b_\beta} = Ax^{ex} + \text{Normal}(0, \underbrace{\text{diag}(Ax^{ex} + \beta \mathbf{1} + \sigma^2 \mathbf{1})}_{=: C_\eta}).$$

Problem to solve:

$$\min_{x \geq 0} \| C_\eta^{-1/2}(b_\beta - Ax)\|_2^2,$$

by the class of MRNSD methods:

- WMRNSD : $C_\eta = \text{diag}(b + \sigma^2 \mathbf{1});$
- $C_\eta^{(k)} = \text{diag}(Ax_k + \beta \mathbf{1} + \sigma^2 \mathbf{1}).$

Incorporating Poisson noise

$$b = \text{Poisson}(Ax^{ex}) + \text{Poisson}(\beta \mathbf{1}) + \text{Normal}(0, \sigma^2 I)$$

[Snyder et al.. Image recovery from CCD, *J. Opt. Soc. Amer. A*, 1993]
[Bardsley and Nagy. Covariance-prec. meth. for NN, *SIMAX*, 2006]

$$\underbrace{b - \beta\mathbf{1}}_{=: b_\beta} = Ax^{ex} + \text{Normal}(0, \underbrace{\text{diag}(Ax^{ex} + \beta\mathbf{1} + \sigma^2\mathbf{1})}_{=: C_\eta}).$$

Problem to solve:

$$\min_{x \geq 0} \| C_\eta^{-1/2}(b_\beta - Ax)\|_2^2,$$

by the class of MRNSD methods:

- WMRNSD : $C_\eta = \text{diag}(b + \sigma^2\mathbf{1})$;
- KWMRNSD : $C_\eta^{(k)} = \text{diag}(Ax_k + \beta\mathbf{1} + \sigma^2\mathbf{1})$.

Introducing CP-MFCGLS

Input: A, b, $x_0^0 \geq 0$, $X^{(0)}$, $C_\eta^{(0)}$.

Initialize: $r_0^k = b - Ax_0^0$, $z_0^0 = A^T(C_\eta^{(0)})^{-1} r_0^0$, $\bar{z}_0^0 = X^{(0)} z_0^0$, $d_0^0 = \bar{z}_0^0$.

- For $k = 0, \ldots,$ till a stopping criterion is satisfied
  - For $m = 0, \ldots,$ till $m_{\max}$ or $\alpha_m = 0$ or a stopping criterion is satisfied
    - Run a FCGLS with x_0^k, preconditioners $X^{(m)}$, $C_\eta^{(k)}$, and NN:
      - Compute $\alpha_m = \frac{(\bar{r}_m^k, (C_\eta^{(k)})^{-1/2} A d_m^k)}{((C_\eta^{(k)})^{-1/2} A d_m^k, (C_\eta^{(k)})^{-1/2} A d_m^k)}$.
      - Update $x_{m+1}^k = x_m^k + \alpha_m d_m^k$.
      - Update $r_{m+1}^k = r_m^k - \alpha_m A d_m^k$.
      - Compute $X^{(m+1)}$.
      - Compute $z_{m+1}^k = A^T (C_\eta^{(k)})^{-1} r_{m+1}^k$.
      - Compute $\bar{z}_{m+1}^k = X^{(m+1)} z_{m+1}^k$.
      - Set $\beta_j = -\frac{((C_\eta^{(k)})^{-1/2} A z_{m+1}^k, (C_\eta^{(k)})^{-1/2} A d_m^k)}{((C_\eta^{(k)})^{-1/2} A d_m^k, (C_\eta^{(k)})^{-1/2} A d_m^k)}$, $j = \hat{m}, \ldots, m$.
      - Update $d_{m+1}^k = \bar{z}_{m+1}^k + \sum_{j=\hat{m}}^{m} \beta_j d_j^k$.
    - Restart with the last approximation, and update $C_\eta^{(k+1)}$.

satellite test problem, Gaussian and Poisson noise

satellite test problem, Gaussian and Poisson noise

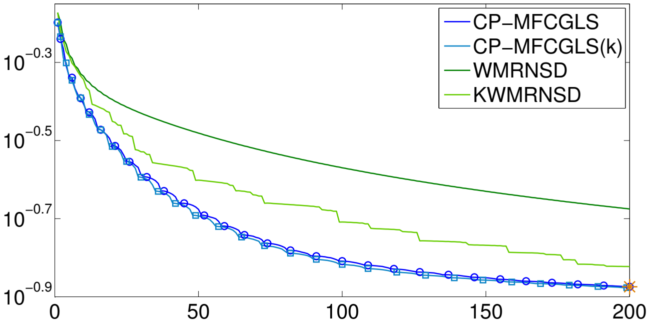
$A \in \mathbb{R}^{65536 \times 65536}$
Gauss: $\sigma = 20$, Poisson: $\beta = 60$, $\widetilde{\varepsilon} \simeq 1.5 \cdot 10^{-2}$

satellite test problem, Gaussian and Poisson noise

$A \in \mathbb{R}^{65536 \times 65536}$

Gauss: $\sigma = 20$, Poisson: $\beta = 60$, $\widetilde{\varepsilon} \simeq 1.5 \cdot 10^{-2}$

Relative Error History



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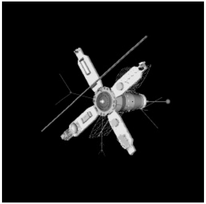
	rel.error	iterations	tot.time	av.time
CP-MFCGLS	1.2785e-01	300.00	31.65	0.08
CP-MFCGLS(k)	1.2778e-01	300.00	32.17	0.08
WMRNSD	1.8201e-01	300.00	28.34	0.09
KWMRNSD	1.3590e-01	300.00	37.19	0.12

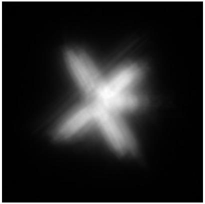
satellite test problem, Gaussian and Poisson noise

$A \in \mathbb{R}^{65536 \times 65536}$, $\tilde{\varepsilon} \simeq 1.5 \cdot 10^{-2}$, 100th iteration

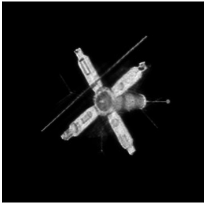
exact



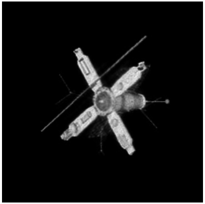
blurred & noisy



KWMRNSD



VP-MFCGLS



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Thanks for your attention!