Homogeneous open convex cones

— recent results —

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In 2014, Gracyzk and Ishi published an interesting paper on Wishart distributions on homogeneous open convex cones.

Purpose of today's talk is to

report some of the recent results about homogeneous open convex cones obtained by Ishi, Nakashima, N. and others individually or with collaborations.

Homogeneous open convex cones (HOCC)

— V: a VS/$\mathbb{R}$, $\dim V < +\infty$, with unique LC topology (e.g. norm topology).

— $\Omega \subset V$ is an open convex cone, regular (proper) in the sense that Ω contains no entire line. (i.e., pointed at the origin like ice cream cones)

— $GL(\Omega) := \{g \in GL(V) \ ; \ g(\Omega) = \Omega\}$: the linear automorphism group of Ω.

— $GL(\Omega)$ is a linear Lie group (as a closed subgroup of $GL(V)$).

— Ω is said to be homogeneous if $GL(\Omega) \curvearrowright \Omega$ transitively:

$$\forall \omega_1, \omega_2 \in \Omega, \ \exists g \in GL(\Omega) \text{ s.t. } g\omega_1 = \omega_2.$$

Example $V = \mathrm{Sym}(r, \mathbb{R})$, $\Omega = \mathscr{P}(r, \mathbb{R})$ (positive definite matrices in V).
We have a group homomorphism $\rho : GL(r, \mathbb{R}) \to GL(\Omega)$ given by

$$\rho(g)x := g\, x\, {}^t g \qquad (x \in V).$$

By Linear Algebra, any $y \in \mathscr{P}(r, \mathbb{R})$ is written as

$$g\, {}^t g \in \rho(GL(r, \mathbb{R})) I_r = GL(\Omega) \cdot I_r,$$

so that Ω is homogeneous. (Note: $GL(\Omega) = \rho(GL(r, \mathbb{R}))$ in this case.)

For $x = \begin{pmatrix} x_{11} & \dots & x_{r1} \\ \vdots & & \vdots \\ x_{r1} & \dots & x_{rr} \end{pmatrix} \in \mathrm{Sym}(r, \mathbb{R})$, put

$$\Delta_1(x) := x_{11}, \quad \Delta_2(x) := \det \begin{pmatrix} x_{11} & x_{21} \\ x_{21} & x_{rr} \end{pmatrix}, \dots, \Delta_r(x) := \det \begin{pmatrix} x_{11} & \dots & x_{r1} \\ \vdots & & \vdots \\ x_{r1} & \dots & x_{rr} \end{pmatrix}.$$

Then, $\Omega = \{x \in \mathrm{Sym}(r, \mathbb{R})\ ;\ \Delta_1(x) > 0,\ \Delta_2(x) > 0, \dots,\ \Delta_r(x) > 0\}$.

For general HOCC $\Omega \subset V$, Vinberg (1963) found a "coordinatization"

$$V = \begin{pmatrix} V_{11} & \dots & V_{r1} \\ \vdots & & \vdots \\ V_{r1} & \dots & V_{rr} \end{pmatrix}, \qquad \begin{array}{l} \text{where } V_{jj} = \mathbb{R}c_j \ (j = 1, 2, \dots,), \\ \text{and } r \text{ is called the rank of } V, \end{array}$$

so that every $v \in V$ can be regarded as a symmetric "matrix" with vector entries. Also a (non-associative) multiplication is introduced in V to view V as a "matrix algebra" without associative law:

an algebra called a clan, a left symmetric algebra with two additional conditions.

You have multiplication rules between the subspaces V_{ji} like the ordinary matrices.

The first subject is about "principal minors" for HOCC.

— Vinberg found polynomials $p_1(x), \ldots, p_r(x)$ on V, so that

$$\Omega = \{x \in V \ ; \ p_1(x) > 0, \ p_2(x) > 0, \ldots, p_r(x) > 0\}.$$

However, these polynomials are, in general, reducible. In fact, for $V = \mathrm{Sym}(r, \mathbb{R})$, we actually have $p_k(x) = \Delta_k(x)$ $(k = 1, 2)$, and for $k \geqq 3$

$$p_k(x) = \Delta_1(x)^{2^{k-3}} \Delta_2(x)^{2^{k-4}} \cdots \Delta_{k-2}(x) \Delta_k(x).$$

In reality, we have $\deg p_k(x) = 2^{k-1}$ for any HOCC Ω.

— Ishi (2001) extracted inductively, through a kind of Euclidean algorithm, *irreducible* polynomials $\Delta_1(x), \ldots, \Delta_r(x)$ from $p_1(x), \ldots, p_r(x)$, so that

$$\Omega = \{x \in V \ ; \ \Delta_1(x) > 0, \ \Delta_2(x) > 0, \ldots, \Delta_r(x) > 0\}.$$

Now there is a closed (and nice) formula of $\Delta_k(x)$ in terms of $p_j(x)$ $(j \leqq k)$, products, powers and quotients of them, due to Nakashima (2014) (presented by poster session).

— $\mathscr{P}(r,\mathbb{R})$ is a typical example of symmetric cones.

— OCC Ω is said to be selfdual if there is $\langle\,\cdot\,|\,\cdot\,\rangle$ s.t.

$$\Omega = \Big\{ v \in V \;;\; \langle\, v \,|\, x \,\rangle > 0 \;(\forall x \in \overline{\Omega} \setminus \{0\} \Big\}$$

— (the RHS is the dual cone Ω^* of Ω w.r.t $\langle\,\cdot\,|\,\cdot\,\rangle$; usually Ω^* is taken in V^*).

— Selfdual HOCC is called a symmetric cone.

Just a digression;

it is an interesting Linear Algebra exercise to give a direct proof for

$$\mathscr{P}(r,\mathbb{R}) = \{ v \in V \;;\; \mathrm{tr}(vx) > 0 \;\text{ for all positive semi-definte } x \neq 0 \}.$$

Here, a direct proof means a proof without using $GL(r,\mathbb{R})$-action.

Symmetric cones $\leftrightarrow$ Euclidean Jordan algebras (up to isomorphisms)

Definition 1

V: a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with a bilinear product $x, y \mapsto xy$.

V is a Jordan algebra $\overset{\text{def}}{\Longleftrightarrow} \begin{cases} (1)\ xy = yx, \\ (2)\ x^2(xy) = x(x^2y). \end{cases}$

Associative law is not assumed.

A real Jordan algebra V with e is Euclidean if V has an associative inner product, i.e., V has a positive definite symmetric bilinear form $\langle \cdot \,|\, \cdot \rangle$ such that

$$\langle\, xy \,|\, z \,\rangle = \langle\, x \,|\, yz \,\rangle \quad (\forall x, y, z \in V).$$

Classification of simple Euclidean Jordan algebras

(1) A Euclidean Jordan algebra is a direct sum of simple ideals.

(2) There are only 5 types of simple Euclidean Jordan algebras (of finite-dim.).

$$\mathrm{Sym}(r,\mathbb{R}), \quad \mathrm{Herm}(r,\mathbb{C}), \quad \mathrm{Herm}(r,\mathbb{H}), \quad \mathrm{Herm}(3,\mathbb{O}), \quad \mathscr{S}(W),$$

where $\mathbb{H} := \{\text{quaternions}\}$, $\mathbb{O} := \{\text{octonions}\}$,

W is a real VS with $\langle\, \cdot \,|\, \cdot \,\rangle_W$, $\mathscr{S}(W) := \mathbb{R} \oplus W$ with

$$(\lambda + w)(\lambda' + w') := \big(\lambda\lambda' + \langle\, w \,|\, w' \,\rangle_W\big) \oplus \big(\lambda w' + \lambda' w\big).$$

$\mathscr{S}(W)$ is called a spin factor, and is the scalar and the linear part of $\mathrm{Cliff}(W)$ associated with $\langle\, \cdot \,|\, \cdot \,\rangle_W$

The corresponding symmetric cones are

$$\mathscr{P}(r,\mathbb{R}), \quad \mathscr{P}(r,\mathbb{C}), \quad \mathscr{P}(r,\mathbb{H}), \quad \mathscr{P}(3,\mathbb{O}),$$

and Lorentz cones: $\{\lambda \oplus w \;;\; \lambda > \|w\|_W\}$

For $\mathrm{Herm}(r, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$): $\Delta_1(x), \ldots, \Delta_r(x)$ are principal minors

(for $\mathbb{K} = \mathbb{H}, \mathbb{O}$, we must pay attention to the determinant; safer to say it's the Jordan algebra determinant),

For $\mathscr{S}(W)$: $\Delta_1(\lambda \oplus w) = \lambda,\ \Delta_2(x) = \lambda^2 - \|w\|_W^2$.

— These are irreducible polynomials, and the notation is compatible with the ones I used for general HOCC.

Anyway, we have $\deg \Delta_j(x) = j$ ($j = 1, 2, \ldots, r$).

Question Is this characteristic of irreducible symmetric cones?

The answer is No!

For any rank $r \geqq 3$, there is an irreducible HOCC, non-selfdual, s.t. $\deg \Delta_j(x) = j$ $(j = 1, 2, \ldots, r)$ (Nakashima–N., 2013, 2014).

$\left(\begin{array}{l}\text{These cones are systematically obtained as the dual cones of HOCC defined} \\ \text{by selfadjoint representations of simple Euclidean Jordan algebras.}\end{array}\right)$

If we take Ω^* into account as well as Ω itself, then we have the following theorem due to Yamasaki and N. (2016).

Theorem 2

If $\Delta_1(x), \ldots, \Delta_r(x)$ for Ω and $\Delta_1^*(x), \ldots, \Delta_r^*(x)$ for Ω^* are of degree $1, 2, \ldots, r$ up to permutations, then Ω is an irreducible symmetric cone.

— Nakashima gave an alternative proof of this theorem (2017, to appear in September or so).

Order defined by an open convex cone

Let S, T be selfadjoint operators on a Hilbert space $\mathfrak{H}$.

$S \geqq T \overset{\text{def}}{\iff} S - T$ is positive semi-definite.

The following is well-known.

Theorem 3

Let S, T be positive definite. Then, $S \geqq T \iff T^{-1} \geqq S^{-1}$.

Proof Clear from

$$T^{-1} - S^{-1} = T^{-1/2}\big(T^{1/2}S^{-1}T^{1/2}\big)^{1/2} T^{-1/2}(S - T)T^{-1/2}\big(T^{1/2}S^{-1}T^{1/2}\big)^{1/2} T^{-1/2}.$$

— Let V be a Euclidean JA, and Ω the corresponding symmetric cone.

For $a, b \in \Omega$, let $a \geqq b \overset{\text{def}}{\iff} a - b \in \overline{\Omega}$.

Theorem 4

Let $a, b \in \Omega$. Then, $a \geqq b \iff b^{-1} \geqq a^{-1}$.

Proof. Just translate the proof of Theorem 3 into JA language. □

Can we generalize Theorem 4 to HOCC?

The answer is NO.

Let Ω be a HOCC, and $\Omega^* \subset V^*$ its dual cone:

$$\Omega^* := \{f \in V^* \,;\, \langle y, f\rangle > 0 \ \ (\forall y \in \overline{\Omega} \setminus \{0\})\}.$$

Let ϕ be the characteristic function of Ω: $\quad \phi(x) := \displaystyle\int_{\Omega^*} e^{-\langle x, f\rangle}\, df \ \ (x \in \Omega).$

The Vinberg $*$-map $\Omega \ni x \mapsto x^* \in \Omega^*$ is defined as $x^* := -\operatorname{grad}\log\phi(x)$, i.e.,

$$\langle v, x^*\rangle := -\frac{d}{dt}\log\phi(x+tv)\Big|_{t=0} \quad (v \in V).$$

Note that $(\lambda x)^* = \lambda^{-1}x^*$ $(\lambda > 0)$.

The following theorem is due to C. Kai (2008).

Theorem 5

Ω is a symmetric cone $\iff$ Ω has the following property:

for $x, y \in \Omega$, one has $x \geqq y$ w.r.t. Ω $\iff$ $y^* \geqq x^*$ w.r.t. Ω^*.

For a symmetric cone, $*$-map is the JA inverse under a suitable identification of V^* with V. The function ϕ can be replaced by a more general semi-invariant function on Ω.

Indeed Kai (2008) showed more.

The pair (x, y), where $x, y \in \Omega$, is said to be Ω-comparable if $x \geqq y$ or $y \geqq x$.

Theorem 6

Ω is a symmetric cone $\iff$ Ω has the following property:

for $x, y \in \Omega$, the pair (x, y) is Ω-comparable

$\iff$ the pair (x^*, y^*) is Ω^*-comparable.

Miscellaneous results

- In any rank $\geqq 3$, $\exists$ irreducible non-selfdual HOCC linearly isomorphic to the dual cone (Ishi–N. 2009).
- Reducible such cones are easy to construct: just take $\Omega \oplus \Omega^*$.

Minimal matrix realization of HOCC

By a **matrix realization** of Ω, we mean a realization as a slice $V_0 \cap \mathscr{P}(N,\mathbb{R})$, i.e., positive definite matrices in a subspace $V_0 \subset \mathrm{Sym}(N,\mathbb{R})$.

• Graczyk–Ishi's presentations of HOCC (2014) based on Ishi (2006)

Take a partition $N = n_1 + \cdots + n_r$, and
consider a system of vector spaces $\mathscr{Z}_{lk} \subset \mathrm{Mat}(n_l \times n_k;\mathbb{R})$ s.t.

(V1) $z \in \mathscr{Z}_{lk},\ z' \in \mathscr{Z}_{kj} \implies zz' \in \mathscr{Z}_{lj}\ \ (1 \leqq j < k < l \leqq r)$,
(V2) $z \in \mathscr{Z}_{lj},\ z' \in \mathscr{Z}_{kj} \implies z\,{}^tz' \in \mathscr{Z}_{lk}\ (1 \leqq j < k < l \leqq r)$,
(V3) $z \in \mathscr{Z}_{lk} \implies z\,{}^tz \in \mathbb{R}I_{n_l} \qquad (1 \leqq k < l \leqq r)$.

With this system we set

$$\mathscr{Z} := \left\{ z = \begin{pmatrix} \lambda_1 I_{n_1} & {}^tz_{21} & \cdots & {}^tz_{r1} \\ z_{21} & \lambda_2 I_{n_2} & & {}^tz_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & \lambda_r I_{n_r} \end{pmatrix} ;\ \begin{array}{l} \lambda_k \in \mathbb{R} \quad (k = 1, 2, \ldots, r), \\ z_{lk} \in \mathscr{Z}_{lk}\ (1 \leqq k < l \leqq r) \end{array} \right\}$$
$$\subset \mathrm{Sym}(N,\mathbb{R}).$$

Take the slice $\mathscr{P}_{\mathscr{Z}} := \mathscr{Z} \cap \mathscr{P}(N,\mathbb{R})$.

- $\mathscr{P}_{\mathscr{Z}}$ is a regular open convex cone in $\mathscr{Z}$.

$$H_{\mathscr{Z}} := \left\{ T = \begin{pmatrix} t_1 I_{n_1} & & & \mathbf{0} \\ T_{21} & t_2 I_{n_2} & & \\ \vdots & \vdots & \ddots & \\ T_{r1} & T_{r2} & \cdots & t_r I_{n_r} \end{pmatrix} ; \begin{array}{l} t_k > 0 \quad (k = 1, 2, \dots, r), \\ T_{lk} \in \mathscr{Z}_{lk} \; (1 \leqq k < l \leqq r) \end{array} \right\}.$$

- The Lie group $H_{\mathscr{Z}}$ acts on $\mathscr{P}_{\mathscr{Z}}$ simply transitively by $z \mapsto T z \,{}^t T$.
- Every HOCC arises in this way.

Recall the coordinatization of HOCC $\Omega \subset V$: $V = \begin{pmatrix} \mathbb{R}c_1 & V_{21} & \cdots & V_{r1} \\ V_{21} & \mathbb{R}c_2 & & V_{r2} \\ \vdots & & \ddots & \vdots \\ V_{r1} & V_{r2} & \cdots & \mathbb{R}c_r \end{pmatrix}$.

We call n_j the repetition number of c_j in the above realization $\mathscr{P}_{\mathscr{Z}}$ of Ω.

In Ishi's LN (2014), one finds the following proposition.

Proposition 7

There is a realization of $\Omega \hookrightarrow \mathrm{Sym}(N, \mathbb{R})$ with $N \leqq \dim V$ for any Ω.

However, the inequality $N \leqq \dim V$ is very rough in general.
For example, if $\Omega = \mathscr{P}(r, \mathbb{R})$, then $\dim V = r + \frac{1}{2}r(r-1)$,
although the N we really need in this case is just r.

Question Can we find the formula for the *minimum* of such N?

The answer is Yes.

- Let $d_{ji} := \dim V_{ji}$ $(j > i)$, and draw a weighted oriented graph by defining the set $\mathscr{V}$ of *vertices* and the set $\mathscr{A}$ of *arcs* by

$$\mathscr{V} := \{1, \ldots, r\}, \quad \mathscr{A} := \{[j \to i]\ ;\ i < j,\ \text{and } d_{ji} > 0\}.$$

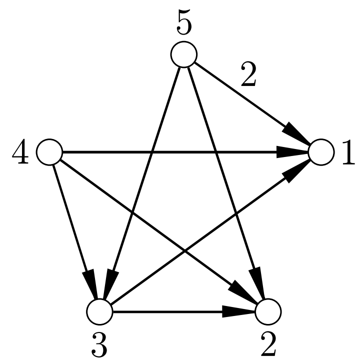
$[j \to i]$ or simply $j \to i$ denotes the arc leaving j and enters i. Thus

$$\overset{j}{\circ} \xrightarrow{d_{ji}} \overset{i}{\circ} \qquad \text{if } \dim V_{ji} > 0.$$

The graph $\Gamma = \Gamma(V) = (\mathscr{V}, \mathscr{A})$ is clearly **oriented**:

we do not have both $j \to i$ and $i \to j$. Moreover no $i \to i$ exists.

Example If $d_{ji} = 1$, we do not write 1 in the graph for simplicity.



$\omega \in \mathscr{V}$ is called a source if there is no $[v \to \omega] \in \mathscr{A}$.
Let $\mathscr{S}$ be the set of sources Γ. Note $\mathscr{S} \neq \varnothing$, since we always have $r \in \mathscr{S}$.
Let $N^{\mathrm{in}}(j) = \{k \,;\, [k \to j] \in \mathscr{A}\}$, $N^{\mathrm{in}}[j] := N^{\mathrm{in}}(j) \cup \{j\}$ for $j = 1, 2, \ldots, r$.
The minimum n_j^0 of the repetition number of c_j is given by

$$n_j^0 = \sum_{\omega \in \mathscr{S} \cap N^{\mathrm{in}}[j]} \dim V_{\omega j},$$

and the minimum N^0 of N is given by $N^0 := n_1^0 + \cdots + n_r^0$.
Thus, if $\omega \in \mathscr{S}$, then $N^{\mathrm{in}}[\omega] = \{\omega\}$, so that $n_\omega^0 = \dim V_{\omega\omega} = 1$.
If $j \notin \mathscr{S}$, then we just have $n_j^0 = \sum_{\omega \in \mathscr{S} \cap N^{\mathrm{in}}(j)} V_{\omega j}$.

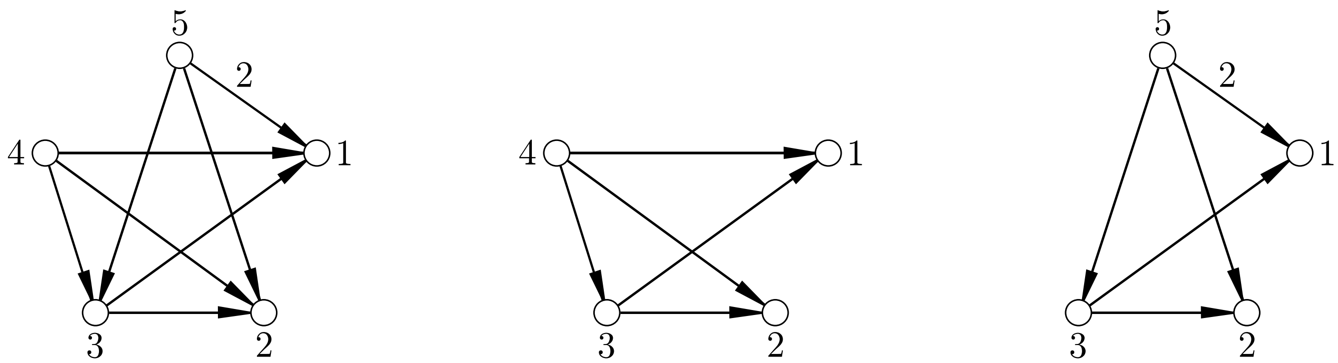
Since $\dim V = \sum_{1 \leqq i \leqq j \leqq r} \dim V_{ji}$, it is a usual phenomenon that

$$|\mathscr{S}| \ll \operatorname{rank} V \implies N^0 \lneqq \dim V \qquad \text{(but not always)}.$$

For example, if $\Omega = \mathscr{P}(r, \mathbb{R})$, then $\mathscr{S} = \{r\}$, and $\mathscr{S} \cap N^{\mathrm{in}}[j] = r$ $(\forall j)$.
Thus $n_j^0 = \dim V_{rj} = 1$ $(\forall j)$, so that $N^0 = r$.
The above claims are based on Yamasaki–N. (2015), and carried out by S. Tanaka in his master thesis (February, 2017).

How to get a realization?

We have $\mathscr{S} = \{4, 5\}$, and $n_1 = 3, \quad n_2 = n_3 = 2, \quad n_4 = n_5 = 1.$



Γ $\quad\quad$ $\Gamma_{[4]}$ $\quad\quad$ $\Gamma_{[5]}$

From $\Gamma_{[5]}$ we proceed as follows:

$V_{[5]} =$

	1	2	3	5	
	$\mathbb{R}c_1$	$\{0\}$	V_{31}	V_{51}	1
	$\{0\}$	$\mathbb{R}c_2$	V_{32}	V_{52}	2
	V_{31}	V_{32}	$\mathbb{R}c_3$	V_{53}	3
	V_{51}	V_{52}	V_{53}	$\mathbb{R}c_5$	5

$\supset E_{[5]} :=$ the shaded part, $\dim E_{[5]} = 5$.

$V_{[5]}$ is a subalgebra of V, and

$E_{[5]}$ is a 2-sided ideal of $V_{[5]}$.

Let $\varphi_{[5]}(x)\eta := R(x)\eta$ $(x \in V_{[5]}, \eta \in E_{[5]})$.

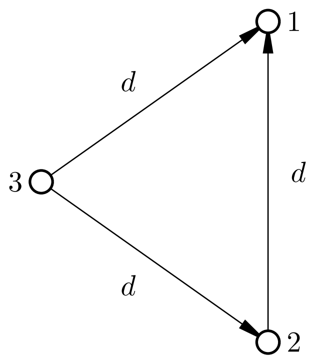
The matrix for $\varphi_{[5]}(x)$ is $\begin{pmatrix} \lambda_1 I_2 & \mathbf{0}_2 & x_{31}\boldsymbol{e}_1 & \boldsymbol{x}_{51} \\ {}^t\mathbf{0}_2 & \lambda_2 & x_{32} & x_{52} \\ x_{31}{}^t\boldsymbol{e}_1 & x_{32} & \lambda_3 & x_{53} \\ {}^t\boldsymbol{x}_{51} & x_{52} & x_{53} & \lambda_5 \end{pmatrix}$ (5×5 matrix), $\boldsymbol{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Similarly, from $\Gamma_{[4]}$, we get the matrix for $\varphi_{[4]}(x)$ is $\begin{pmatrix} \lambda_1 & 0 & x_{31} & x_{41} \\ 0 & \lambda_2 & x_{32} & x_{42} \\ x_{31} & x_{32} & \lambda_3 & x_{43} \\ x_{41} & x_{42} & x_{43} & \lambda_4 \end{pmatrix}$.

Put these two matrices in a direct sum form $\rightsquigarrow$ 9×9 matrix acting on $\mathbb{R}^9$.
Carry out a base permutation in $\mathbb{R}^9$ to get to the Graczyk–Ishi presentation.

$$\Omega = \left\{ \left(\begin{array}{ccc|cc|cc|c|c}
\lambda_1 & 0 & 0 & 0 & 0 & x_{31} & 0 & x_{41} & 0 \\
0 & \lambda_1 & 0 & 0 & 0 & 0 & x_{31} & 0 & x_{51}^{(1)} \\
0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & x_{51}^{(2)} \\
\hline
0 & 0 & 0 & \lambda_2 & 0 & x_{32} & 0 & x_{42} & 0 \\
0 & 0 & 0 & 0 & \lambda_2 & 0 & x_{32} & 0 & x_{52} \\
\hline
x_{31} & 0 & 0 & x_{32} & 0 & \lambda_3 & 0 & x_{43} & 0 \\
0 & x_{31} & 0 & 0 & x_{32} & 0 & \lambda_3 & 0 & x_{53} \\
\hline
x_{41} & 0 & 0 & x_{42} & 0 & x_{43} & 0 & \lambda_4 & 0 \\
\hline
0 & x_{51}^{(1)} & x_{51}^{(2)} & 0 & x_{52} & 0 & x_{53} & 0 & \lambda_5
\end{array} \right) \gg 0 \right\}$$

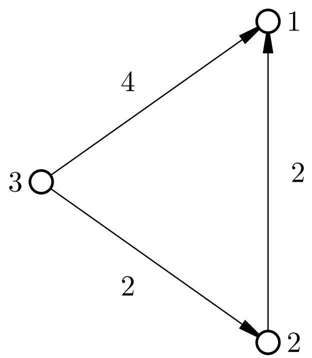
• We have $\Omega \mapsto \Gamma(V)$: HOCC $\mapsto$ weighted oriented graph.
(1) Not every oriented graph comes from a HOCC.



This comes from a HOCC $\iff$ $d = 1, 2, 4, 8$.

These are $\mathscr{P}(3, \mathbb{K})$; $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

(2) We have continuously many linearly inequivalent HOCCs of dim $= 11$ with the same oriented graph.



There are still other theorems we have obtained for homogeneous tube domains $\Omega + iV$, and for the homogeneous Siegel domains $D(\Omega, Q)$.