On the approximation of positive definite Hankel matrices

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Outline

1. Motivation: fast multiplication with structured matrices
2. Our structure: small displacement rank
3. Decay of singular values and the Zolotarev problem
4. Estimates for the Zolotarev problem
5. Consequences for
   - Cauchy matrices
   - Pick/Löwner matrices
   - Vandermonde matrices
   - real positive definite Hankel matrices.

Fast matrix-vector multiplication

Given $X \in \mathbb{C}^{m\times n}$, we define for $k \geq 1$ the singular numbers

$$\sigma_k(X) = \min\{\|X - B\| : \mathrm{rank}(B) < k\},$$

attained for matrix X_k (with euclidean/spectral norm).

In case $m = n$ of a square matrix, we can approach the matrix-vector product Xc by $X_{k+1}c$ in complexity $\mathcal{O}(kn)$, with precision

$$\sup_{c\in\mathbb{C}^n} \frac{\|Xc - X_{k+1}c\|}{\|X\|\,\|c\|} = \frac{\sigma_{k+1}(X)}{\sigma_1(X)}.$$

With ϵ-rank

$$\mathrm{rank}_\epsilon(X) = \min\{k \geq 0 : \frac{\sigma_{k+1}(X)}{\sigma_1(X)} \leq \epsilon\}$$

we get complexity $\mathcal{O}(n\,\mathrm{rank}_\epsilon(X))$ for precision ϵ.

Fast Hadamard matrix-vector multiplication

Hadamard product $T \odot X = \left(T_{j,k} X_{j,k}\right)_{j,k}$.

Observation [Townsend, Webb & Olver'16]: Suppose that matrix-vector product Tc has complexity $\mathcal{O}(n \log(n))$ (e.g., T Toeplitz, Hankel, circulant matrix).

Then we can approach $(T \odot X)c$ by precision ϵ in complexity $\mathcal{O}(n \log(n) \operatorname{rank}_\epsilon(X))$.

Idea of proof: If $X = uv^T$ is of rank 1 then
$(T \odot X)c = \operatorname{diag}(u) T \operatorname{diag}(v) c$ has complexity $\mathcal{O}(n \log(n))$. □

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From $\|T \odot Y\|_F \leq \|T\| \, \|Y\|_F$ we get error estimate

$$\frac{\|T \odot X - T \odot X_{k+1}\|_F}{\|T\| \, \|X\|_F} \leq \sup_{j \geq 0} \frac{\sigma_{k+1+j}(X)}{\sigma_{1+j}(X)}.$$

Matrix structure through small displacement rank

Given fixed $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, the quantity

$$\rho = \mathrm{rank}(AX - XB)$$

is called (A, B)-displacement rank of $X \in \mathbb{C}^{m\times n}$ [Heinig & Rost'84].

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EX1: Cauchy matrix $X = \left(\frac{1}{a_j - b_k}\right)_{j,k}$, $e = (1, 1, ..., 1)^T$

$$\mathrm{diag}(a_j)X - X\,\mathrm{diag}(b_k) = ee^T \quad \text{of rank 1}$$

EX2: Cauchy matrix pre/post multiplied by diagonal matrix
$X = \left(\frac{f_j g_k}{a_j - b_k}\right)_{j,k}$,

$$\mathrm{diag}(a_j)X - X\,\mathrm{diag}(b_k) = fg^T \quad \text{of rank } \rho = 1$$

EX3: Loewner $\left(\frac{f_j - g_k}{a_j - b_k}\right)_{j,k}$: same A, B, but of rank $\rho = 2$.
EX4: Pick=Loewner with $b_k = -\overline{a_k}$, $f_k = -\overline{g_k}$.

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EX5: Vandermonde matrix $X = \left(a_j^{k-1}\right)_{j,k}$,

$$\operatorname{diag}(a_j)X - XS(\varphi) \quad \text{of rank } \rho = 1, \quad S(\varphi) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \varphi \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

EX6: Krylov matrix $X = \left(A^k b\right)_k$ (including diagonal times Vandermonde)

$$AX - XS(\varphi) = fe_n^T \quad \text{of rank } 1.$$

If $m = n$ and $A = A^*$ then condition number grows [BB'00]

$$\frac{\sigma_1(X)}{\sigma_n(X)} = \|X\|\,\|X^{-1}\| \geq \frac{\exp(\frac{2\text{Catalan}}{\pi}(n-1))}{4\sqrt{n-1}}.$$

Other Examples: Hankel, Toeplitz, block versions,...

Decay of singular values

THM1: [BB, Cortona'08]
Let A, B be normal, with spectra included in $E, F \subset \mathbb{C}$.
If X has (A, B)-displacement rank ρ then for $j, k = 1, 2, \ldots$

$$\frac{\sigma_{j+\rho k}(X)}{\sigma_j(X)} \leq Z_k(E, F)$$

with the Zolotarev number

$$Z_k(E, F) := \inf_{r \in \mathcal{R}_{k,k}} \sup_{z \in E} |r(z)| \sup_{z \in F} \left| \frac{1}{r(z)} \right|.$$

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$$X - Y = r(A) X_j r(B)^{-1}.$$

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COR1: Same bound up to C^2 if E, F are C-spectral for A resp. B.
COR2: Same bound up to $2C_{Crouzeix}$ if $E = W(A)$, $F = W(B)$ [BB'11].

Some facts on Zolotarev numbers

$$Z_k(E,F) := \inf_{r\in\mathcal{R}_{k,k}} \sup_{z\in E} |r(z)| \sup_{z\in F} \left|\frac{1}{r(z)}\right|.$$

1. for all $k \geq 1$
$$\exp(\frac{-k}{\mathrm{cap}(E,F)}) \leq Z_k(E,F) \leq Z_1(E,F)^k,$$
2. Asymptotically,
$$\lim_{k\to\infty} Z_k(E,F)^{1/k} = \exp(\frac{-1}{\mathrm{cap}(E,F)}).$$
3. For any Moebius transform T we have $Z_k(E,F) = Z_k(T(E),T(F))$, $\mathrm{cap}(E,F) = \mathrm{cap}(T(E),T(F))$.
4. If $E = [-1,-\lambda]$, $F = [\lambda, 1]$ for some $\lambda \in (0,1)$ then [Zolotarev'1877]
$$Z_k(E,F) \leq 4\exp(\frac{-k}{\mathrm{cap}(E,F)}) \leq 4\exp(\frac{-k\pi^2}{2\log(4/\lambda)}).$$

Some more facts on Zolotarev numbers

With the decreasing Groetsch modulus

$$\mu(\lambda) = \frac{\pi}{2}\frac{K(\sqrt{1-\lambda^2})}{K(\lambda)}, \quad K(\lambda) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\lambda^2 t^2)}}dt,$$

Zolotarev found out that

$$\mu(Z_k([-1,-\lambda],[\lambda,1])) = \frac{k}{\mathrm{cap}([-1,-\lambda],[\lambda,1])},$$

it remains to apply a formula for the inverse of μ

$$\kappa = 4\sqrt{q}\prod_{j=1}^{\infty}\frac{(1+q^{2j})^4}{(1+q^{2j-1})^4}, \quad q = \exp(-2\mu(\kappa)).$$

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Mistake in [Lebedev'76], reproduced in [Medovikov & Lebedev'05], [Osedelets'07], [Druskin, Knizhnerman, Zaslavsky'09], [Güttel et.al.'14], [Nakatsukasa & Freund'15], [Bini, Massei & Robol,16],....

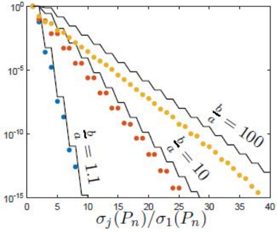
Numerical rank for Cauchy matrices ($\rho = 1$), Pick and Loewner matrices ($\rho = 2$)

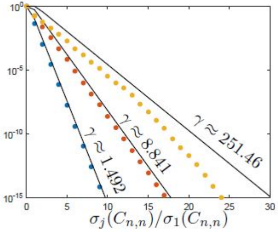
Here E, F do not depend on dimensions m, n of X.

THM2: If $a_j \in T([-1, -\lambda])$ and $b_k \in T([\lambda, 1])$ for some $\lambda \in (0, 1)$ and some Moebius transform T and X like in EX1–EX4 then

$$\mathrm{rank}_\epsilon(X) \le r := \rho \left\lceil \frac{1}{\pi^2} \log(\frac{4}{\epsilon}) \log(\frac{4}{\lambda}) \right\rceil$$

and more precisely $\sigma_{j+r}(X) \le \epsilon\, \sigma_j(X)$ for all $j \ge 1$.





Numerical rank for Vandermonde/Krylov ($\rho = 1$)

Problem with Vandermonde with real abscissa or Krylov with hermitian A: here $\sigma(A) \subset E = \mathbb{R}$ but

$$\sigma(B) = \sigma(S(-1)) = \{\exp(\frac{\pi}{n}(2j-1)) : j = 1, ..., n\} =: \Lambda_n$$

depends on n. Here n even !
First Approach : Use asymptotic results from [Gryson,BB'10] giving $\lim_{n,k\to\infty,\frac{k}{n}\to t>0} Z_k(\mathbb{R}, \Lambda_n)^{1/n}$. Problem: k gets too large.
Our approach

$$\sigma(B) \subset F_{\pi/n} \cup F_{\pi+\pi/n}, \quad F_\varphi = \{e^{it} : \varphi \le t \le \pi - \varphi\}$$

Lemma: [BB & Townsend'16] For even n,

$$\begin{aligned} Z_{2k}(\mathbb{R}, F_{\pi/n} \cup F_{\pi+\pi/n}) &\le 2\sqrt{Z_k(F_{\pi/n}, F_{\pi+\pi/n})} \\ &= 2\sqrt{Z_k([-1, -\tan^2(\frac{\pi}{2n})], [\tan^2(\frac{\pi}{2n}), 1])}. \end{aligned}$$

Numerical rank for Vandermonde/Krylov (bis)

THM3: With $X \in \mathbb{C}^{m \times n}$ Vandermonde/Krylov like in EX5–EX6

$$\text{rank}_\epsilon(X) \leq r \quad := \quad 2 + 2 \left\lceil \frac{4}{\pi^2} \log(\frac{4}{\epsilon}) \log(\frac{8\lfloor \frac{n}{2} \rfloor}{\pi}) \right\rceil$$

and more precisely $\sigma_{j+r}(X) \leq \epsilon\, \sigma_j(X)$ for all $j \geq 1$.

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and more precisely $\sigma_{j+r}(X) \le \epsilon \, \sigma_j(X)$ for all $j \ge 1$.

Idea of proof: If n is odd then denote by $\widetilde{X} \in \mathbb{C}^{m \times (n-1)}$ (also Vandermonde/Krylov) the first $(n-1)$ columns of X. Then interlacing of singular values gives $\operatorname{rank}_\epsilon(X) \le \operatorname{rank}_\epsilon(\widetilde{X}) + 1$ and it remains to discuss the case of even n. THM1 and Lemma give bound with $\lambda = \tan^2(\frac{\pi}{4\lfloor n/2 \rfloor})$. □

Numerical rank for real semi pos. def. Hankel

THM3: With $X \in \mathbb{C}^{m \times n}$ Vandermonde/Krylov like in EX5–EX6

$$\mathrm{rank}_\epsilon(X) \leq r \quad := \quad 2 + 2\left\lceil \frac{4}{\pi^2} \log(\frac{4}{\epsilon}) \log(\frac{8\lfloor \frac{n}{2} \rfloor}{\pi}) \right\rceil$$

and more precisely $\sigma_{j+r}(X) \leq \epsilon\, \sigma_j(X)$ for all $j \geq 1$.

We use Fiedler factorization $Y = X^H X$ with X as in THM3 and obtain:

COR3: With $Y \in \mathbb{R}^{n \times n}$ semi pos. def. Hankel

$$\mathrm{rank}_\epsilon(Y) \leq r \quad := \quad 2 + 2\left\lceil \frac{2}{\pi^2} \log(\frac{16}{\epsilon}) \log(\frac{8\lfloor \frac{n}{2} \rfloor}{\pi}) \right\rceil$$

and more precisely $\sigma_{j+r}(Y) \leq \epsilon\, \sigma_j(Y)$ for all $j \geq 1$.

Further reading: arXiv:1609.09494

MERCI !

Further details about Zolotarev (1)

If $E = [-1, -\lambda]$, $F = [\lambda, 1]$ for some $\lambda \in (0, 1)$ then [Zolotarev'1877]

$$\mu(Z_k(E, F)) = k \frac{\pi^2}{2\mu(\lambda)} = \frac{k}{\text{cap}(E, F)}$$

with the decreasing Groetsch modulus

$$\mu(\lambda) = \frac{\pi}{2} \frac{K(\sqrt{1 - \lambda^2})}{K(\lambda)}, \quad K(\lambda) = \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - \lambda^2 t^2)}} dt,$$

in particular

$$Z_k(E, F) \leq 4 \exp\left(\frac{-k}{\text{cap}(E, F)}\right) \leq 4 \exp\left(\frac{-k\pi^2}{2 \log(4/\lambda)}\right).$$

since $\mu(\lambda) \leq \log(4/\lambda)$.

Proof of Lemma

Set $E = [-1, -\lambda]$, $F = [\lambda, 1]$, $\lambda = \tan^2(\frac{\pi}{2n})$, n even.
There exists $R \in \mathcal{R}_{k,k}$ real-valued on $\mathbb{R}$ with $R(-z) = 1/R(z)$ and thus $|R(z)| \leq 1$ for $z \in i\mathbb{R}$ extremal for $Z_k(E, F)$.

Thus there exists $r \in \mathcal{R}_{k,k}$ real-valued on $\partial\mathbb{D}$ with $r(1/z) = 1/r(z)$ and $|r(z)| \leq 1$ for $z \in \mathbb{R}$ extremal for $Z_k(F_{\pi/n}, F_{\pi+\pi/n})$. Hence

$$
\begin{aligned}
&Z_{2k}(\mathbb{R}, F_{\pi/n} \cup F_{\pi/n+\pi}) \\
&\leq \max_{z\in\mathbb{R}} \left|\frac{r(w) + 1/r(w)}{2}\right| \max_{w\in F_{\pi/n}\cup F_{\pi/n+\pi}} \left|\frac{2}{r(w) + 1/r(w)}\right| \\
&= \frac{2\sqrt{Z_k(E, F)}}{1 + Z_k(E, F)} \leq 2\sqrt{Z_k(E, F)}
\end{aligned}
$$

Further details about Zolotarev (2)

Denote $\widetilde{F}_{\phi} = \{e^{it} : \varphi \leq t \leq 2\pi - \varphi\} = F^2_{\varphi/2} = F^2_{\varphi/2+\pi}$ then by considering only rational functions in z^2

$$Z_{2k}(\mathbb{R}, F_{\pi/n} \cup F_{\pi/n+\pi}) \leq Z_k([0, +\infty), \widetilde{F}_{2\pi/n})$$

Sending the circle to the imaginary axis and -1 to ∞

$$Z_{2k}([0, +\infty), \widetilde{F}_{\pi/n}) = Z_{2k}([-1, 1], i\mathbb{R} \setminus [-\tan^{-1}(\frac{\pi}{2n}), \tan^{-1}(\frac{\pi}{2n})])$$
$$\leq Z_k([0, 1], (-\infty, -\tan^{-2}(\frac{\pi}{2n})])$$

...