Shape Optimization and Isoperimetric and Functional Inequalities

November 21 - 25, 2016

José M. Arrieta: Thin domains with a locally periodic highly oscillatory boundary.

We consider a two dimensional thin domain where the boundary has a highly oscillatory behavior but the oscillations are not purely periodic. For instance, we may consider the case where the thin domain is of the type $R_\epsilon = \{(x,y) : 0 < x < 1; 0 < y < \epsilon G(x, x/\epsilon)\}$ where the function $G(x,\cdot)$ is periodic of period $L(x)$, for some function $L(\cdot)$. Observe that we are allowing that the period and amplitude of the oscillations varies in space. We will analyze the homogenized limit as the thickness of the domain goes to 0. We are interested in understanding how the varying amplitude and period appear in the homogenized limit problem.
This is a joint work with Manuel Villanueva-Pesquera (UCM- Madrid) and Marcone Pereira (USP-Brazil).

Mark Ashbaugh: A Sharp Lower Bound for the First Eigenvalue of the Vibrating Clamped Plate under Compression.

We give a sharp lower bound to the fundamental frequency of a vibrating clamped plate under compression in the context of plates of different shapes of fixed area. Mathematically, the problem is that of bounding the first eigenvalue of a certain 4th-order partial differential operator with leading term the bi-Laplacian from below by a positive constant over the square of the area of the domain area. We give a Rayleigh-Faber-Krahn-type result for this problem for small compressions. Thus, our lower bound is saturated for a disk, and the constant appearing in our inequality is that for the disk under the appropriate compression. Our results apply only in two dimensions. Time permitting, possibilities and impediments for the analogs of our results in higher dimensions will be discussed.
(This is joint work with R. Benguria and R. Mahadevan.)

Catherine Bandle and Alfred Wagner: On an eigenvalue problem with infinitely many positive and negative eigenvalues: Rayleigh-Faber-Krahn inequalities for the principal eigenvalues.

The existence theory for the solutions of a parabolic problem with dynamical boundary conditions leads to the spectral theory of an associated elliptic problem with the eigenvalue parameter both in the equation and on the boundary. We shall discuss this spectrum which, in certain cases leads to an infinite sequence of positive and an infinite sequence of negative eigenvalues. Special attention is given to those eigenvalues, the eigenfunctions of which are of constant sign. In certain cases a Rayleigh-Faber-Krahn inequality holds which follows from the one for the membrane with Robin boundary conditions.

Rafael Benguria: The Brezis-Nirenberg Problem for the Laplacian with a singular drift in $\mathbb{R}^n$ and also in $\mathbb{S}^n$

In this talk I aill consider the Brezis-Nirenberg problem for the Laplacian with a singular drift in $\mathbb{R}^n$ and also in $\mathbb{S}^n$, and determine the values of the spectral parameter taht characterzies the existence of positive solutions.
This is joint work with Soledad Benguria (U. Wisconsin, Madison).

Chiara Bianchini: Wulff Shape characterization for anisotropic capacitary potentials.

In the space R^N andowed with a (regular) norm H, we call "Wulff shape of H" the sublevel sets of the dual norm H_0, that is, the anisotropic balls. In this talk we will present a class of overdetermined elliptic problems involving the so called Finsler Laplacian (eventually p-Laplacian) and we will prove symmetry results showing that if such problems admits a solutions in a set D or $R^N \setminus D$, then the domain D must be Wulff shape of H.
This talk is based on some joint works with Giulio Ciraolo (Università di Palermo) and Paolo Salani (Università di Firenze).

Virginie Bonnaillie-Noël: Spectral minimal partitions for a family of tori.

We study partitions of the two-dimensional flat torus $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$, into k domains, with b a real parameter in $(0, 1]$ and k an integer. We look for partitions which minimize the energy, defined as the largest first eigenvalue of the Dirichlet Laplacian on the domains of the partition. We are in particular interested in the way these minimal partitions change when b is varied. We present here an improvement, when k is odd, of the results on transition values of b established by Helffer-Hoffmann–Ostenhof and state a conjecture on those transition values. We establish an improved upper bound of the minimal energy by explicitly constructing hexagonal tilings of the torus. These tilings are close to the partitions obtained from a systematic numerical study based on an optimization algorithm adapted from Bourdin-Bucur-Oudet. This is a joint work with C. Léna.

Friedemann Brock: Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^\alpha$

We solve a class of isoperimetric problems on$\mathbb{R}^N$ with respect to weights that are powers of the distance to the origin. For instance, we show that, if $k[0, 1]$, then among all smooth sets with fixed Lebesgue measure, its perimeter with weight $|x|^k$ achieves its minimum for a ball centered at the origin. Our results also imply a weighted Polya-Szego principle. In turn, we establish the radiality of optimizers in some Caffarelli-Kohn-Nirenberg inequalities, and we obtain sharp bounds for eigenvalues of some nonlinear problems.
This is joint work with A. Alvino, F. Chiacchio, A. Mercaldo and M.R. Posteraro.

Bruno Colbois: Bounds for the spectrum of the magnetic Laplacian.

We consider a compact Riemannian manifold and the magnetic Laplacian on it (with Neumann type boundary condition if the boundary is not empty). We first establish a family of upper bounds for all the eigenvalues, compatible with the Weyl law. When the potential of the magnetic field is a closed 1-form, we get a sharp upper bound for the first eigenvalue. In the second part, we consider only closed potentials, and we establish a sharp lower bound for the first eigenvalue when the manifold is a 2-dimensional Riemannian cylinder. The equality case characterizes the situation where the metric is a product. We also look at the case of doubly convex domains in the Euclidean plane. This is a joint work with Alessandro Savo.

Gisella Croce: On the selection of solutions to a nonlinear PDE system.

In this talk we will present a recent result obtained in collaboration with G. Pisante about the solutions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ to the vectorial system

$$\begin{cases} Du \in \mathcal{O}(2) & \Omega \\ u = 0 & \partial\Omega\,, \end{cases}$$

where Ω is bounded and open. It is known that this system has a infinite number of solutions. We will propose a selection principle based on the minimization of the set of irregularities of the gradient of the solutions.

Gianni Dal Maso: Existence and uniqueness of dynamic evolutions for a peeling test in dimension one.

We present a one-dimensional model of a dynamic peeling test for a thin film, where the wave equation is coupled with a Griffith criterion for the propagation of the debonding front. Our main results provide existence and uniqueness for the solution to this coupled problem under different assumptions on the data.

Marc Dambrine: Taking uncertainties into account in numerical shape optimization.

Shape optimization problems usually depends on parameters like applied loadings, physical coefficients that are usually not perfectly known. Like in any optimization problem, the solution of a shape optimization problem may depend of the value of these parameters. As a model, we consider that these parameters are random following a given law. Describing the law the solution of the shape optimization problem with random input seems out of reach. Hence, we adress the question of computing in a realistic way a reasonable shape while taking the uncertainties into account.

Guido De Philippis: Allard's rectifiability theorem for anisotropic energies.

Allard's rectifiability theorem asserts that every d-varifold with bounded first variation and positive d-dimensional density is rectifiable. In this talk I will show that there exists a necessary and sufficient condition on the integrand in order to extend this result to

varifolds with bounded first variation with respect to anisotropic integrands.
Joint work with A. De Rosa and F. Ghiraldin.

Ilaria Fragala: Some new inequalities for the Cheeger constant.

We discuss some recently proved inequalities for the Cheeger constant in dimension two, including: - a polygonal version of Faber-Krahn inequality; - a reverse isoperimetric inequality for convex bodies; - a Mahler-type inequality in the axisymmetric
Joint work with D.Bucur.

Pedro Freitas: Asymptotic behaviour of optimisers of Laplace eigenvalues.

We will discuss several issues concerning the asymptotic behaviour of optimisers of eigenvalues of the Laplacian, present some new results and highlight some of the specific difficulties appearing in this class of problems.

Nicola Fusco: A stability result for the first eigenvalue of the p-Laplacian.

We present a sharp quantitative form of the Faber-Krahn inequality for the first eigenvalue of the p-Laplacian. This extends to the case p>1 a recent result proved by Brasco, De Philippis and Velichkov for the Laplacian.

Filippo Gazzola: A minimaxmax problem for improving the torsional stability of rectangular plates.

We introduce a new function which measures the torsional instability of a partially hinged rectangular plate. By exploiting it, we compare the torsional performances of different plates reinforced with stiffening trusses. This naturally leads to a shape optimization problem which can be set up through a minimaxmax procedure.

Alessandro Giacomini: Shape optimization with Robin conditions and free discontinuity problems.

In this talk I will describe an approach to the proof of the Faber-Krahn inequality for the Robin-Laplacian based on functions of bounded variation and free discontinuity problems. The technique can be adapted to cover variants of the first eigenvalue, such as the torsional rigidity, and to deal also with the study of optimal constants for some Poincaré inequality with traces. The results are obtained in collaboration with Dorin Bucur.

Alexandre Girouard: Discretization and Steklov eigenvalues of compact manifold with boundary.

Let M be a compact n-dimensional Riemannian manifolds with cylindrical boundary, Ricci curvature bounded below by k and injectivity radius bounded below by r. We introduce a notion of discretization of the manifold M, leading to a graph with boundary which is roughly isometric to M , with constant depending only on k, r, n. In this context, we prove a uniform spectral comparison inequality between the Steklov eigenvalues of the manifold M and those of its discretization. Some applications to the construction of

sequences of surfaces with boundary of fixed length and with large Steklov eigenvalues are given. In particular, we obtain such a sequence for surfaces with connected boundary. These applications are based on the construction of graph-like surfaces which are modeled using sequences of graphs with good expansion properties.
This talk is based on joint work with Bruno Colbois (Neuchatel) and Binoy Raveendran.

Katie Gittins: Asymptotic optimal sets for the eigenvalues of the Laplacian.

We consider shape optimisation problems for the eigenvalues of the Laplacian with boundary conditions among certain collections of open sets in $\mathbb{R}^m$, $m \geq 2$. The sets in these collections have either prescribed measure or prescribed perimeter. For $k \in \mathbb{N}$, if an optimal set $\Omega_k^* \subset \mathbb{R}^m$ exists for the k'th eigenvalue in the chosen collection, then the aim is to determine the asymptotic optimal set as $k \to \infty$. That is, the limit of a sequence of optimal sets $(\Omega_k^*)_k$ as $k \to \infty$. We focus our attention on the optimisation of the Dirichlet and Neumann eigenvalues on cuboids in $\mathbb{R}^3$ and rectangles in $\mathbb{R}^2$ respectively. We review the current state of affairs for these problems and compare with similar questions which remain unresolved to date.

Bernd Kawohl: Two dimensions are easier.

I present a handful of geometrical inequalities which are easier to prove in two than in higher dimensions. In fact, most of them appear to be open in three or more dimensions.

David Krejcirik: Isoperimetric versus isochoric spectral optimisation for the Robin problem.

We give an expository talk on recent progress in the optimisation of eigenvalues of the Robin Laplacian as regards the geometry of the domain under perimeter and area constraints.

Jimmy Lamboley: Regularity for functionals involving perimeter.

In this talk, we explain how to study existence and regularity of optimal shapes for the problem

$$\min\Big\{P(\Omega) + \mathcal{G}(\Omega) : \ \Omega \subset D, \ |\Omega| = m\Big\},$$

where P denotes the perimeter, $|\cdot|$ is the volume, and the functional $\mathcal{G}$ is either one of the following:

- the Dirichlet energy E_f, with respect to a (possibly sign-changing) function $f \in L^p$;
- a spectral functional of the form $F(\lambda_1, \ldots, \lambda_k)$, where λ_k is the kth eigenvalue of the Dirichlet Laplacian and $F : \mathbb{R}^k \to \mathbb{R}$ is locally Lipschitz continuous and increasing in each variable.

The domain D is the whole space $\mathbb{R}^d$ or a bounded domain.
We will review the previous results on the topic, and insist on the new ideas developped to fully deal with these very general cases. We also explain in what way our approach

could be used for other functionals.
This is a joint work with G. De Philippis, M. Pierre and B. Velichkov.

Richard Laugesen: Optimal stretching for lattice points and eigenvalues.

What shape of domain minimizes the n-th eigenvalue of the Dirichlet Laplacian, for large n? (Here we normalize the area to equal 1.) Does the minimizer approach a disk as n tends to infinity? Supporting this idea is the discovery by Antunes and Freitas that among rectangles, the minimizer approaches a square in the limit. Their result for rectangles relies on lattice point counting in ellipses. In joint work with Shiya Liu (University of Illinois), we extend to more general lattice counting problems, proving again that the almost balanced situation is optimal in the limit. We similarly generalize a Neumann eigenvalue maximization result of van den Berg, Bucur and Gittins.

Dario Mazzoleni: Regularity of the optimal sets for spectral functionals. Part II, some generalizations.

We start from the variational problem

$$\min\left\{\langle_1() + \cdots + \langle_k() \ : \subset \mathbb{R}^d,\ || = 1\right\},$$

where the variable is the domain , $|\cdot|$ denotes the Lebesgue measure and the cost functional is the sum of the first k Dirichlet eigenvalues on .

In Susanna Terracini's talk it was shown how to prove that the optimal sets have C^∞ regular boundary up to a set of zero $\mathcal{H}^{d-1}-$measure, using techniques strongly related to the study of the free boundary $\partial\{|U| > 0\}$ of the local minima of the functional

$$H^1_{loc}(\mathbb{R}^d, \mathbb{R}^k) \ni W \mapsto \int |\nabla W|^2 + |\{|W| > 0\}|.$$

In this talk we will focus our attention to some technical details of the proof, mostly regarding the optimality condition at the boundary, and we will show how to extend this approach to more general functionals of eigenvalues $F\colon \mathbb{R}^k \to \mathbb{R}$ Lipschitz continuous, non-decreasing in each variable and strictly increasing in the first one. In particular we have in mind, for example, $\langle_1 + \langle_k$ and $\langle_1^2 + \langle_k$, which give rise to different issues in the proofs.

The talk is based on joint works with Susanna Terracini and Bozhidar Velichkov.

Frank Morgan: Isoperimetry with Density.

In 2015 Chambers proved the Log-convex Density Conjecture, which says that for a radial density f on R^n, spheres about the origin are isoperimetric if and only if log f is convex (the stability condition). We discuss recent progress and open questions for other densities, unequal perimeter and volume densities, and other metrics.

Nikolai Nadirashvili: Isoperimetric inequalities for spectrum of Laplacian on surfaces.

We discuss generalizations of classical inequalities of Hersch and Li-Yau on higher eigenvalues of the Laplacia.

Carlo Nitsch: Symmetry breaking for a problem in optimal insulation.

We consider the problem of optimally insulating a given domain; this amounts to solve a nonlinear variational problem, where the optimal thickness of the insulator is obtained as the boundary trace of the solution. We deal with two different criteria of optimization: the 1rst one consists in the minimization of the total energy of the system, while the second one involves the 1rst eigenvalue of the related differential operator. Surprisingly, the second optimization problem presents a symmetry breaking in the sense that for a ball the optimal thickness is nonsymmetric when the total amount of insulator is small enough.

Edouard Oudet: Convex relaxation and variational approximation of the Euclidean Steiner

TBA

Iosif Polterovich: Nodal geometry of Steklov eigenfunctions.

I will present an overview of some recent progress on the study of the nodal sets of Steklov eigenfunctions. In particular, I will discuss sharp estimates on the nodal length of Steklov eigenfunctions on real-analytic Riemannian surfaces with boundary obtained in my joint work with D. Sher and J. Toth.

Yannick Privat: Geometrical properties of resources optimal arrangements for species survival

n this work, we are interested in the analysis of optimal resources configurations (typically foodstuff) necessary for a species to survive. For that purpose, we use a logistic equation to model the evolution of population density involving a term standing for the heterogeneous spreading (in space) of resources. The principal issue investigated in this talk writes: "How to spread in an optimal way resources in a closed habitat?" This problem can be recast as the one of minimizing the principal eigenvalue of an operator with respect to the domain occupied by resources, under a volume constraint. By using symmetrization techniques, as well as necessary optimality conditions, we prove new qualitative results on the solutions. In particular, we investigate the optimality of balls. This is a joint work with Jimmy Lamboley (univ. Paris Dauphine), Antoine Laurain (univ. Sao Paulo) and Grégoire Nadin (univ. Paris 6).

Paolo Salani: About the stability of Borell-Brascamp-Lieb inequalities.

I will present recent results, obtained in collaboration with Andrea Rossi (Univ. Firenze), about the stability of the so called Borell-Brascam-Lieb inequalities, without any concavity assumption on the involved functions.

Kathrin Stollenwerk: Optimal shape of a domain which minimizes the buckling load of a clamped plate.

We prove the existence of an optimal domain which minimizes the buckling load of a clamped plate among all bounded domains with given measure. Instead of treating this constrained variational problem, we will us a formulation due to Alt and Caffarelli and introduce a penalized problem. Assuming that a smooth optimal domain exists, we show that this domain must be a ball
Joint work with A. Wagner.

Susanna Terracini: Regularity of the optimal sets for spectral functionals. Part I: sum of eigenvalues.

In this talk we deal with the regularity of optimal sets for a shape optimization problem involving a combination of eigenvalues, under a fixed volume constraints. As a model problem, consider

$$\min\Big\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ \text{open}\ ,\ |\Omega|=1\Big\},$$

where $\langle_i(\cdot)$ denotes the eigenvalues of the Dirichlet Laplacian and $|\cdot|$ the d-dimensional Lebesgue measure. We prove that any minimizer $_{opt}$ has a regular part of the topological boundary which is relatively open and C^∞ and that the singular part has Hausdorff dimension smaller than $d-d^*$, where $d^*\geq 5$ is the minimal dimension allowing the existence of minimal conic solutions to the blow-up problem.

We mainly use techniques from the theory of free boundary problems, which have to be properly extended to the case of vector-valued functions: nondegeneracy property, Weiss-like monotonicity formulas with area term; finally through the properties of non tangentially accessible domains we shall be in a position to exploit the "viscosity" approach recently proposed by De Silva.

This is a joint work with Dario Mazzoleni and Bozhidar Velichkov.

Cristina Trombetti: On the Stability of the Bossel-Daners Inequality.

The Bossel-Daners is a Faber-Krahn type inequality for the first Laplacian eigenvalue with Robin boundary conditions. We prove a stability result for such inequality.

Michiel Van den Berg: On Polya inequality for torsional rigidity and first Dirichlet eigenvalue.

An inequality of Polya asserts that for all open sets in Euclidean space with finite measure the product of torsional rigidity and first Dirichlet eigenvalue is bounded by its measure. We discuss the sharpness of this inequality and present some improvements for convex sets. This is joint work with Enzo Ferone, Carlo Nitsch and Cristina Trombetti. Time permitting we will also discuss new bounds for the maximum of the torsion function in terms of the bottom of the spectrum of the Dirichlet Laplacian.

Bozhidar Velichkov: An epi-perimetric inequality approach to the regularity of the free boundaries.

In this talk we will prove the $C^{1,\alpha}$ regularity of the free boundary in the two dimensional case for the classical one-phase Alt-Caffarelli problem. Our result is achieved by proving an epiperimetric inequality for the local minimizers of the functional

$$u \mapsto \int |\nabla u|^2 + |\{u > 0\}| \tag{1}$$

by producing a good competitor and then applying standard techniques introduced in the context of free boundary problems by Weiss, Focardi-Spadaro and Garofalo-Petrosyan-Garcia. Our proof is direct: that is we construct a competitor whose energy is strictly better than the 1-homogeneous extension of the boundary datum. We also obtain the regularity of the flat free boundary in the vector-valued case $u : \mathbb{R}^2 \to \mathbb{R}^k$ without any assumption on the local minimizer u.
Joint work with Luca Spolaor.