A block Lanczos algorithm for colour images

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Outline

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Image restoration as inverse problem

The within-channel blurring process of a digital RGB color image can be formulated as a 3 Fredholm integral equation of the first kind which have the following classic form :

$$g^{(k)}(x,y) = \int\int_{\Omega} K^{(k)}(x,y,s,t) f^{(k)}(s,t) ds dt, \quad k \in \{r,g,b\} \quad (1)$$

where

- $f^{(k)}$ represents the true k channel,
- $g^{(k)}$ the blurred k channel,
- $K^{(k)}$ is a given Point Spread Function (PSF).

Linear discrete ill-posed problems

- Assuming that the blurring is spatially invariant, equations (1) can be discretized to form three independent deblurring problems

$$Ax_r = b_r, \quad Ax_g = b_g \quad \text{and} \quad Ax_b = b_b \tag{2}$$

or, using Kronecker product notation,

$$(I_3 \otimes A)x = b, \tag{3}$$

where,

$$b = \begin{bmatrix} b_r \\ b_g \\ b_b \end{bmatrix} \quad x = \begin{bmatrix} x_r \\ x_g \\ x_b \end{bmatrix}$$

multichannel images as a linear system of equation with multiple right hand sides

- The goal is to model the blurring as a linear system of equations with multiple right-hand sides
$$AX = B \tag{4}$$
where
$$B = \left[\begin{array}{ccc} b_r & b_g & b_b \end{array}\right] \quad X = \left[\begin{array}{ccc} x_r & x_g & x_b \end{array}\right]$$

- Generally, the block linear system of is contaminated by an error E,
$$AX = B, \quad B = \widehat{B} + E$$
equations

- Intuitively, when dealing with the problem (4), a simple solution will be $X = A^{-1}B$, provided that A^{-1} exists

We use Tikhonov regularization, and solve nonlinear least squares (NLLS) problem :

$$X_\mu = \underset{X}{\operatorname{argmin}} \left(\|AX - B\|_F^2 + \mu^{-1} \|X\|_F^2 \right). \tag{5}$$

The normal equations associated with (5) are given by

$$\left(A^T A + \mu^{-1} I \right) X = A^T B, \tag{6}$$

It follows that (6) has the unique solution

$$X_\mu = \left(A^T A + \mu^{-1} I \right)^{-1} A^T B. \tag{7}$$

Goals : Use a block iterative scheme to choose μ and the corresponding regularized solution X_μ

Parameter choice method : the discrepany principle

Let D_μ be the discrepancy principle defined as follows.

$$D_\mu := B - AX_\mu. \tag{8}$$

In this work, we assume that the quantity

$$\epsilon = ||E||_F, \tag{9}$$

is available. That is, the regularization parameter μ is commonly chosen so that

$$||D_\mu||_F = \eta\epsilon, \tag{10}$$

for some user-specified $\eta > 1$ and compute an approximation of X_μ

Parameter choice method : the discrepancy principle

Introduce the function

$$\phi(\mu) := ||B - AX_\mu||_F^2. \tag{11}$$

By substituting the expression of X_μ in (11) $\phi(\mu)$ can be expressed as

$$\phi(\mu) = \mathrm{tr}\left(B^T(\mu AA^T + I)^{-2}B\right), \tag{12}$$

the expression $\phi(\mu)$ can be now expressed as $\mathrm{tr}(S)$, where

$$S := B^T f_\mu\left(AA^T\right)B. \tag{13}$$

where $f_\mu(t) := (\mu t + 1)^{-2}$. We now write S as a Riemann-Stieltjes integral

Introduce the following spectral factorization

$$AA^T = W\Lambda W^T, W \in \mathbb{R}^{n\times n}, W^T W = I_n, \Lambda = \mathrm{diag}[\lambda_1, ..., \lambda_n],$$

with $\lambda_1 \leq ... \leq \lambda_n$. Defining $\Gamma = [\Gamma_1, ..., \Gamma_n] = B^T W \in \mathbb{R}^{k\times n}$, where $\Gamma_i \in \mathbb{R}^k$, it follows that

$$S = \Gamma f_\mu(\Lambda)\Gamma^T = \sum_{i=1}^{n} f_\mu(\lambda_i)\Gamma_i\Gamma_i^T = \int f_\mu(\lambda) d\Gamma(\lambda) := \mathcal{I}f_\mu, \qquad (14)$$

where $\Gamma : \mathbb{R} \to \mathbb{R}^{k\times k}$ is discrete matrix distribution with a jump of size $\Gamma_i\Gamma_i^T$ at each eigenvalue λ_i of AA^T.

We define the following inner product induced by the measure $\Gamma(\lambda)$,

$$< p, q >= \int p(\lambda) d\Gamma(\lambda) q(\lambda)^T,$$

and let $p_i, i = 1, 2, ...$ be a sequence of matrix polynomials orthonormal with respect to this inner product, i.e.,

$$\langle p_i, p_j \rangle = \delta_{ij} I_k,$$

The sequence of matrix orthonormal polynomials p_i satisfy a block three-term recurrence of the form

$$\lambda p_{j-1}(\lambda) = p_j(\lambda)\Gamma_j + p_{j-1}(\lambda)\Omega_j + p_{j-2}(\lambda)\Gamma_{j-1}^T, \quad p_0(\lambda) := I_k, \quad p_{-1}(\lambda) := 0.$$

it follows that

$$\lambda P_l(\lambda) = P_l(\lambda) J_{kl} + P_l(\lambda)\Gamma_l E_l^T, \tag{15}$$

where

$P_l(\lambda) := [p_0(\lambda), ..., p_l(\lambda)] \in \mathbb{R}^{k\times kl}$, $E_i := [e_{(i-1)k+1}, ..., e_{ik}]$ and

$$J_{kl} := \begin{bmatrix} \Omega_1 & \Gamma_1^T & & & \\ \Gamma_1 & \Omega_2 & \Gamma_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & \Gamma_{l-2} & \Omega_{l-1} & \Gamma_{l-1}^T \\ & & & \Gamma_{l-1} & \Omega_l \end{bmatrix} \in \mathbb{R}^{kl\times kl}.$$

The matrix J_{kl} is computed via a partial block Lanczos tridiagonalization of the matrix AA^T without explicit knowledge of the measure $d\Gamma$.
Let $B = QR$ be the QR factorization of B. We set $P_1 = Q$ and then use the following algorithm

1. Let $P_1 \in \mathbb{R}^{n\times k}$ be an initial matrix satisfying $P_1^T P_1 = I_k$
2. Set $P_0 := 0 \in \mathbb{R}^{n\times k}$, $\Gamma_0 = 0 \in \mathbb{R}^{k\times k}$
3. for $j = 1, 2, \ldots, l$
    1. $\Omega_j = P_j^T AA^T P_j$
    2. $R_j = AA^T P_j - P_j\Omega_j - P_{j-1}\Gamma_{j-1}^T$
    3. $P_{j+1}\Gamma_j = R_j$ (QR factorization), P_{j+1} is orthogonal and Γ_j is upper triangular
4. endfor

If the algorithm does not break down before step l, then it is easy to verify the following relation

$$AA^T P_l^{(k)} = P_l^{(k)} J_{kl} + P_{l+1} \Gamma_l E_l^T, \tag{16}$$

where $P_l^{(k)} = [P_1, ..., P_l]$, and J_{kl} is the matrix from (15). Moreover, the vector -columns $P_l^{(k)}$ form an orthonormal basis of the block Krylov subspace

$$\mathbb{K}_l(AA^T, P_1) = \text{Range}[P_1, AA^T P_1, (AA^T)^2 P_1, ..., (AA^T)^{l-1} P_1].$$

We now want to approximate $\mathcal{I}f_\mu = \int f_\mu(\lambda)d\Gamma(\lambda)$ by using block Gauss quadrature and block anti-Gauss quadrature. The most general quadrature formula is of the form

$$\mathcal{G}_l f_\mu = \sum_{i=1}^{l} W_i f_\mu(T_i) W_i^T, \tag{17}$$

By diagonalizing each T_i , we can obtain the following simpler formula

$$\mathcal{G}_l f_\mu = E_1^T f_\mu(J_{kl}) E_1, \tag{18}$$

whose remainder formula is given by the following $k \times k$ matrix

$$\mathcal{I}f_\mu - \mathcal{G}_l f_\mu = \frac{f_\mu^{(2l)}(\eta)}{(2l)!} \int s(\lambda) d\Gamma(\lambda), \tag{19}$$

- The idea of block anti-Gauss quadrature rule is to construct a quadrature rule whose error is equal but of opposite sign to the error of block Gauss rule
- The $(l+1)$-block anti-Gauss quadrature rule $\mathcal{H}_{l+1} f_\mu$ is then characterized by

$$(\mathcal{I} - \mathcal{H}_{l+1})f = -(\mathcal{I} - \mathcal{G}_l)f, \quad f \in \mathbb{P}^{2l+1}. \tag{20}$$

which also implies that

$$\mathcal{H}_{l+1} f = (2\mathcal{I} - \mathcal{G}_l)f, \quad f \in \mathbb{P}^{2l+1}.$$

- We can demonstrate as above that the $(l+1)$-block anti-Gauss quadrature rules can be expressed as

$$\mathcal{H}_{l+1} f_\mu = E_1^T f_\mu(\tilde{J}_{k(l+1)}) E_1 \tag{21}$$

where,

$$\tilde{J}_{k(l+1)} := \begin{bmatrix} \tilde{\Omega}_1 & \tilde{\Gamma}_1^T & & & \\ \tilde{\Gamma}_1 & \tilde{\Omega}_2 & \tilde{\Gamma}_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{\Gamma}_{l-1} & \tilde{\Omega}_l & \tilde{\Gamma}_l^T \\ & & & \tilde{\Gamma}_l & \tilde{\Omega}_{l+1} \end{bmatrix} \in \mathbb{R}^{k(l+1)\times k(l+1)},$$

with,

$$\begin{aligned} \tilde{\Omega}_i &= \Omega_i, \quad 1 \leq i \leq l \\ \tilde{\Gamma}_i &= \Gamma_i, \quad 1 \leq i \leq l-1 \\ \tilde{\Gamma}_l &= \sqrt{2}\Gamma_l, \\ \tilde{\Omega}_{l+1} &= \Omega_{l+1}; \end{aligned}$$

C. Fenu, D. Martin, L. Reichel, and G. Rodriguez, *Block Gauss and anti-Gauss quadrature with application to networks,*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 1655-1684

- We recall that determining the regularization parameter μ_ϵ can be achieved by solving the following nonlinear equation,

$$\phi(\mu) = \eta^2\epsilon^2, \tag{22}$$

- By using block Gauss and anti-Gauss quadrature, $\phi(\mu)$ can be approximated by $\phi_l(\mu) = \mathrm{tr}(\mathcal{L}_{2l+1} f_\mu)$,

$$\mathcal{L}_{2l+1} f_\mu := \frac{1}{2}(\mathcal{G}_l f_\mu + \mathcal{H}_{l+1} f_\mu), \tag{23}$$

  for l small
- Thus, instead of solving (22), we solve following small problem

$$\phi_l(\mu) = \eta^2\epsilon^2 \tag{24}$$

- For each l, approximations of $[\mathcal{I}f_\mu]_{ij}$ are given by $[\mathcal{G}_l f_\mu]_{ij}$ and $[\mathcal{H}_{l+1} f_\mu]_{ij}$, $1 \leq i, j \leq k$.
- The purpose of our proposed numerical method is to keep the number of block Lanczos algorithm steps l small.
- In order for $\mathcal{L}_{2l+1} f_\mu$ to be a good approximation of $\mathcal{I}f_\mu$, the following stopping criteria is used,

$$E_{l,\mu} := \frac{1}{2}\| G_l f_\mu - H_{l+1} f_\mu \|_{\max} < \tau \tag{25}$$

where τ is an absolute tolerance and $\|B\|_{\max} = \max_{1\leq i,j\leq k} |B_{ij}|$

- We assume that for some l, the above stopping criteria is satisfied for $\mu = \mu_l^{(p)}$
- We now want to solve the normal equation

$$(A^T A + \mu_l^{(p)} I)X = A^T B,$$

- Let $B = QR$ be the QR factorization of B. We want to compute a sequence of approximations solutions

$$X_l = Q_l^{(k)} Y_l, \quad l = 1, 2, \ldots \tag{26}$$

where $Q_l^{(k)}$ is the orthonormal matrix defined in the following decompositions

$$\begin{aligned} A^T P_l^{(k)} &= Q_l^{(k)} R_{kl}^T & (27)\\ A Q_l^{(k)} &= P_l^{(k)} R_{kl} + F_k E_k^T, & (28) \end{aligned}$$

where $P_l^{(k)} \in \mathbb{R}^{n \times lk}$, $Q_l^{(k)} = [Q_1, \ldots, Q_l] \in \mathbb{R}^{n \times lk}$,

- $P_l^{(k)^T} P_l^{(k)} = Q_l^{(k)^T} Q_l^{(k)} = I_{lk}$. We refer to $F_k \in \mathbb{R}^{n\times k}$ as the residual matrix. It satisfies
$$P_l^{(k)^T} F_k = 0.$$
The matrix
$$R_{kl} := \begin{bmatrix} S_1 & L_2 & & & \\ & S_2 & L_2 & & \\ & & S_3 & \ddots & \\ & & & \ddots & L_l \\ & & & & S_l \end{bmatrix} \in \mathbb{R}^{kl\times kl}, \tag{29}$$
is upper triangular
- We refer to this decompositions as a partial block Lanczos bidiagonalization of A with initial matrix $P_1 = Q$

The approximate solution X_l is then determined by the following Galerkin equations

$$\begin{array}{rcl} Q_l^{(k)^T}(A^TA+\mu_l^{(p)}I)Q_l^{(k)}Y_l & = & Q_l^{(k)^T}A^TB, \\ & = & Q_l^{(k)^T}A^TP_1R, \\ & = & Q_l^{(k)^T}Q_1S_1R, \\ & = & E_1S_1R. \end{array}$$

We have

$$Q_l^{(k)^T}(A^TA+\mu_l^{(p)}I)Q_l^{(k)}Y_l=(R_{kl}^TR_{kl}+E_lL_{l+1}L_{l+1}^TE_l^T+\mu_l^{(p)}I_{kl})Y_l$$

We then compute the solution Y_l by solving

$$\min_{Y_l}\left\|\begin{bmatrix}\bar{R}_{kl}\\ {\mu_l^{(p)}}^{1/2}I_{kl}\end{bmatrix}Y_l-{\mu_l^{(p)}}^{-1/2}\begin{bmatrix}0\\ S_1R\end{bmatrix}\right\|_F^2, \qquad (30)$$

where $\bar{R}_{kl}=\left[R_{kl},L_{l+1}^TE_l^T\right]$

We summarize our approach by the following the algorithm,

1. Input : A, B, ϵ, τ, η, μ : initial guess for the zero-finding method
2. Compute $B = QR$ and set $P_1 := Q \in \mathbb{R}^{n \times k}$,
3. Fro $l = 1, 2, \ldots$ until $E_{l,\mu} < \tau$
    1. Determine the matrix with orthonormal columns $Q_l^{(k)}$ and the block-bidiagonal matrix R_{kl} by the block Lanczos bidiagonalization
    2. Update the value μ by solving $\phi_l(\mu) = \eta^2 \epsilon^2$ with the zero-finding method
4. Determine Y_l by solving

$$\min_{Y_l} \left\| \begin{bmatrix} \bar{R}_{kl} \\ {\mu_l^{(p)}}^{1/2} I_{kl} \end{bmatrix} Y_l - {\mu_l^{(p)}}^{-1/2} \begin{bmatrix} 0 \\ S_1 R \end{bmatrix} \right\|_F^2,$$

and then X_l by

$$X_l = Q_l^{(k)} Y_l$$

Example 1 : The blurring matrix A is given by

$$A = (2\pi\sigma^2)^{-1} A_1 \otimes A_2$$

where $A_1 = A_2 = [a_{ij}]$ and $[a_{ij}]$ is a Toeplitz matrix given by

$$a_{ij} = \begin{cases} \frac{1}{\sigma\sqrt{(2\pi)}} \exp\left(-\frac{(i-j)^2}{2\sigma^2}\right), |i-j| \leq r, \\ 0 \qquad \text{otherwise} \end{cases}$$

We set $r = 8$ and $\sigma = 2$. We then consider a 1% noise level. We construct the corrupted image, in block form, as

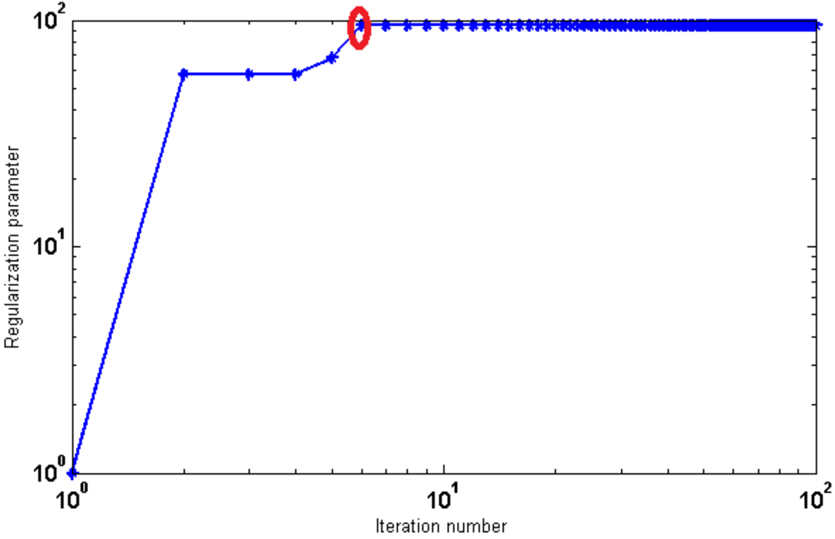
$$B = AX + E$$

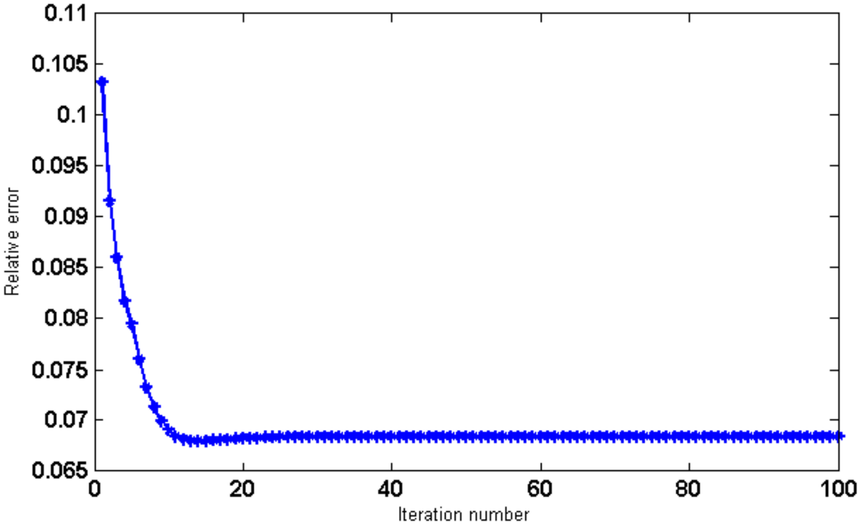
.

Original $256 \times 256 \times 3$ fruits image and image contaminated by 1% noise and Gaussian blur.

Restoration by BLA with $\eta = 2$ and $\tau = 10^{-2}$ determined by 18 block Lanczos steps

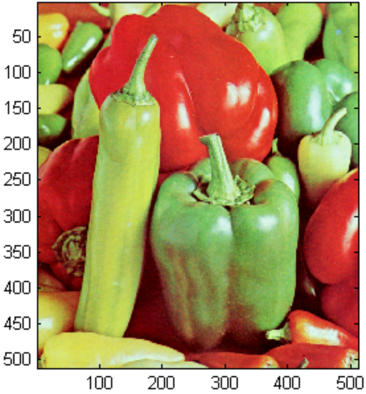
- The computed optimal value $\mu^{-1} = 1.05 \times 10^{-2}$
- The signal to noise ratio (SNR) is given by $\mathrm{SNR}(X_\mu) = 27.50$

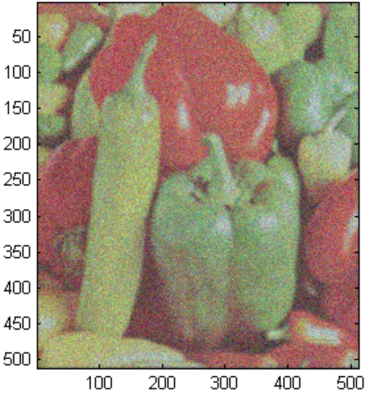

10^2
10^1
10^0
Regularization parameter
10^0
10^1
10^2
Iteration number



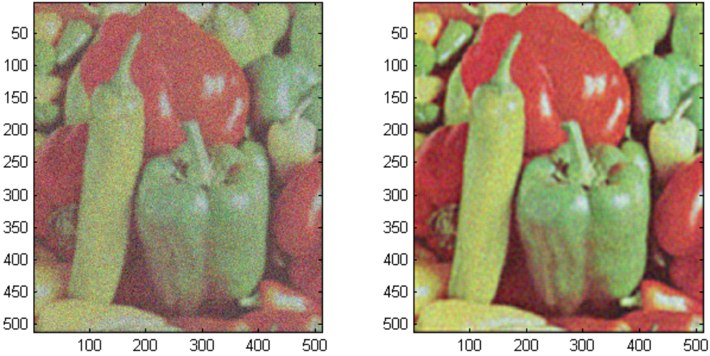
0.11
0.105
0.1
0.095
0.09
0.085
0.08
0.075
0.07
0.065
Relative error
0
20
40
60
80
100
Iteration number

Example 2 : Original $512 \times 512 \times 3$ peppers image and image contaminated by 30% noise and Gaussian blur($\sigma = 4$, $r = 4$).





Restoration by BLA with $\eta = 2$ and $\tau = 10^{-3}$ determined by 3 block Lanczos steps



- The computed optimal value $\mu^{-1} = 2.43 \times 10^{-2}$
- The signal to noise ratio (SNR) is given by $\text{SNR}(X_\mu) = 17.8345$

- We now compare our method with solving the linear least squares problem using the SVD of A, where the latter matrix is given as a Kronecker product.
- For large problems, the SVD of A can be obtained from the SVD of the Kronecker factors A_1 and A_2
- If $A_1 = U_1 \Sigma_1 V_1^T$ and $A_2 = U_2 \Sigma_2 V_2^T$, then

$$A = A_1 \otimes A_2 = (U_1 \Sigma_1 V_1^T) \otimes (U_2 \Sigma_2 V_2^T) = (U_1 \otimes U_2)(\Sigma_1 \otimes \Sigma_2)(V_1 \otimes V_2)^T$$

- In order to find a good regularization parameter, Newton's method is utilized to find the solution μ of $\|Ax_\mu - b\|_2 = \eta \epsilon$.
- We refer to the method utilizing the SVD decomposition of the Kronecker factors for computing the regularization parameter and the corresponding regularized solution as KSVD

Table: Results for Example 2.

Method	μ^{-1}	PSNR	CPU time(sec)
BLA	2.43×10^{-2}	17.83	0.82
KSVD	2.04×10^{-2}	16.71	2.12

Concluding remarks

- We presented in this work a method for inexpensively compute a suitable ragularization parameter for large ill-posed linear system of multiple right hand sides
- The proposed method is based on block Lanczos algorithm and block Gauss quadrature
- We applied the method for the restoration of a real digital noisy and blurred image by using the Tikhonov regularization
- The numerical tests show that the method is effective