Solution of inverse hyperbolic problems via data-driven ROMs

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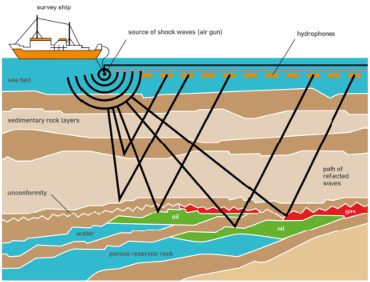
Outline

Motivation: seismic oil&gas exploration

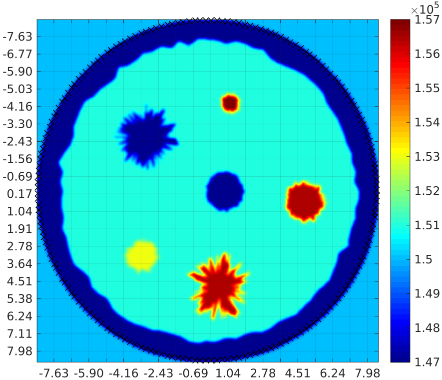


- **Full waveform inversion** (FWI): inverse problem for a hyperbolic PDE with no approximations
- Seismic waves induced by sources (shots)
- Measurements of seismic signals on the surface or in a well bore
- Determine the acoustic or elastic parameters of the subsurface

Conventional approach: FWI by nonlinear fitting of the simulated data via multiple forward problem solutions. Challenges: computational grids $> 10^9$; huge Jacobians; non-convex misfit functional.

Other possible applications: ultrasound tomography

Malignant breast tumors can be distinguished by irregular boundary



- Ultrasound screening for early detection of breast cancer
- Conventional ultrasound imaging techniques are rather crude, advanced methods originating in geophysics are in demand

ROM as fast proxy for forward solver

- One way to address high computational cost of the forward solvers is to use their fast proxies constructed via *parametric* ROMs, e.g., reduced basis method. Mainly applied for parabolic problems, in application to diffusive optical tomography (DOT), hydro-geology, control-source electromagnetic method (CSEM) in exploration geophysics, etc. [3]
- Wave problems: Mikhail Zaslasvsky yesterday, Jörn Zimmerling on Friday.

[3] Ghattas,Lieberman, Wilcox, Daniel, White, de Sturler, Kilmer, Gugercin, Beattie, Dr., Simoncini, Zaslavsky, etc.

ROM as direct imaging tool

- We pursue another approach, first emerged for the solution of the electrical impedance tomography (EIT) problems: We construct the reduced order model from the data (a.k.a. data driven ROM), and then estimate the unknown PDE coefficients directly via the matrix of the ROM state equation.

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- We will show that this is possible for a special class of ROMs mimicking finite-volumes discretization of the underlying PDEs.

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- We will show that this is possible for a special class of ROMs mimicking finite-volumes discretization of the underlying PDEs.
- Illustrate *forgotten* connections between classical algorithms of linear algebra and inverse scattering.

Outline

1D wave problem

- We start with 1D wave equation on $[0,1] \times t$

$$-v^2 u_{xx} + u_{tt} = 0, \quad u_t|_{t=0} = 0, \ u|_{t=0} = b, \quad u_x|_{x=0} = 0, \ u|_{x=1} = 0.$$

with regular enough variable wave speed $v(x) > 0$. After transition to the travel time coordinates and symmetrization of the Laplace operator (Liouville transform), we obtain

$$Au + u_{tt} = 0, \quad u_t|_{t=0} = 0, \ u|_{t=0} = b,$$

with Schödinger operator $Au = -u_{xx} - qu$, where $q = v^{0.5} \frac{d^2}{dx^2} v^{-0.5}$.[4]

[4] Here and below we abuse notation by using the same variable notations for the transformed equation.

Continuous inverse problem

- The solution can be written as $u = \cos(t\sqrt{A})b$.
- We consider single input/single output (SISO) data (transfer function)

$$f(t) = \langle u(t), b\rangle = \langle \cos(t\sqrt{A})b, b\rangle$$

assuming that b is smooth approximation of $\delta(x+0)$, here $\langle ., .\rangle$ is the $L_2[0,1]$-inner product.

Problem

Inverse problem: $f \mapsto v$.

Discrete-time data

- We choose time discretization step τ (corresponding the Nyquist sampling rate of the signal) and consider discrete data

$$f(j\tau), j = 0, \ldots, 2n-1$$

.

Problem

To image the wave-speed distribution on the scale of the minimal wavelength.

Outline

The discrete-time dynamic system

- Let $P = \cos(\tau\sqrt{A})$ be so-called transition operator, $u_i = u(i\tau) = \cos(i\tau\sqrt{A})b$ the snapshots of the solution, then the data can be formally computed via discrete-time dynamic system

$$\begin{aligned} \tilde{A}u_i + \frac{u_{i-1} - 2u_i + u_{i+1}}{\tau^2} &= 0, \\ u_0 = b, \quad u_{-1} &= u_0, \end{aligned}$$

  with $\tilde{A} = \frac{2}{\tau^2}(I - P) = A + O(\tau^2)$,
  so $b^* u_j = f(j\tau), \qquad j = 0, \ldots, 2n-1.$
- The discrete-time dynamic system up to $O(\tau^2)$ coincides with the standard second-order time stepping scheme.

Formulation in terms of Chebyshev polynomials

- The snapshots can be written as $u_i = u(i\tau) = \cos(i\tau\sqrt{A})b = T_i(P)b$, T_j are the Chebyshev polynomials of the first kind, $P = \cos(\tau\sqrt{A})$.
- The Chebyshev polynomials appear from identities $\cos(\tau j\sqrt{A}) \equiv \cos[j \arccos(P)]b \equiv T_j(P)$.
- So, the data can be written in terms of Chebyshev moment problem

$$f(\tau j) = \langle \cos(i\tau\sqrt{A})b, b\rangle = \langle T_j(P)b, b\rangle, \quad j = 0, \ldots, 2n-1.$$

Data-driven discrete-time ROM

- Let $\mathbf{U} \in \mathbb{R}^{\infty,n}$ be the semiinfinite snapshot matrix with columns u_i, and $\mathbf{U}^*\mathbf{U}, \mathbf{U}^*\tilde{A}\mathbf{U} \in \mathbb{R}^{n\times n}$ are resp. mass and stiffness matrices with elements $\langle u_i, u_j\rangle$, $\langle u_i, \tilde{A}u_j\rangle$ resp.

Proposition (Dr., Mamonov, Thaler, Zaslavsky, SIIMS 2016)

The discrete-time ROM

$$\mathbf{U}^*\tilde{A}\mathbf{U}\tilde{u}_i + \mathbf{U}^*\mathbf{U}\frac{\tilde{u}_{i-1} - 2\tilde{u}_i + \tilde{u}_{i+1}}{\tau^2} = 0,$$
$$\tilde{u}_0 = \tilde{b}, \quad \tilde{u}_{-1} = \tilde{u}_0,$$

satisfies data matching condition

$$\tilde{b}^*\tilde{u}_j = f(j\tau), \qquad j = 0, \ldots, 2n-1,$$

here $\tilde{b} = \mathbf{U}^*\mathbf{U}e_1$.

Hankel + Toeplitz algorithm for matrix pencil

- For i, j element of $\mathbf{U}^*\mathbf{U}$ we obtain

$$
\begin{aligned}
u_i^* u_j = b^* T_i(P) T_j(P) b = b^* \cos(i\tau\sqrt{A}) \cos(j\tau\sqrt{A}) b &= \\
0.5 \left\{ b^* \cos[(i+j)\tau\sqrt{A})] b + b^* \cos[(i-j)\tau\sqrt{A})] b \right\} &= \\
0.5 \left\{ \overbrace{f[(i+j)\tau]}^{Hankel} + \overbrace{f[(i-j)\tau]}^{Toeplitz} \right\}
\end{aligned}
$$

- Similar formula for $\mathbf{U}^*\tilde{A}\mathbf{U} = \mathbf{U}^*\frac{2}{\tau^2}(I-P)\mathbf{U}^*$.

Resistivity imaging from reduced order system?

- Mapping from true v to the ROM can be summarized via sequence:

$$v \overset{measurement}{\mapsto} f(j\tau), j = 1, \ldots, 2n-1 \overset{data-drivenROM}{\mapsto} \mathbf{U}^* \tilde{A} \mathbf{U}, \mathbf{U}^* \mathbf{U}$$

- **Approx. inverse mapping?**

$$\mathbf{U}^* \tilde{A} \mathbf{U}, \mathbf{U}^* \mathbf{U} \mapsto \tilde{v} \approx v$$

.

- Need to infuse some geometric interpretation in the ROM. For that we transform it to a form mimicking finite-difference time domain discretization (FDTD) in both time and space.

QR tri-diagonalization

Let R be the upper triangular matrix obtained from the Cholesky decomposition $RR^* = (\mathbf{U}^*\mathbf{U})^{-1}$ with natural row indexes from 1 to n. It is the QR transform of $\mathbf{U}$, yielding $\mathbf{V} = \mathbf{U}R \in \mathbb{R}^{\infty,n}$, $\mathbf{V}^*\mathbf{V} = I$.

Proposition

The QR transform generates tridiagonal matrix

$$\mathcal{T} = R^*(\mathbf{U}^*\tilde{A}\mathbf{U})R = \mathbf{V}^*\tilde{A}\mathbf{V},$$

so the ROM can be written in the FD form both in time and space

$$\begin{aligned} \mathcal{T}\tilde{u}_i + \frac{\tilde{u}_{i-1} - 2\tilde{u}_i + \tilde{u}_{i+1}}{\tau^2} &= 0, \\ \tilde{u}_0 = e_1, \quad \tilde{u}_{-1} &= \tilde{u}_0, \end{aligned}$$

with data matching condition $e_1^*\tilde{u}_j = f(j\tau), \qquad j = 0, \ldots, 2n-1.$

Finite-difference inversion

- Tridiagonal $\mathcal{T}$ can be interpreted as the finite-difference approximation of A, so we can write the ROM state equation in the FDTD form[5] as $\frac{v_i}{\hat{h}_i}\left(\frac{w_{i+1,j}-w_{i,j}}{\hat{v}_i h_i} - \frac{w_{i,j}-w_{i-1,j}}{\hat{v}_i h_{i-1}}\right) - \frac{w_{i,j-1}-2w_{i,j}+w_{i,j+1}}{\tau^2} = 0, \; i = 1, \ldots,$ subject to some initial and boundary conditions, where $h_i, \hat{h}_i$ are respectively primary and dual grid steps, and $\hat{v}_i$, v_i are the values of the FD wave speed at respectively dual and primary nodes . Special choice of grid (a.k.a. optimal grid) allowed earlier to obtain reasonably good images for 1D and 2D EIT problems and works well for 1D wave problems. [6]. However, we have not been able to produce competitive inversion results with the FD approach for 2D seismic problems due to poor lateral resolution.

[5]The concept of the FD realization of ROMs originated in 1950s by Mark Krein.
[6]Borcea, Dr., Guevara Vasquez, Mamonov, Inside Out, 2010; Dr.,Mamonov, Thaler, Zaslavsky, SIIMS 2016

Back-projection approach

- Recall that $\tilde{A} = A + O(\tau^2)$. So, neglecting $O(\tau^2)$ term, $\mathcal{T}$ can be viewed as Galerkin projection of A on basis $\mathbf{V}$, i.e.,

$$\mathcal{T} = \mathbf{V}^* \tilde{A} \mathbf{V} \approx \mathbf{V}^* A \mathbf{V},$$

so

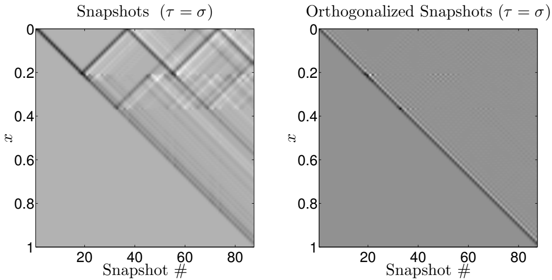
$$A \approx \mathbf{V} \mathcal{T} \mathbf{V}^*.$$

- We can compute $\mathcal{T}$ directly from the data (Hankel+Toeplitz,then Cholesky), but don't know $\mathbf{V}$.
- Let assume that $\mathbf{V} \approx \mathbf{V}_0$, where $\mathbf{V}_0$ is computed for some background wave-speed distribution v_0. If this holds, then from the data and known background v_0 we can compute

$$A \approx \mathbf{V}_0 \mathcal{T} \mathbf{V}_0^*.$$

- This assumption was originally motivated by similarity of $\mathcal{T}$ to the stiffness matrix of piece-wise linear FE method. It turns out, that it is the core of famous Marchenko-Gelfand-Levitan inversion algorithm.

Marchenko-Gelfand-Levitan-Krein in nutshell



- Left. Matrix **U** for a layered model. It is upper triangular due to causality, with pronounced multiple reflections.
- Right. Matrix **V** obtained after the Gram-Schmidt orthogonalization of **U**'s columns (QR). The orthogonalization suppresses multiples.
- Basic linear algebra: full rank upper triangular matrix + QR = identity

Galerkin back-projection algorithm

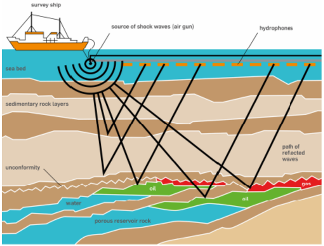
- We denote $\mathcal{T}_0 = \mathbf{V}_0^* A_0 \mathbf{V}_0$, where $A_0 = \Delta + q_0 = \Delta + v_0^{0.5} \frac{d^2}{dx^2} v_0^{-0.5}$ is the Schroedinger operator corresponding the background wave-speed v_0. Then we define the Galerkin back-projection algorithm as

$$\delta q = \mathrm{diag}^{-1} \left[\mathbf{V}_0 \mathbf{V}_0^*\right] \mathrm{diag} \left[\mathbf{V}_0 (\mathcal{T} - \mathcal{T}_0) \mathbf{V}_0^*\right].$$

- By definition $A - A_0 = q - q_0 = v^{0.5} \frac{d}{dx} v^{-0.5} - v_0^{0.5} \frac{d}{dx} v_0^{-0.5}$, so

$$\delta q \approx q - q_0 = v^{0.5} \frac{d^2}{dx^2} v^{-0.5} - v_0^{0.5} \frac{d^2}{dx^2} v_0^{-0.5}.$$

2D setting



- We consider 2D inverse problem for acoustic wave eq. with an array of m receivers. The shots are fired by moving the transmitter consequently at the receiver positions, so the data are the elements of the matrix-valued multi-input/multi-output (MIMO) transfer function

$$F(t) = F(t)^* \in \mathbb{R}^{m \times m},$$

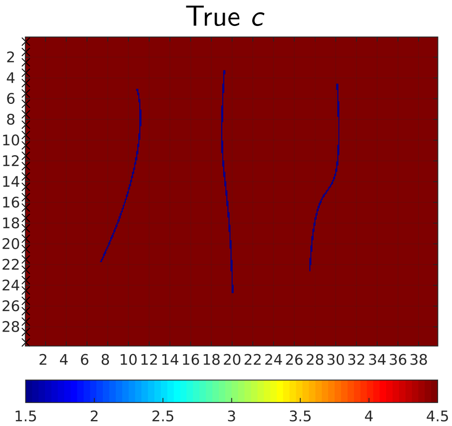
sampled at $t = j\tau$, $j = 0, \ldots, 2n - 1$.

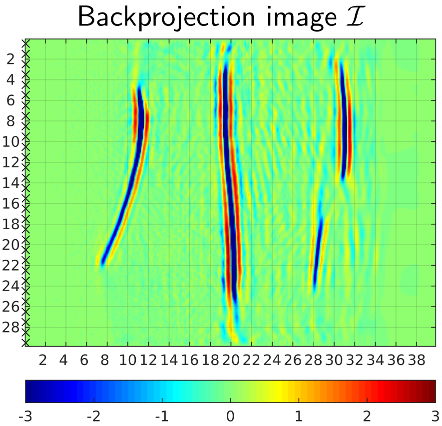
Block-generalization

- All SISO linear algebra is automatically extended to the MIMO case by using $m \times m$ matrices instead of numbers (with some instability treatment), i.e., instead of tridiagonal $\mathcal{T} \in \mathbb{R}^{n \times n}$ we will have block-tridiagonal matrix $\mathcal{T} \in \mathbb{R}^{mn \times mn}$ with $m \times m$ blocks, etc.

Outline

High contrast example: hydraulic fractures





- Important application: acoustic monitoring of hydraulic fracturing
- Multiple thin fractures (down to $1cm$ in width, here $10cm$)
- Very high contrasts: $c = 4500m/s$ in the surrounding rock, $c = 1500m/s$ in the fluid inside fractures

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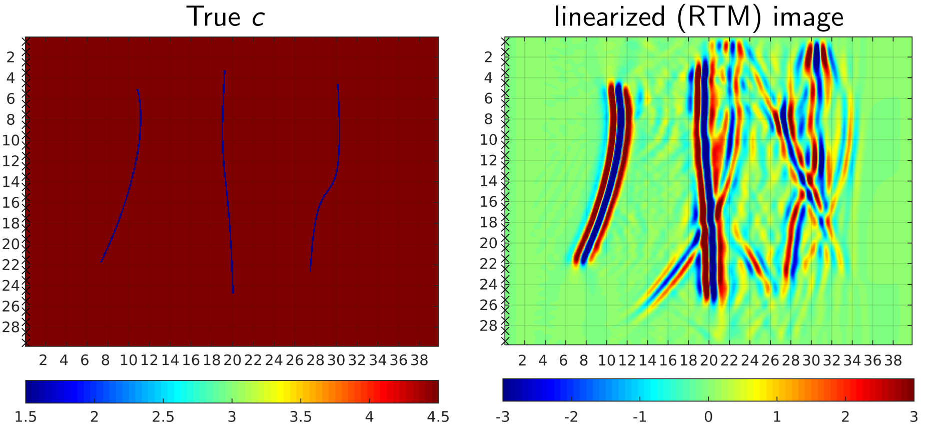

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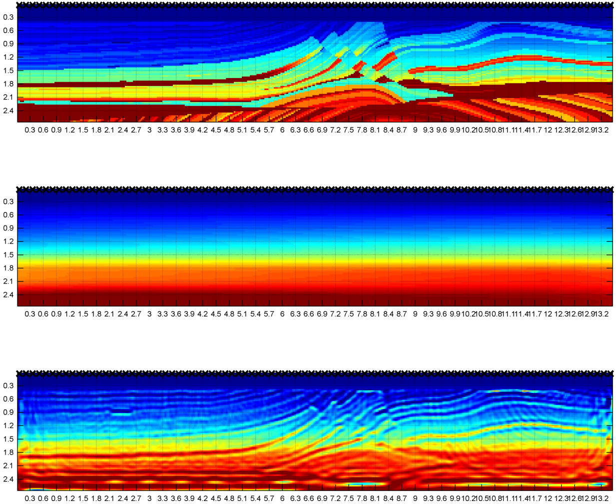
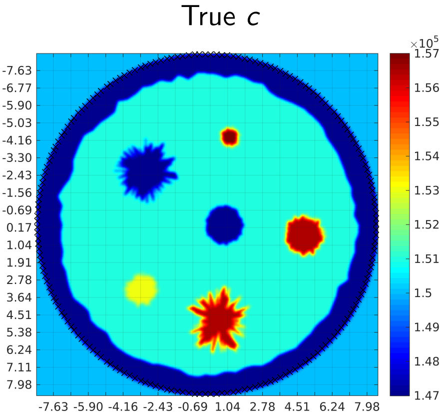
- Strong reflections, any linearized image dominated with multiples

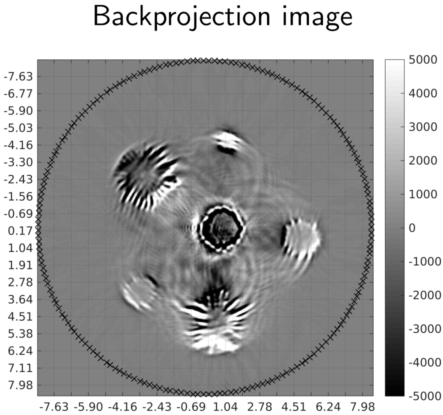
Large scale model: 2D Marmousi; 1D background



From top to bottom: 2D Marmousi model (section); background (1D model); image.

Other possible applications: ultrasound tomography





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Summary

- We presented a linear algebraic framework of the solution of the inverse hyperbolic problem by constructing ROM from data and then imaging media directly via the the equivalent reduced order state variable problem. Its sparsity pattern mimics the finite-volumes discretization of the underlying PDE, so corresponding projection formulation yields localizes basis weakly dependent on the PDE coefficients.
- Beautiful connection of reduced order models, QR tri-diagonalization and celebrated approach of Marchenko-Gelfand-Levitan-Krein.
- POD/balanced truncation type of approach for data compression/regularization in progress, promising results.
- Possible generalizations to other interpolatory formulations, e.g., frequency interpolation of diffusion problems via Löwner type formulas.