Global Dynamics for symmetric planar maps

Begoña Alarcón

Universidade Federal Fluminense
Brazil

joint work with: Isabel Labouriau, Sofia Castro and Javier Ribón

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Global Dynamics for symmetric planar maps vs Periodic non autonomous differential equations

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Non-autonomous differential equations

Consider the system

$$\dot{x} = F(t, x) \qquad (1)$$

where

- $F : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ is $T-$periodic in t and
- the solutions $\varphi(t; \tau, x_0)$ exist, are unique, can be extended indefinitely into the future and depend continuously on initial condition.

The solutions of (1) define a semiflow π on $S^1 \times \mathbf{R}^2$, given by

$$\varphi(s, (\tau, x_0)) = (\tau + s \bmod T, \varphi(\tau + s; \tau, x_0)), \quad \forall s \geq 0$$

where $\tau \in S^1 = [0, T]$ with 0 and T identified.
Consider $\tau = 0$ for simplicity and let $P(x_0) = \varphi(T; 0, x_0)$, the first return map under the semiflow π.
The map P is called the $[0, T]$ Poincaré map of the Equation (1).

Non-autonomous differential equations

- the solutions $\varphi(t;\tau,x_0)$ exist, are unique, can be extended indefinitely into the future,
  - ✓ $P:\mathbf{R}^2\to\mathbf{R}^2$ is well defined and injective. Observe that $P(\mathbf{R}^2)$ can not equal $\mathbf{R}^2$.
- the solutions $\varphi(t;\tau,x_0)$ are depend continuously on initial condition,
  - ✓ $P:\mathbf{R}^2\to\mathbf{R}^2$ continuous.

So $P\in\mathrm{Emb}^+(R^2)$, where $\mathrm{Emb}(\mathbf{R}^2)$ is the set of planar continuous and injective self maps.
Furthermore, $T-$periodic solutions of Equation (1) correspond to fixed points of P and $nT-$periodic solutions of (1) are periodic points of P.

Non-autonomous differential equations

If in addition...

- all solutions can be also extended indefinitely into the past,
  - ✓ $P \in \mathbf{Homeo}^{+}(R^2)$. Observe that $P^{-1}(p) = \varphi(-T; 0, p)$
- and have differentiable dependence on initial condition,
  - ✓ $P : \mathbf{R}^2 \to \mathbf{R}^2$ is differentiable and the Jacobi-Liouville Formula holds

$$0 < detP'(p) = \exp\{\int_0^T div_x F(t, \varphi(t, p))dt\}$$

  - ✓ $P \in \mathbf{Diff}^{+}(R^2)$
  - ✓ if $div_x F(t, x) < 0$ then P **is area-contracting**.

Non-autonomous differential equations

Definition

The system $\dot{x} = F(t, x)$ given by (1) is said to be dissipative if there exists a real number $B > 0$ such that $\|\varphi(t, x)\| \leq B$, $\forall t \geq \tau$, where τ may depend on x and B.

Theorem (Yoshizawa, 1975)

If the system $\dot{x} = F(t, x)$ *is dissipative, then Equation* (1) *has a* $T-$*periodic solution.*

Remark

If the system (1) *is dissipative, the corresponding Poincaré map is also dissipative.*

Definition

A system $\dot{x} = F(t, x)$ given by (1) is said to be convergent if there exists a unique $T-$periodic solution and it is globally asymptotically stable (GAS).

Let's play topological dynamics

Theorem (P. Murthy, 1998)

If U is an open simply connected subset of $\mathbf{R}^2$ and $g : U \to U$ is a continuous, $1-1$, orientation preserving self-map such that $\Omega(g) \neq \emptyset$, then g has a fixed point in U.

Lemma (Alarcón,Guiñez, Gutierrez 2008)

Let $f \in Emb(\mathbf{R}^2)$ such that $f(0) = 0$. Suppose that one of the following hold:

- *f is orientation preserving and $Fix(f) = \{0\}$*
- *f is orientation reversing and $Fix(f^2) = \{0\}$*

If there exists a $f-$invariant ray, γ, then $\Omega(f) \subset \gamma$ and either $\omega(p) = \{0\}$ or $\omega(p) = \emptyset$, for all $p \in \mathbf{R}^2$.

Let's play topological dynamics

Theorem (Alarcón-Guiñez-Gutierrez, 2008)

Let $f \in Emb(\mathbf{R}^2)$ *such that* $f(0) = 0$ *and* f **is dissipative**.
Suppose that one of the following hold:

- f *is orientation preserving and* $Fix(f) = \{0\}$
- f *is orientation reversing and* $Fix(f^2) = \{0\}$

If there exists a $f-$*invariant ray,* γ*, then* $\Omega(f) \subset \gamma$ *and either* $\omega(p) = \{0\}$ *or* ~~$\omega(p) = \emptyset$~~*, for all* $p \in \mathbf{R}^2$*.*
Additionally, 0 *is globally asymptotically stable (GAS) provided by* 0 **is locally stable**.

Remark

Ortega and Ruiz del Portal applied in 2011 *this result to Population Dynamics.*

Go back to non-autonomous systems

Theorem

Consider the Equation (1) *and suppose that* $F(t,0) = 0$, $\forall t$ *and the following assumptions hold:*

- *the system is dissipative,*
- *the linearized system* $\dot{y} = \frac{\partial F}{\partial x}(t,0)y$ *is asymptotically stable,*
- **there exists a ray invariant by the Poincaré map.**

Then, the system (1) *is convergent if and only if there are no other* $T-$*periodic solutions.*

Symmetries

Definition

We say that $\gamma \in GL(2)$ is a symmetry of Equation (1) if $F(t, \gamma x) = \gamma F(t, x)$, $\forall x$ and $\forall t$.

Theorem (Alarcón-Castro-Labouriau, 2013)

If $\gamma \in O(2)$ is a symmetry of Equation (1), then $P(\gamma x) = \gamma P(x)$, $\forall x \in \mathbf{R}^2$.

Lemma (Alarcón-Castro-Labouriau, 2013)

Let $f \in Emb(\mathbf{R}^2)$ such that the linear reflection κ is a symmetry of f and $Fix(f) = \{0\}$. Suppose that one of the following hold:

- *f is orientation preserving and does not interchange connected components of $\mathbf{R}^2 \setminus Fix\langle\kappa\rangle$.*
- *$Fix(f^2) = \{0\}$.*

Then for all $p \in \mathbf{R}^2$ either $\omega(p) = \{0\}$ or $\omega(p) = \emptyset$.

Symmetry group

Definition

We define the symmetry group of f as the biggest closed subset of $GL(2)$ containing all the symmetries of f. It will be denoted by $\Gamma_f(g)$.

Remark

- *f is always Γ_f-equivariant*
- *we only consider the symmetry groups $O(2), SO(2)$, $D_n, \mathbf{Z}_n$ for $n \geq 2$ and $\mathbf{Z}_2\langle\kappa\rangle$.*
- *Observe that all considered symmetry groups Γ has $Fix(\Gamma) = \{0\}$, so $f(0) = 0$.*

Symmetric planar maps

Symmetry group	$Df(0)$	hyperbolic local dynamics
$O(2)$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ $\alpha \in \mathbf{R}$	attractor / repellor
$SO(2)$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\alpha, \beta \in \mathbf{R}$	attractor / repellor
$D_n,\ n \geq 3$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ $\alpha \in \mathbf{R}$	attractor / repellor
$\mathbf{Z}_n,\ n \geq 3$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\alpha, \beta \in \mathbf{R}$	attractor / repellor
$\mathbf{Z}_2$	any matrix	saddle / attractor / repellor
$\mathbf{Z}_2\langle\kappa\rangle$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\alpha, \beta \in \mathbf{R}$	saddle / attractor / repellor
$D_2 = \mathbf{Z}_2\langle-\kappa\rangle \oplus \mathbf{Z}_2\langle\kappa\rangle$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\alpha, \beta \in \mathbf{R}$	saddle / attractor / repellor

Saddles only appear with symmetry group $D_2, \mathbf{Z}_2\langle\kappa\rangle$ and $\mathbf{Z}_2$.

Let's play topological dynamics with symmetry

Proposition (Alarcón-Castro-Labouriau, 2013)

Let $f \in Emb(\mathbf{R}^2)$ *such that* $Fix(f) = \{0\}$ *and its group of symmetry is either* $O(2)$ *or* $\mathbf{Z}_2\langle\kappa\rangle$ *or* D_n*. Suppose that one of the following hold:*

- f *is orientation preserving and does not interchange connected components of* $\mathbf{R}^2 \setminus Fix\langle\kappa\rangle$.
- $Fix(f^2) = \{0\}$.

Then for all $p \in \mathbf{R}^2$ *either* $\omega(p) = \{0\}$ *or* $\omega(p) = \emptyset$.

Remark

- *also holds if* $\Gamma_f = SO(2)$ *but we need* f *is area-contracting.*
- *its false for* $\Gamma_f = \mathbf{Z}_n$*. We construct counter-examples with irrational prime ends rotation number.*

Topological global saddles

Definition

We say that 0 is a topological global saddle if

(i) 0 is a hyperbolic saddle,

(ii) there are no homoclinic contacts and the curves $W^s(0, f)$ and $W^u(0, f)$ are unbounded. Moreover, $W^s \cup W^u$ separates the plane into exactly four connected components.

(iii) for all $p \notin W^s(0, f) \cup W^u(0, f) \cup \{0\}$ both $\|f^n(p)\| \to \infty$ and $\|f^{-n}(p)\| \to \infty$ as n goes to ∞.

In case of 0 is a direct (twisted) saddle, 0 is called direct (twisted) topological global saddle.

Remark

Observe that topological global saddles are still far from the global conjugation to the linear saddle.

What about symmetric saddles?

Theorem (Alarcón-Castro-Labouriau, to appear)

Let $f \in Homeo(\mathbf{R}^2)$ *of class* C^1 *such that* $Fix(f) = \{0\}$ *and* 0 *is a hyperbolic saddle. Suppose that* $\Gamma_f = D_2$ *and one of the following holds:*

- 0 *is a direct saddle*
- $Fix(f^2) = \{0\}$

Then, the origin is a topological global saddle.

Remark

- f *is a free homeomorphism of the plane.*
- *Observe that both* $W^s \subset Fix\langle\kappa_1\rangle$ *and* $W^u \subset Fix\langle\kappa_2\rangle$ *which are two invariant line.*
- *For symmetric group* $\mathbf{Z}_2\langle\kappa_2\rangle$ *we have only one reflection and for* $\mathbf{Z}_2$ *we have no reflections.*

Go back to non-autonomous systems

Example (Alarcón-Castro-Labouriau, to appear)

As an illustration of such a transformed system, consider:

$$\begin{cases} \dot{x} = \alpha x + f_1(x,y) \\ \dot{y} = -\beta y + f_2(x,y) \qquad\qquad \alpha, \beta > 0 \\ \dot{z} = 1 \end{cases}$$

such that $f_i(x,y) = O(|(x,y)|^2)$ and $f = (f_1, f_2)$ is D_2-equivariant, and either $\dot{x} \neq 0$ or $\dot{y} \neq 0$ for $(x,y) \neq (0,0)$.

The linear part of P is given by $(x,y) \mapsto \left(e^{\alpha}x, e^{-\beta}y\right)$ and, by previous theorem, the origin is a topological global saddle.

What about saddles without symmetry?

Theorem (Hirsch, 2000)

Let $f \in \mathit{Diff}^+(\mathbf{R}^2)$ be such that every fixed point is isolated and has index ≤ 0. Then the following statements hold:

i) *For every x, as n goes to $\pm\infty$, either $f^n(x)$ goes to a fixed point or $\|f^n(x)\| \to \infty$.*

ii) *For each direct saddle p, every homoclinic contact is a fixed point different from p and each branch at p is homeomorphic to $[0, \infty)$.*

iii) *If the only fixed point is a direct saddle p, then there are no homoclinic contacts and every branch of $W^s(p)$ and of $W^u(p)$ is unbounded.*

Partial results: Weak global saddles

Definition

We say that 0 is a weak global saddle if

(i) 0 is a hyperbolic saddle,

(ii) there are no homoclinic contacts and the curves $W^s(0,f)$ and $W^u(0,f)$ are unbounded. Moreover, ~~$W^s \cup W^u$ **separates the plane into exactly four connected components**~~.

(iii) for all $p \notin W^s(0,f) \cup W^u(0,f) \cup \{0\}$ both $\|f^n(p)\| \to \infty$ and $\|f^{-n}(p)\| \to \infty$ as n goes to ∞.

Proposition (Alarcón-Castro-Labouriau, to appear)

Let $f \in Diff(\mathbf{R}^2)$ be such that the only fixed point p is a hyperbolic saddle. Suppose that one of the following holds:

- *0 is a direct saddle*
- $Fix(f^2) = \{0\}$

Then, the origin is a weak global saddle.

Partial results

Remark

Observe that the curves W^s and W^u for a weak global saddle can be badly behaved, so the set

$$(\mathbf{R}^2 \setminus \overline{W^s \cup W^u}) \cup (W^s \cup W^u)$$

may have lots of connected components.

Proposition (Alarcón-Ribón, work in progress)

Consider the simply connected subset

$$\Delta = cc((\mathbf{R}^2 \setminus \overline{W^s \cup W^u}) \cup (W^s \cup W^u), 0)$$

then 0 is a topological global saddle for $f|_\Delta$.

Weak global saddles in forced Lienard Equations

Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t), \qquad (2)$$

where $f, g : \mathbf{R} \to \mathbf{R}$ are locally Lipschitz maps of class C^1.
Suppose in addition that the following assumptions holds:

(A1) $p : \mathbf{R} \to \mathbf{R}$ is continuous and periodic with minimal period $T > 0$;

(A2) f is bounded and $f(x) \geq 0$, for all $x \in \mathbf{R}$;

(A3) g is a strictly decreasing homeomorphism;

(A4) $\exists c, d \geq 0$ such that $|g(x)| \leq c + d\,|x|$, for all $x \in \mathbf{R}$.

Theorem (Alarcón-Castro-Labouriau, to appear)

The unique $T-$ periodic solution of (2) is a weak global saddle for the associated Poincaré map.

Proof: Weak saddles in forced Lienard Equations

Theorem (Campos and Torres, 1999)

There exists exactly one $T-$periodic solution of (2).

Poincaré map

- Assumptions on (2) imply that the Poincaré map P is an orientation preserving diffeomorphism of the plain.
- Theorem of Campos and Torres implies that $\mathrm{Fix}(P) = \{p\}$.
- Assumptions (A2) and (A3) imply that the unique fixed point of P is a direct saddle.

Global saddles conjugated to the linear saddle

Theorem (Kerékjártó, 1934)

An orientation preserving homeomorphism of the plane h is conjugated to the topological translation $T(x, y) = (x + 1, y)$ *if and only if, for all* $p \in \mathbf{R}^2$, $\|h^n(p)\| \to +\infty$, *as* $|n|$ *goes to* $+\infty$, *and the convergence is uniform on compact sets.*

Remark

Bonatti and Kolev presented in 1997 an alternative proof considering the quotient space given by the orbits.

Theorem (Alarcón-Ribón, in progress)

Let $f \in \mathit{Diff}^+(\mathbf{R}^2)$ *be such that the only fixed point 0 is a direct saddle. Suppose that the following hold:*

- *both* W^s *and* W^u *are closed*
- *for all* $p \notin W^s \cup W^u \cup \{0\}$, $\|f^n(p)\| \to +\infty$, *as* $|n|$ *goes to* $+\infty$, *and the convergence is uniform on compact sets in* $\mathbf{R}^2 \setminus W^s \cup W^u \cup \{0\}$.

Then, the map f is globally conjugated to the linear saddle.

Global saddles in forced Lienard Equations

Consider the differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t),$$

where $f, g : \mathbf{R} \to \mathbf{R}$ are locally Lipschitz maps of class C^1.
Suppose in addition that the following assumptions holds:

(A1) $p : \mathbf{R} \to \mathbf{R}$ is continuous and periodic with minimal period $T > 0$;

(A2) f is bounded and $f(x) \geq 0$, for all $x \in \mathbf{R}$;

(A3) g is a strictly decreasing homeomorphism;

(A4) $\exists c, d \geq 0$ such that $|g(x)| \leq c + d\,|x|$, for all $x \in \mathbf{R}$.

We hope to prove that

The Poincaré map associated to the Linear equation (plus perhaps some extra condition) is globally conjugated to the linear saddle.

Let Maths be with you :-)