On the dynamics of minimal homeomorphisms of $\mathbb{T}^2$

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Dimension 2

$\mathbb{T}^2$ is the only closed surface supporting minimal homeos

Minimal homeos on $\mathbb{T}^2$

1. **Ergodic rotations:** $(\alpha, \beta) \in \mathbb{R}^2$,

$$R_{\alpha,\beta} : \mathbb{T}^2 \ni (x, y) \mapsto (x + \alpha, y + \beta),$$

s.t. $m\alpha + n\beta \in \mathbb{Z}$ with $m, n \in \mathbb{Z} \implies m = n = 0$

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2. **Time-t reparametrizations of linear flows:** $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R}_+)$, s.t. $\forall c \in \mathbb{R}$, $\nexists u \in C^0(\mathbb{T}^2, \mathbb{R})$ s.t. $\partial_x u + \alpha \partial_y u = \psi - c$.
   Then for $X := \psi \cdot (1, \alpha)$, Φ_X^t is minimal for generic t

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3. [Furstenberg '61]: $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\phi \colon \mathbb{T}^1 \to \mathbb{R}$ t.q. $\nexists u \in C^0(\mathbb{T}^1, \mathbb{R})$, $u \circ R_\alpha - u = \phi$:

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4. **Irrational Dehn twists:** $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $m \in \mathbb{Z} \backslash \{0\}$

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Invariant foliations?

All previous examples have an invariant foliation.

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Theorem [Fathi-Herman '77]

Generic diffeos in

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have **no invariant continuous foliation.**

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$f\colon \mathbb{T}^2 \circlearrowleft$ a homeo isotopic to the identity and $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ a lift

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Rotation set [Misiurewicz-Ziemian '89]

$\Delta_{\tilde{f}} := \tilde{f} - id_{\mathbb{R}^2} \in C^0(\mathbb{T}^2, \mathbb{R}^2)$ and the **rotation set**

$$\rho(\tilde{f}) := \left\{ \lim \frac{\tilde{f}^{n_i}(z_i) - z_i}{n_i} : n_i \to \infty, z_i \in \mathbb{R}^2 \right\} = \left\{ \int_{\mathbb{T}^2} \Delta_{\tilde{f}} \, \mathrm{d}\mu : \mu \in \mathfrak{M}(f) \right\}.$$

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[Jonker-Zhang '98 + K.-Koropecki '08]:
$\mathrm{Per}(f) = \varnothing$ and $\Omega(f) = \mathbb{T}^2 \implies \rho(\tilde{f}) \cap \mathbb{Q}^2 = \varnothing$

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Conjecture [Franks-Misiurewicz '90]

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Theorem [Avila, '13]

There is $f \in \mathrm{Diff}_0^{\infty}(\mathbb{T}^2)$ minimal s.t. $\rho(\tilde{f})$ is **irrational slope segment**

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Main results

Theorem A [K. 2015]

If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is minimal and $\rho(\tilde{f})$ is not a point, then there exists an f-invariant pseudo-foliation

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Corollary

If f is minimal and $\rho(\tilde{f})$ is not a point, then

1. either $\rho(\tilde{f})$ is a rational slope segment and f is a topological extension of an irrational circle rotation
2. or $\rho(\tilde{f})$ is an irrationa slope segment and f is topologically mixing

Pseudo-foliations

Plane pseudo-foliation

It's a partition $(\tilde{\mathscr{F}}_z)_{z\in\mathbb{R}^2}$ of $\mathbb{R}^2$ s.t. every atom

- is closed, connected and has empty interior
- and separates $\mathbb{R}^2$ in exaclty 2 connected components

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Torus pseudo-foliation

It's a partition of $(\mathscr{F}_x)_{x\in\mathbb{T}^2}$ of $\mathbb{T}^2$ s.t. there exists a plane psuedo-foliation $(\tilde{\mathscr{F}}_z)_{z\in\mathbb{R}^2}$ satisfying

$$\pi(\tilde{\mathscr{F}}_z) = \mathscr{F}_{\pi(z)}, \quad \forall z \in \mathbb{R}^2.$$

Rotational deviations

- [Poincaré, ∼1900] $f \in \mathrm{Homeo}_0(\mathbb{T}^1)$, $\tilde{f}\colon \mathbb{R} \circlearrowleft$ a lift and $\rho = \rho(\tilde{f}) \in \mathbb{R}$ rotation number, then

$$\left|\tilde{f}^n(z) - z - n\rho\right| \leqslant 1, \quad \forall n \in \mathbb{Z},\ \forall z \in \mathbb{R}$$

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- In dim 2, if $\rho(\tilde{f}) \subset \ell_\alpha^v = \{z \in \mathbb{R}^2 : \langle z, v\rangle = \alpha\}$, then we say f has **uniformly bounded v-deviations** iff $\exists C > 0$ s.t.

$$\sup_{z\in\mathbb{R}^2} \sup_{n\in\mathbb{Z}} \left|\langle \tilde{f}^n(z) - z, v\rangle - n\alpha\right| \leqslant C$$

Bounded v-deviations vs. invariant pseudo-foliations

Theorem B [K.-Pereira Rodrigues '15]

If $\mathrm{Per}(f) = \varnothing$ and $\Omega(f) = \mathbb{T}^2$, then f leaves invariant a pseudo-foliation iff $\exists v \in \mathbb{S}^1$ s.t. f exhibits uniformly bounded v-deviations

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Theorem [Beguin-Crovisier-Jäger '15]

"Pseudo" is sharp: there exists a diffeo which is a topological extension of an irrational circle rotation whose fibers are pseudo-circles.

Proof of Thm A

We will prove the following

Theorem A'

If f is minimal and $\rho(\tilde{f}) = \{\alpha\} \times [a, b]$, with $a < 0 < b$, then there exists $M > 0$ s.t.

$$|\mathrm{pr}_1(f^n(z) - z) - n\alpha| \leqslant M, \quad \forall z \in \mathbb{R}^2, \ \forall n \in \mathbb{Z}$$

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Our strategy to prove Thm A':

1. Construct two families of vertical $\tilde{f}$-invariant sets (top and bottom)
2. Construct two families of horizontal $\tilde{f}$-invariant sets (left anf right)
3. Get a contradiction showing that they don't intersect

Vertical invariant sets

1. We define $\mathbb{H}_r^T := \{z \in \mathbb{R}^2 : \mathrm{pr}_2(z) > r\}$ and $\mathbb{H}_r^B := \mathbb{R}^2 \setminus \overline{\mathbb{H}_r^T}$; and

$$\Lambda_r^T := \mathrm{cc}\left(\bigcap_{n\in\mathbb{Z}} \tilde{f}^n(\mathbb{H}_r^T), \infty\right)$$

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3. $\Lambda_r^T \cap \Lambda_s^B = \emptyset$, $\forall r, s$ (Gottschalk-Hedlund)

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4. **Large horizontal oscilations:** Assuming unbounded v-deviations,

$$\mathrm{pr}_1\left(\Lambda_r^T \setminus \mathbb{H}_s^T, z\right) \to [-\infty, \infty],$$

as $s \to +\infty$ and $\forall z \in \Lambda_r^T$

Horizontal sets... ρ-centralized skew-product

Given $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, $\tilde{f} \colon \mathbb{R}^2 \circlearrowleft$ a lift and any $\rho \in \rho(\tilde{f})$, define $F \colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$ by

$$F(t,z) := \Big(R_\rho(t), R_t^{-1} \circ \big(R_\rho^{-1} \circ \tilde{f}\big) \circ R_t(z) \Big)$$

In our case we take $\rho = (\alpha, 0)$

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Main properties:

- If $\tilde{f} = id + \Delta_{\tilde{f}}$, then

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- For any $n \in \mathbb{Z}$,

$$F^n(0,z) = \big(n\rho, \tilde{f}^n(z) - n\rho\big), \quad \forall z \in \mathbb{R}^2$$

Fibered horizontal invariant sets

1. Define $\mathbb{V}_r^L := \{z \in \mathbb{R}^2 : \mathrm{pr}_1(z) < r\}$, $\mathbb{V}_r^R := \mathbb{R}^2 \setminus \overline{\mathbb{V}_r^L}$ and

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4. $\Lambda_r^L(t) \cap \Lambda_s^R(t) = \varnothing \iff f$ exhibits unbounded v-deviations
5. **Strong spreading:** If unbounded v-deviations, then for every open $U, V \subset \mathbb{R}^2$, $\forall t \in \mathbb{T}^2$, $\exists N \in \mathbb{N}$ s.t.

$$F^n(t, U) \cap \mathbb{T}^2 \times V \neq \varnothing, \forall n \geqslant N$$

Fibered horizontal invariant sets

1. Define $\mathbb{V}_r^L := \{z \in \mathbb{R}^2 : \mathrm{pr}_1(z) < r\}$, $\mathbb{V}_r^R := \mathbb{R}^2 \setminus \overline{\mathbb{V}_r^L}$ and
$$\Lambda_r^L(t) := \mathrm{cc}\left(\{t\} \times \mathbb{R}^2 \cap \bigcap_{n\in\mathbb{Z}} F^n(\mathbb{T}^2 \times \mathbb{V}_r^L), \infty\right),$$
2. $\Lambda_r^L(t) \neq \emptyset$ and $\Lambda_r^R(t) \neq \emptyset$ (proof à la Birkhoff)
3. $$\overline{\bigcup_{r\in\mathbb{R}} \Lambda_r^L(t)} = \{t\} \times \mathbb{R}^2$$
4. $\Lambda_r^L(t) \cap \Lambda_s^R(t) = \emptyset \iff f$ exhibits unbounded v-deviations
5. **Strong spreading:** If unbounded v-deviations, then for every open $U, V \subset \mathbb{R}^2$, $\forall t \in \mathbb{T}^2$, $\exists N \in \mathbb{N}$ s.t.
$$F^n(t, U) \cap \mathbb{T}^2 \times V \neq \emptyset, \forall n \geqslant N$$
6. So, $\Lambda_r^T(t)$, $\Lambda_{r'}^B(t)$, $\Lambda_s^R(t)$, $\Lambda_{s'}^L(t)$ are disjoint. Contradiction!

Merci!