Cauchy-Stieltjes families with polynomial variance functions

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NEF versus CSK families

Two examples of "kernel families" $k(x,\theta)\mu(dx)$

$$\mathcal{K} = \{P_\theta(dx) : \theta \in \Theta\}$$

- Natural exponential families (NEF) :

$$P_\theta(dx) = \frac{1}{Z_\theta} e^{\theta x} \mu(dx)$$

  where μ is a (probability) measure with (some) exponential moments, $\Theta = (\theta_-, \theta_+)$.

- Cauchy-Stieltjes kernel families (CSK):

$$P_\theta(dx) = \frac{1}{Z_\theta} \frac{1}{1 - \theta x} \mu(dx)$$

  where μ is a probability measure with support bounded from above. The "generic choice" for Θ is $\Theta = (0, \theta_+)$, or (θ_-, θ_+) if μ is compactly supported...

A specific example of CSK

Noncanonical parameterizations

Let $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ be the Bernoulli measure.

$Z_\theta = \int \frac{1}{1-\theta x}\mu(dx) = 1/2 + \frac{1/2}{1-\theta} = \frac{2-\theta}{2(1-\theta)}$

- "Noncanonical" parametrization:
- $P_\theta = \frac{1-\theta}{2-\theta}\delta_0 + \frac{1}{2-\theta}\delta_1$, $\theta \in (-\infty, 1)$.
- "Canonical" parametrization: $p = \frac{1}{2-\theta}$
- $Q_p := P_{2-\frac{1}{p}} = (1-p)\delta_0 + p\delta_1, \;\; p \in (0,1)$
- Bernoulli family parameterized by probability of success p.
- $p = \int x Q_p(dx)$ (parametrization by the mean)

Parametrization by the mean

$$m(\theta) = \int x P_\theta(dx) = \begin{cases} \frac{1}{Z_\theta} \frac{d}{d\theta} Z_\theta & \text{NEF} \\ \\ \frac{1}{Z_\theta} \frac{Z_\theta - 1}{\theta} & \text{CSK} \end{cases}$$

- For non-degenerate measure μ, function $\theta \mapsto m(\theta)$ is strictly increasing and has inverse $\theta = \theta(m)$.
- $\theta \mapsto m(\theta)$ maps $(0, \theta_+)$ onto (m_0, m_+), "the domain of means".
- Parameterizations by the mean:

$$\{Q_m(dx) : m \in (m_0, m_+)\}$$

where $Q_m(dx) = P_{\theta(m)}(dx)$

Variances for NEF or CSK

$$V(m) = \int (x-m)^2 Q_m(dx)$$

- Variance function $m \mapsto V(m)$

  always exists for NEF.

  exists for CSK when $\mu(dx)$ has the first moment.
- Variance function $V(m)$ together with the domain of means $m \in (m_-, m_+)$ determines NEF uniquely (Morris (1982)).
- Variance function $V(m)$ of CSK family together with $m_0 = m(0) \in \mathbb{R}$, the mean of μ, determines measure μ uniquely

  Hence $V(m)$ determines CSK uniquely

CSK/NEF are determined uniquely by $V(m)$

(some details are left out!)

$Q_m(dx) = f(x,m)\mu(dx)$

- NEF: (Wedderburn (1974))

$$\frac{\partial}{\partial m} f(x,m) = \frac{x-m}{V(m)} f(x,m) \tag{1}$$

- CSK when $m_0 = \int x\mu(dx) = 0$:

$$\frac{f(x,m) - f(x,0)}{m} = \frac{x-m}{V(m)} f(x,m); \quad f(x,0) = 1, \tag{2}$$

- In particular, (2) has solution $f(x,m) = \frac{V(m)}{V(m)+m(m-x)}$.
- When $f(x,m)\mu(dx)$ is a probability measure?

8m

Skip free cumulants

CSK is determined uniquely by the variance function

More precisely, by $V(\cdot)$ and $m_0 = \int x\mu(dx)$

Theorem (WB-Ismail(2005))

The free cumulants of compactly supported μ are

- $c_1 = m_0$,
- *for* $n \geq 1$,

$$c_{n+1} = \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}\left(V(x)\right)^n\Big|_{x=m_0}.$$

Recall that free cumulants $c_k(\mu)$ are polynomials in the moments $\int x d\mu, \int x^2 d\mu, \cdots \int x^k d\mu, \dots$ which linearize free convolution: $c_k(\mu \boxplus \nu) = c_k(\mu) + c_k(\nu)$.

All NEF with quadratic variance functions are known

Morris class. Meixner laws

- The NEF with the variance function $V(m) = 1 + am + bm^2$ was described by Morris (1982), Ismail-May (1978). Eg:
  1. $\mathcal{K}$ is the family of Gaussian laws (of unit variance) iff $V(m) = 1$
  2. $\mathcal{K}$ is the family Poisson-type laws iff $V(m) = 1 + am$ with $a \neq 0$
  3. $\mathcal{K}$ is the family of binomial type laws (affine transformations of the convolution powers of a Bernoulli law) iff $V(m) = 1 + am + bm^2$ with $-1 \leq b = -1/n < 0$
- Letac-Mora (1990): cubic $V(m)$

  Eg., $V(m) = m^3$ corresponds to the family of 1/2-stable laws
- Various other classes Kokonendji, Letac, ...

All CSK with quadratic variance functions are known

Suppose $m_0 = 0$, $V(m) = 1 + am + bm^2$.

Theorem (WB-Ismail (2005))

Examples of quadratic variance functions (3 of 6 cases):

1. μ *is the Wigner's semicircle (free Gaussian) law iff* $V(m) = 1$
   $\mathcal{K}(\mu)$ *are the (atomless) Marchenko-Pastur (free Poisson) type laws*
2. μ *is the Marchenko-Pastur (free Poisson) type law iff* $V(m) = 1 + am$ *with* $a \neq 0$
3. μ *is the free binomial type law (Kesten law, McKay law) iff* $V(m) = 1 + am + bm^2$ *with* $-1 \leq b < 0$

18m ⏩ Skip to cubic (part II) 25m ⏩ End now

Cubic variance functions

part I

- In [WB-Hassairi (2011]) we consider $f(x, m) = \frac{\mathbb{V}(m)}{\mathbb{V}(m)+m(m-x)}$ with
$$\mathbb{V}(m) = m(am^2 + bm + c)$$
- Probability measure μ which generates CSK family $Q_m(dx) = f(x, m)\mu(dx)$ has no mean, so the variance of Q_m is infinite!
- This difficulty does not arise for NEF!

16 ⏩ Skip pseudo Variance functions 19 m ⏩ Go to cubic (part II) 25 m ⏩ End now

Pseudo-Variance function for CSK

- The variance
$$V(m) = \frac{1}{Z_{\theta(m)}} \int \frac{(x-m)^2}{1-\theta(m)x} \mu(dx)$$
is undefined if $m_0 = \int x\mu(dx) = -\infty$.
- Consider
$$\mathbb{V}(m) = m\left(\frac{1}{\theta(m)} - m\right) \tag{3}$$
where $\theta(\cdot)$ is the inverse of $\theta \mapsto m(\theta) = \int xP_\theta(dx)$ on $(0, \theta_+)$.
- This defines a "pseudo-variance" function $\mathbb{V}(m)$ that is well defined for all non-degenerate probability measures μ with support bounded from above.
- When $V(m)$ exists, then
$$\mathbb{V}(m) = \frac{m}{m-m_0} V(m)$$

Example: CSK family with cubic pseudo-variance function

Measure μ generating CSK with $\mathbb{V}(m) = m^3$ has density

$$\mu(dx) = \frac{\sqrt{-1-4x}}{2\pi x^2} 1_{(-\infty,-1/4)}(x)dx \tag{4}$$

Measure μ is 1/2-stable with respect to $\boxplus$, a fact already noted before: [Bercovici and Pata, 1999, page 1054], [Pérez-Abreu and Sakuma, 2008]

$$\left\{ Q_m(dx) = \frac{m^2\sqrt{-1-4x}}{2\pi(m^2+m-x)x^2} 1_{(-\infty,-1/4)}(x)dx : m \in (-\infty, m_+) \right\}$$

What is m_+? 19m ⏩ Skip domain of means 25m ⏩ End now

Domain of means: $\{Q_m : m \in (-\infty, m_+)\}$

Answers for $\mathbb{V}(m) = m^3$ (WB-Fakhfakh-Hassairi -2014)

1. $m \in (-\infty, -1)$, because $\lim_{\theta \nearrow \theta_{\max}} m(\theta) = -1$.
2. $m \in (-\infty, -1/2)$, because $\frac{1}{1-\theta x} 1_{(-\infty,-1/4)}(x)$ is positive for $\theta \in (0, \infty) \cup (-\infty, -4)$, and $\lim_{\theta \nearrow -4} m(\theta) = -1/2$.
3. $m \in (-\infty, -1/2) \cup (-1/2, \infty)$, because $f(x, m) = \frac{m^2}{m^2+m-x} 1_{(-\infty,-1/4)}(x) \geq 0$ for all $m \neq -1/2$.
   - Unfortunately, $\int Q_m(dx) < 1$ for $m > -1/2$.
   - But $Q_m(dx) := \frac{m^2}{(m^2+m-x)} \mu(dx) + \frac{(1+2m)_+}{(m+1)^2} \delta_{m+m^2}$ is well defined and parameterized by the mean for all $m \in (-\infty, \infty)$.

Similar situation arises for $V(m) = 1$, where $Q_m(dx)$ is a Marchenko-Pastur law. By adding an atom at $m + \frac{V(m)}{m}$ we can extend the domain of means to $(-\infty, \infty)$.

Polynomial variance functions

Part II: finite mean $m_0 = 0$, $V(0) = 1$

Theorem (WB-Fakhfakh-2017)

$V(m) = 1 + am + bm^2 + cm^3$ is a variance functions for any real c and a if $b^3 = 27c^2$.

Remark

$V(m) = 1 + m^2 + m^3$ is not a variance function.
$V(m) = m + m^2 + m^3$ is a pseudo-variance function but measure μ has infinite mean.

This is $p = 3$ of the following more general result.

Theorem (WB-Fakhfakh-2017)

Fix real $p \geq 1$, and real a, c. Then for m close enought to 0, function $V(m) = (1 + cm)^p + am$ is a variance function of a CSK family generated by a compactly supported centered ($\boxplus$-infinitely divisible) probability measure.

A lemma on variance functions

The following seems to have no analogue for NEFs.

Theorem (WB-Fakhfakh-2017)

If $V(m)$ is a variance function corresponding to a compactly supported centered probability measure μ_0, then for any real a function

$$V_a(m) := am + V(m)$$

for m close enough to 0 is a variance function for a CSK family generated by some (uniquely determined) compactly supported centered probability measure μ_a.

Polynomial variance functions

Part II: finite mean $m_0 = 0$, $V(0) = 1$

The density of Q_m has series expansion

$$f(x, m) = \frac{V(m)}{V(m) + m(m - x)} = \sum_{n=0}^{\infty} P_n(x) m^n.$$

Polynomials $\{P_n(x)\}$ are monic and solve the recursion

$$xP_n(x) = P_{n+1}(x) + P_{n-1}(x) + \sum_{k=1}^{n} \frac{V^{(k)}(0)}{k!} P_{n+1-k}(x), \; n \geq 0$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$.

Theorem (WB-Fakhfakh-2017)

Suppose V is a variance function for CSK family generated by centered compactly supported measure μ, with $V(0) > 0$. Then the following are equivalent.

1. *$V(m)$ is a polynomial of degree at most $d+1$;*
2. *There exist constants $\{b_k : k = 0, 1, \ldots, d+1\}$ with $b_0 > 0$ such that polynomials $\{P_n\}$ satisfy finite recursion*

$$xP_n(x) = \sum_{k=0}^{(d+1)\wedge n} b_k P_{n+1-k}(x), n \geq 2 \tag{5}$$

   with initial conditions $P_0(x) = 1$, $P_1(x) = x$.

3. *Polynomials $P_n(x), P_k(x)$ are orthogonal in $L_2(d\mu)$ for $n \geq 2 + (k-1)d$.*
4. *Polynomial $P_2(x)$ is orthogonal in $L_2(d\mu)$ to all polynomials $\{P_n(x) : n \geq 2 + d\}$.*

Summary

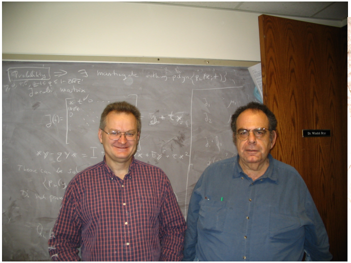
Kernels $e^{\theta x}$ and $1/(1-\theta x)$ generate NEF and CSK families

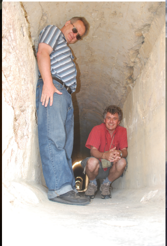
Similarities

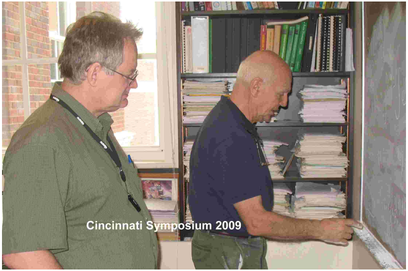
- parameterizations by the mean
- Quadratic variance functions determine interesting laws
- Convolution/free convolutuion affects (pseudo) variance functions for NEF/CSK in a similar way. Eg. $\mathbb{V}(m) = m^3$ is 1/2-stable with respect to $*/\boxplus$.
- When V is cubic, polynomials from expansions of the density are related to "generalized orthogonality".

Differences

- The generating measure of a NEF is not unique.
- The variance function of CSK family may be undefined.
- A CSK family may be well defined beyond the "domain of means".







Thank you

Thank you

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References

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