$SU(1,1)$ Covariant integral quantization of the unit disk

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Contents

Introduction

- SU$(1,1)$, the two-fold covering of $SO_0(1,2)$, can be interpreted as
  - a dynamical group for the $(1+1)$ Anti-de-Sitter as a space-time and
  - the unit disk $\mathcal{D}$ as a phase space, i.e. the set of free motions with a fixed "energy" at rest
- Therefore, a comprehensive program of quantization of the unit disk as a phase space by using the covariant integral quantization is appealing.
- This program has to different parts
  - To analyze the different choices for the weight function which is fundamental ingredient of the IQ
  - To study as well the semi-classical return to the original phase space through the construction of the lower (Lieb) or covariant (Berezin) symbols, which have a true probabilistic interpretation when the OQ is based on normalized positive operator valued measures (POVM).
- We present here the first part of the project

The unit disk as a Kählerian manifold

$$\mathcal{D} = \{z \in \mathbb{C}\,,\ |z| < 1\}$$

two-dimensional Kählerian manifold equipped with the (Poincaré) metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1-|z|^2)^2}\,.$$

The corresponding surface element is given by the two-form:

$$\Omega = \frac{i}{2}\,\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1-|z|^2)^2} = \frac{\mathrm{d}(\Re z)\,\mathrm{d}(\Im z)}{(1-|z|^2)^2} \equiv \mu(\mathrm{d}^2 z)\,.$$

These quantities are both issued from a *Kählerian potential* $\mathcal{K}_{\mathcal{D}}$:

$$\mathcal{K}_{\mathcal{D}}(z,\bar{z}) : -\pi^{-1}(1-|z|^2)^{-2}\,,$$

$$\mathrm{d}s^2 = \frac{1}{2}\,\frac{\partial^2}{\partial z\,\partial\bar{z}}\,\ln\mathcal{K}_{\mathcal{D}}(z,\bar{z})\,dz\,d\bar{z}\,,$$

$$\mu(\mathrm{d}^2 z) = \frac{i}{4}\,\frac{\partial^2}{\partial z\,\partial\bar{z}}\,\ln\mathcal{K}_{\mathcal{D}}(z,\bar{z})\,dz\wedge d\bar{z}\,.$$

The Lie group SU(1, 1)

$$\mathrm{SU}(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \ \alpha, \beta \in \mathbb{C}\,,\ \det g = |\alpha|^2 - |\beta|^2 = 1 \right\}$$

Three basis elements as elements of the Lie algebra $\mathfrak{su}(1,1)$ are chosen as

$$N_0 = \frac{\mathrm{i}}{2}\sigma_3\,, \qquad N_1 = \frac{1}{2}\sigma_1\,, \qquad N_2 = \frac{1}{2}\sigma_2\,,$$

(σ_i Pauli matrices) with the commutation relations

$$[N_0, N_1] = N_2\,, \quad [N_0, N_2] = -N_1\,, \quad [N_1, N_2] = -N_0\,.$$

Cartan factorization of $SU(1,1) = PH$ is associated with the

$$i_{ph}\ :\ g \quad \xrightarrow{\text{(Cartan) involution}} \quad (g^\dagger)^{-1}\,.$$

$H = \{g \in SU(1,1)\ \text{ s.t. } i_{ph}(g) = g\} = U(1)$ (maximal compact subgroup)

$P = \{g \in SU(1,1)\ \text{ s.t. } i_{ph}(g) = g^{-1}\}$ (Hermitian matrices)

The factorization $\mathrm{SU}(1,1) = PH$ reads explicitly

$$\mathrm{SU}(1,1) \ni g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = p(z)\, h(\theta)\,,$$

with

$$P\ : \qquad p(z) = \begin{pmatrix} \delta & \delta z \\ \delta \bar{z} & \delta \end{pmatrix}\,, \quad z = \beta \bar{\alpha}^{-1}\,, \quad \delta = (1 - |z|^2)^{-1/2}\,,$$

and

$$H\ : \qquad h(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\,, \quad \theta = 2\arg\alpha\,, \quad 0 \leq \theta < 4\pi\,.$$

The unit disk as a coset of $SU(1,1)$

$$z \in \mathcal{D} \xrightarrow{\text{bundle section}} p(z) \in P \quad \Rightarrow \quad \mathcal{D} \equiv SU(1,1)/H$$

Note that

$$p^2 = gg^\dagger\,, \qquad (p(z))^{-1} = p(-z)\,.$$

Haar measure (normalized for the H part) on SU(1, 1) from Cartan decomp.

$$d_{haar}(g) = \frac{\mathrm{d}^2 z}{(1-|z|^2)^2}\,\frac{\mathrm{d}\theta}{2\pi}$$

Cartan factor. allows to make SU(1, 1) act on $\mathcal{D}$ through a left action on P

$$g : p(z) \mapsto p(z') \quad \text{defined by} \quad g\,p(z) = p(z')\,h'\,.$$

Hence $\mathcal{D}$ is invariant under Möbius transformations

$$\mathcal{D} \ni z \mapsto z' \equiv g \cdot z = \frac{\alpha\, z + \beta}{\bar{\beta}\, z + \bar{\alpha}} \in \mathcal{D}$$

Inversely

$$z = g^{-1} \cdot z' = \frac{\bar{\alpha}\, z' - \beta}{-\bar{\beta}\, z' + \alpha}$$

SU(1, 1) leaves invariant the boundary $\mathbb{S}^1 \simeq U(1)$ of $\mathcal{D}$ under the above transf. The invariance of $\mathcal{D}$ under Möbius transf. also holds for metric quantities issued from the invariant Kählerian potential $\mathcal{K}_{\mathcal{D}}$

The unit disk as a AdS phase space

Since the unit disk is Kählerian, it is symplectic and so can be given a phase space structure and interpretation.
The 2-form Ω determines the Poisson bracket

$$\{f,g\} = \frac{\mathrm{i}}{2}(1-|z|^2)^2 \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}}\frac{\partial g}{\partial z}\right)$$

Basic observables generating the $\mathrm{SU}(1,1)$ symmetry on this classical level:

$$\mathcal{D} \ni z \mapsto \; k_0(z) = \frac{1+|z|^2}{1-|z|^2}\,,\; k_1(z) = \mathrm{i}\frac{z-\bar{z}}{1-|z|^2}\,,\; k_2(z) = \frac{z+\bar{z}}{1-|z|^2}\,.$$

They obey the Poisson commutation rules

$$\{k_0,k_1\} = k_2\,, \quad \{k_0,k_2\} = -k_1\,, \quad \{k_1,k_2\} = -k_0\,,$$

which are consistent with the Lie commutators of $SU(1,1)$.
Also the two combinations

$$k_+ = k_2 - \mathrm{i}k_1 = \frac{2z}{1-|z|^2}\,, \quad k_- = k_2 + \mathrm{i}k_1 = \frac{2\bar{z}}{1-|z|^2}$$

SU(1, 1) UIR

For a given $\eta > 1/2$, consider the Fock-Bargmann Hilbert space $\mathcal{FB}_\eta$ of all analytic functions $f(z)$ on $\mathcal{D}$ that are square integrable with scalar product

$$\langle f_1 | f_2 \rangle = \frac{2\eta - 1}{2\pi} \int_{\mathcal{D}} \overline{f_1(z)}\, f_2(z)\, (1 - |z|^2)^{2\eta - 2}\, \mathrm{d}^2 z$$

An orthonormal basis is made of powers of z suitably normalized:

$$e_n(z) \equiv \sqrt{\frac{(2\eta)_n}{n!}}\, z^n\,, \qquad n \in \mathbb{N},$$

where $(2\eta)_n := \Gamma(2\eta + n)/\Gamma(2\eta)$ Pochhammer symbol.

For $\eta \in (1/2, \infty)$ one defines the UIR U^η of the universal covering of $SU(1,1)$ on $\mathcal{FB}_\eta$

$$(U^\eta(g)\, f)\,(z) = (-\bar{\beta}\, z + \alpha)^{-2\eta}\, f\left(\frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha}\right)$$

where $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$

Matrix elements of $U^\eta(g)$

with respect to the orthonormal basis $\{e_n\}$ are

$$U^\eta_{nn'}(g) = \langle e_n | U^\eta(g) | e_{n'}\rangle = \left(\frac{n_>!\,\Gamma(2\eta+n_>)}{n_<!\,\Gamma(2\eta+n_<)}\right)^{1/2} \alpha^{-2\eta-n_>}\,\bar{\alpha}^{n_<} \times$$
$$\times \frac{(\gamma(\beta,\bar{\beta}))^{n_>-n_<}}{(n_>-n_<)!}\,{}_2F_1\left(-n_<\,,\; n_>+2\eta\,;\; n_>-n_<+1\,;\; \frac{|\beta|^2}{|\alpha|^2}\right)\,,$$

where

$$\gamma(\beta,\bar{\beta}) = \begin{cases} -\beta & n_> = n' \\ \bar{\beta} & n_> = n \end{cases}\,, \quad n_{\gtrless} = \begin{cases} \max \\ \min \end{cases} (n,n') \geq 0\,.$$

Since $\frac{|\beta|^2}{|\alpha|^2} = 1 - \frac{1}{|\alpha|^2}$, this expression is given in terms of Jacobi polynomials

$$U^\eta_{nn'}(g) = \left(\frac{n_<!\,\Gamma(2\eta+n_>)}{n_>!\,\Gamma(2\eta+n_<)}\right)^{1/2} \alpha^{-2\eta-n_>}\,\bar{\alpha}^{n_<} \times$$
$$\times \frac{(\gamma(\beta,\bar{\beta}))^{n_>-n_<}}{\sqrt{(n_>-n_<)!}}\,P_{n_<}^{(n_>-n_<\,,\,2\eta-1)}\left(\frac{1-|\beta|^2}{1+|\beta|^2}\right)\,.$$

Diagonal elements

$$U^{\eta}_{nn}(g) = \alpha^{-2\eta-n}\,\bar{\alpha}^{n}\,{}_2F_1\left(-n\,,\,n+2\eta\,;\,1\,;\,\frac{|\beta|^2}{|\alpha|^2}\right)\,.$$

For the elements $g = h(\theta)$ in U(1), we have

$$U^{\eta}_{nn'}(h(\theta)) = \delta_{nn'}\,e^{-\mathrm{i}(\eta+n)\theta}\,,$$

whereas for the elements $g = p(z)$ in P,

$$U^{\eta}_{nn'}(p(z)) = \left(\frac{n_>!\,\Gamma(2\eta+n_>)}{n_<!\,\Gamma(2\eta+n_<)}\right)^{1/2}(1-|z|^2)^{\eta}\,\frac{|z|^{n_>-n_<}}{(n_>-n_<)!}\,e^{\mathrm{i}(n'-n)\phi}\times$$
$$\times(\mathrm{sgn}(n-n'))^{n-n'}\,{}_2F_1\left(-n_<\,,\;n_>+2\eta\,;n_>-n_<+1\,;\,|z|^2\right)\,,$$

with $z = |z|e^{\mathrm{i}\phi}$, and if $n = n'$,

$$U^{\eta}_{nn}(p(z)) = (1-|z|^2)^{\eta}\,{}_2F_1\left(-n\,,n+2\eta\,;1\,;\,|z|^2\right)$$
$$= (1-|z|^2)^{\eta}\,P^{(0\,,\,2\eta-1)}_{n}\left(1-2|z|^2\right)\,.$$

Covariant integral quantization with UIR of a Lie group

U : UIR of Lie group G in a Hilbert space $\mathcal{H}$; M : bounded operator on $\mathcal{H}$

Family of "displaced" version of M under the action of the $U(g)$'s

$$\{M(g) := U(g)\, M\, U^{\dagger}(g)\,,\ g \in G\}$$

POVM

$$R = \int_G M(g)\, d_{haar}(g)\,,$$

From the left-invariance of $d_{haar}(g)$

$$RU(g) = U(g)R \quad \forall g \in G \quad \xrightarrow{\text{Schur's Lemma}} \quad R = c_M\, I$$

i.e., we have the "resolution" of the unity up to a constant c_M

$$c_M = \int_G \mathrm{tr}\left(\rho_0\, M(g)\right)\, d_{haar}(g)$$

The unit trace positive operator ρ_0 is chosen to make the integral convergent.

If c_M is finite and positive, the true resolution of the identity follows:

$$\int_G M(g)\, d\nu(g) = I\,, \qquad d\nu(g) := d_{haar}(g)/c_M\,.$$

CIQ with sq. integrable UIR

Let us consider a UIR U for which M is an "admissible" operator, i.e.,

$$c_M = \int_G d_{haar}(g)\, \mathrm{tr}[\rho_0 M(g)] < \infty \qquad \text{for a certain } \rho_0$$

or for square-integrable UIR U for which $M = \rho$ is an admissible density oper.

$$c(\rho) = \int_G d_{haar}(g)\, \|\rho U(g)\|^2_{\mathcal{HS}} < \infty\,,$$

where $\|A\|_{\mathcal{HS}} = \mathrm{tr}\left(AA^\dagger\right)$ is the Hilbert-Schmidt norm. Then

$$M(g) = U(g) M U^\dagger(g) \;\; \forall g \in G \quad \Rightarrow \quad \text{resolution of the identity}$$

This allows

covariant integral quantization of complex-valued funct. on G

$$f \mapsto A_f = \int_G f(g)\, M(g)\, d\nu(g)\,.$$

Covariance means that

$$U(g)A_f U^{\dagger}(g) = A_{U_{reg}(g)f}\,,$$

where

$$(U_{reg}(g)f)(g') := f(g^{-1}g')\,, \quad f \in L^2(G, d_{haar}(g))$$

CIQ through Cartan decomp.

$K \subset G$: maximal compact subgroup $\subset$ a connected semi-simple Lie group,
The homogeneous coset space

$$P = G/K$$

is symmetric, diffeomorphic to a Euclidean space, and the Cartan decomp.

$$G = P\,K \Leftrightarrow \forall g \in G \, \exists \, p \in P \, , \, k \in K \, , \; g = pk = kp' \, , \; p' = k^{-1}pk \, ,$$

holds. The action of G on P

$$g : p \mapsto g \cdot p = p' \quad \text{carried out through the left action} \;\; gp = p'k'$$

Hence, the subgroup K is the stabilizer of a point in P.
From the maximal abelian subgroup of $A \subset P$ we get the decomposition

$$G = KAK \, .$$

Since G is unimodular, the Haar measure L and R-invariant and factorizes

$$d_{haar}(g) = d\mu_P(p)\, d_{haar}(k)$$

with the invariance property for $d\mu_P$

$$d\mu_P(kpk') = d\mu_P(p) \quad \forall \, k, k' \in K \, .$$

Integral quantization of Cartan symmetric space

Let $a \mapsto w(a)$ be a function on A left and right K-invariant, i.e.

$$w(kak') = w(a), \qquad \forall\, k, k' \in K\,.$$

and U a UIR of G. Suppose that w allows to define

$$M^w := \int_P d\mu_P(p)\, w(a(p))\, U(p)\,.$$

as an operator bounded

Displaced version of M^w under the action of U

$$M^w(g) = U(g)\, M^w\, U^\dagger(g)\,,$$

and supposing that the Haar measure on K is normalized, one derives that

$$I = \int_G \frac{d_{haar}(g)}{C^w}\, M^w(g) \;\Rightarrow\; \int_P \frac{d\mu_P(p)}{C^w}\, M^w(p) = I\,,$$

where the density operator ρ_0 has been suitably chosen.

with

$$C^w = \int_P \mathrm{d}\mu_P(p)\, \mathrm{tr}\left(\rho_0\, M^w(p)\right) \,,$$

Integral quantization of functions (or distributions)

$$f(p) \mapsto A_f = \int_P \frac{d\mu_P(p)}{C^w}\, f(p)\, M^w(p)\,.$$

In relation with $SU(1,1)$:

$$K = U(1)\,, \qquad A = \left\{ \begin{pmatrix} \delta & \delta|z| \\ \delta|z| & \delta \end{pmatrix},\quad z \in \mathcal{D} \right\}$$

with $\delta = \sqrt{1-|z|}$

$SU(1,1)$ integral quantizations for the unit disk

Let us pick $\eta > 1/2$ and a (weight) function

$$[0,1] \ni u \equiv |z|^2 \mapsto w(u)$$

and define the operator

$$M^{w;\eta} := \int_{\mathcal{D}} \frac{d^2 z}{(1-|z|^2)}\, w(|z|^2)\, U^{\eta}(p(z))\,,$$

and its transported versions

$$M^{w;\eta}(p(z)) = U^{\eta}(p(z)\, M^{w;\eta}\, U^{\eta}(p(-z))$$

if it is properly defined by the above integral, is expected to resolve the unity with respect to a measure on $\mathcal{D}$ proportional to $\mathrm{d}^2 z/(1-|z|^2)^2$

$$I = \frac{1}{C^w} \int_{\mathcal{D}} \frac{\mathrm{d}^2 z}{(1-|z|^2)^2}\, M^{w;\eta}(p(z))\,.$$

Moreover, one imposes $M^{w;\eta}$, through an appropriate factor in the expression of the weight w, to have unit trace in the case it is traceclass.

Isotropy of the weight funct. ($w(|z|^2) = w(u)$) $\Rightarrow$ $M^{w;\eta}$ is diagonal in the basis $\{e_n\}$

$$M^{w;\eta}_{nn'} = \delta_{nn'}\, 2^{1-\eta}\,\pi \int_{-1}^{1} \mathrm{d}v\, w\left(\frac{1-v}{2}\right)\,(1+v)^{\eta-2}\, P_n^{(0,2\eta-1)}(v)\,, \quad v = 1-2u$$

w will have a constant factor such as the unit trace condition holds

$$\mathrm{tr}\, M^{w;\eta} = 2^{2-\eta}\,\pi \sum_{n=0}^{\infty} \int_{-1}^{1} \mathrm{d}v\, w\left(\frac{1-v}{2}\right)\,(1+v)^{\eta-2}\, P_n^{(0,2\eta-1)}(v) = 1\,.$$

One then computes C^w with the simplest $\rho_0 = |e_0\rangle\langle e_0|$

$$C^w = \int_{\mathcal{D}} \frac{d^2 z}{(1-|z|^2)^2}\,\langle e_0|M^{w;\eta}(p(z)|e_0\rangle\,.$$

This yields the relation of C^w with $\mathrm{tr}\, M^{w;\eta}$

$$C^w = \frac{\pi}{2\eta-1}\,\mathrm{tr}\, M^{w;\eta} = \frac{\pi}{2\eta-1}\,.$$

Finally the resolution of the identity holds with the measure

$$I = \frac{2\eta-1}{\pi} \int_{\mathcal{D}} \frac{\mathrm{d}^2 z}{(1-|z|^2)^2}\, M^{w;\eta}(p(z))\,.$$

Particular family of weight functions w

$$w(u) := \frac{1}{D^{w_s}}\,(1-u)^s = \frac{1}{D^{w_s}}\,\left(\frac{1+v}{2}\right)^s \equiv w_s(u), \quad u = |z|$$

we have for the matrix elements of $M^{w_s;\eta}$

$$\begin{aligned}
M^{w_s;\eta}_{nn'} &= \delta_{nn'}\,\frac{1}{D^{w_s}}\,\frac{2\pi}{s+\eta-1}\,\frac{\Gamma(\eta+s)\Gamma(s-\eta)}{\Gamma(\eta+s+n)\Gamma(s-\eta-n)} \\
&= (-1)^n\,\delta_{nn'}\,\frac{1}{D^{w_s}}\,\frac{2\pi}{s+\eta-1}\,\frac{\Gamma(\eta+s)\Gamma(\eta-s+n+1)}{\Gamma(\eta+s+n)\Gamma(\eta-s+1)} \qquad \text{for } s\neq\eta+1\,, \\
&= \delta_{nn'}\,\delta_{n0}\,\frac{1}{D^{w_s}}\,\frac{\pi}{\eta} \qquad \text{for } s=\eta+1\,.
\end{aligned}$$

The unit trace condition imposes

$$\begin{aligned}
D^{w_s} &= \frac{2\pi}{s+\eta-1}\,{}_2F_1(\eta-s+1,1;\eta+s;-1) \qquad \text{for } s\neq\eta+1\,, \\
&= \frac{\pi}{\eta} \qquad \text{for } s=\eta+1\,.
\end{aligned}$$

From

$$M^{w_s;\eta}_{nn'} = \delta_{nn'} \, \frac{1}{D^{w_s}} \, \frac{2\pi}{s+\eta-1} \, \frac{\Gamma(\eta+s)\Gamma(s-\eta)}{\Gamma(\eta+s+n)\Gamma(s-\eta-n)}$$

we infer that $M^{w_s;\eta}$ is a density operator for all

$$s = \eta + p\,, \qquad p = 1, 2, \ldots$$

which corresponds to a finite rank $= p$ operator

Particular cases

1) $s = \eta + 1$

$$M_{nn'}^{w_s;\eta} = \delta_{nn'}\,\delta_{n0}\,\frac{1}{D^{w_s}}\,\frac{\pi}{\eta}$$

corresponds to Perelomov SU(1, 1) coherent states (with $\bar{z}$ instead of z)

$$M^{w_{\eta+1};\eta} = |e_0\rangle\langle e_0|\,, \qquad M^{w_{\eta+1};\eta}(p(\bar{z})) = |z;\eta\rangle\langle z;\eta|\,.$$

2) $s = 1/2$

Parity oper. (bounded but not traceclass) obtained fixing $D^{w_{1/2}} = 4\pi/(2\eta - 1)$

$$\mathcal{P} := \sum_{n=0}^{\infty}(-1)^n|e_n\rangle\langle e_n| = \frac{2\eta-1}{4\pi}\int_{\mathcal{D}}\frac{\mathrm{d}^2 z}{(1-|z|^2)^{3/2}}\,U^{\eta}(p(z))\,.$$

3) $s \neq \eta + 1$

$$M^{w_s;\eta}_{nn'} = (-1)^n \, \delta_{nn'} \, \frac{1}{D^{w_s}} \, \frac{2\pi}{s+\eta-1} \, \frac{\Gamma(\eta+s)\Gamma(\eta-s+n+1)}{\Gamma(\eta+s+n)\Gamma(\eta-s+1)}$$

can be written as

$$M^{w_s;\eta}_{nn'} = (-1)^n \, \delta_{nn'} \, \frac{1}{D^{w_s}} \, \frac{2\pi}{s+\eta-1} \, \frac{(\eta-s+1)_n}{(\eta+s)_n}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol.

$$\begin{aligned} M^{w_s;\eta} &= \sum_{n,n'} M^{w_s;\eta}_{nn'} \, |e_n\rangle\langle e_n| \\ &= \frac{1}{D^{w_s}} \, \frac{2\pi}{s+\eta-1} \, \sum_n (-1)^n \, \frac{(\eta-s+1)_n}{(\eta+s)_n} \, |e_n\rangle\langle e_n| \end{aligned}$$

$$M^{w_{\eta+1};\eta}(p(\bar{z})) = \frac{1}{D^{w_s}} \, \frac{2\pi}{s+\eta-1} \, \sum_n (-1)^n \, \frac{(\eta-s+1)_n}{(\eta+s)_n} \, |z;\eta;n\rangle\langle z;\eta;n| \, .$$

$SU(1,1)$ Coherent states

Let η be a real parameter such that $\eta > 1/2$ and let us equip the unit disk with a measure proportional to (4):

$$\mu_\eta(d^2z) := \frac{2\eta - 1}{\pi}\,\mu(d^2z) = \frac{2\eta - 1}{\pi}\,\frac{d^2z}{(1-|z|^2)^2}\,.$$

Consider now the Hilbert space $L^2_\eta = L^2(\mathcal{D}, \mu_\eta)$ of all functions $f(z,\bar{z})$ on $\mathcal{D}$ that are square integrable with respect to μ_η. Select all functions of the form

$$\phi(z,\bar{z}) = (1-|z|^2)^\eta g(\bar{z})\,,$$

where $g(z)$ is holomorphic on $\mathcal{D}$. The closure of the linear span of such functions is a Hilbert subspace of L^2_η. An orthonormal basis of it is given by the countable set of functions

$$\phi_n(z,\bar{z}) \equiv \sqrt{\frac{(2\eta)_n}{n!}}(1-|z|^2)^\eta \bar{z}^n \qquad n \in \mathbb{N},$$

where $(2\eta)_n = \frac{\Gamma(2\eta+n)}{\Gamma(2\eta)}$ is the Pochhammer symbol.

Coherent states

$$|z;\eta\rangle := \sum_{n=0}^{\infty} \overline{\phi_n(z,\bar{z})}|e_n\rangle = (1-|z|^2)^{\eta} \sum_{n=0}^{\infty} \sqrt{\frac{(2\eta)_n}{n!}} z^n |e_n\rangle \,.$$

By construction these states are normalized and solve the identity $I_{\mathcal{H}}$ in $\mathcal{H}$:

$$\langle z;\eta|z;\eta\rangle = 1\,, \qquad \int_{\mathcal{D}} \mu_{\eta}(d^2 z)\, |z;\eta\rangle\langle z;\eta| = I_{\mathcal{H}}\,.$$

$$\langle z';\eta|z;\eta\rangle = (1-|z|^2)^{\eta}\,(1-\bar{z}'\,z)^{-2\eta}\,(1-|z'|^2)^{\eta}\,.$$

It is also a reproducing kernel, for which the Hilbert subspace is a Fock-Bargmann space.

Coherent states as a transported vacuum

Group theoretical content of the coherent states $|z;\eta\rangle$

$$\forall z \in \mathcal{D} \quad \Leftrightarrow \quad p(\bar{z}) \in SU(1,1)$$

Let us now apply to the lowest state $|e_0\rangle$ the operators of the representation U^η restricted to the set P of such matrices, and expand the "transported" state in terms of the Fock-Bargmann basis:

$$U^\eta(p(\bar{z}))\,|e_0\rangle = \sum_{n=0}^{\infty} U^\eta_{n0}(p(\bar{z}))\,|e_n\rangle = (1-|z|^2)^\eta \sum_{n=0}^{\infty} \sqrt{\frac{(2\eta)_n}{n!}}\, z^n |e_n\rangle = |z;\eta\rangle.$$

Discretely indexed set of families of coherent states

$$|z;\eta;m\rangle := U^\eta(p(\bar{z}))\,|e_m\rangle = \sum_{n=0}^{\infty} U^\eta_{nm}(p(\bar{z}))\,|e_n\rangle$$

General results

We now establish general formulas for the quantization issued from a weight function $w(u)$ yielding the operator $M^{w;\eta}$

$$M^{w;\eta} = \int_{\mathcal{D}} \frac{d^2 z}{(1-|z|^2)^2}\, w(|z|^2)\, U^{\eta}(p(z))\,,$$

Let us first establish the nature of $M^{w;\eta}$ as an integral operator in the Fock-Bargmann Hilbert space $\mathcal{FB}_\eta$. Thus the action on ϕ in $\mathcal{FB}_\eta$ of $M^{w;\eta}$

$$(M^{w;\eta}\phi)(z) = \frac{2\eta-1}{2\pi} \int_{\mathcal{D}} \mathrm{d}^2 z' \,(1-|z'|^2)^{2\eta-2}\, \mathcal{M}^{w;\eta}(z,z')\, \phi(z')\,,$$

where the kernel $\mathcal{M}^{w;\eta}$ is given by

$$\mathcal{M}^{w;\eta}(z,z') = \frac{2\pi}{2\eta-1} \frac{(1-|z|^2|z'|^2)}{|1+\bar{z}z'|^2} \left(\frac{|1+\bar{z}z'|}{1+\bar{z}z'}\right)^{2\eta} \frac{\left(1-|z|^2|\right)^{-\eta}}{\left(1-|z'|^2|\right)^{\eta}}\, w(z,z'),$$

where

$$w(z,z') = w\left(\left|\frac{(\delta')^{-2} z - (\delta)^{-2} z'}{(1-|z|^2|z'|^2)}\right|^2\right)\,,$$

with $\delta = \left(1-|z|^2\right)^{-1/2}$, $\delta' = \left(1-|z'|^2\right)^{-1/2}$.

If we impose $M^{w;\eta}$ to be symmetric oper., we get the resulting symmetry of the kernel

$$\mathcal{M}^{w;\eta}(z,z') = \overline{\mathcal{M}^{w;\eta}(z',z)}$$

When $M^{w;\eta}$ is a pure state $|\psi\rangle\langle\psi|$ (as it is for the construction of coherent states) the corresponding weight function is given by

$$w(z,z') = \frac{2\eta-1}{2\pi}\,\frac{|1+\bar{z}z'|^2}{(1-|z|^2|z'|^2)}\,\left(\frac{1+\bar{z}z'}{|1+\bar{z}z'|}\right)^{2\eta} \times \psi(z)\,\overline{\psi(z')}\left(1-|z|^2|\right)^{\eta}\left(1-|z'|^2|\right)^{\eta}.$$

In particular, we have the relation for the modulus of ψ

$$w(z,z) = \frac{2\eta-1}{2\pi}\,(1+|z|^2)\left(1-|z|^2\right)^{2\eta-1}|\psi(z)|^2\,.$$

which is immediate from

$$\mathcal{M}^{w;\eta}(z,z') = \psi(z)\,\overline{\psi(z')}$$

IQ for $SU(1,1)$ and $M = M^{w;\eta}$ of functions

$$f \mapsto A_f^w = \frac{1}{C^w} \int_{\mathcal{D}} \frac{d^2 z}{(1-|z|^2)^2}\, f(z)\, M^{w;\eta}(p(z))\,.$$

By construction, the quantization map is covariant with respect to the unitary action U^η of $SU(1,1)$, i.e.

$$U^\eta(g_0)_f\, A_f^w\, U^{\eta\,\dagger}(g_0) = A^w_{\mathfrak{U}(g_0)f}\,,$$

$\mathfrak{U}$ being the left regular representation of $SU(1,1)$.
The action on $\phi \in \mathcal{FB}_\eta$ of t A_f^w defined by the IQ map is given by

$$(A_f^w \phi)(z) = \frac{2\eta - 1}{2\pi} \int_{\mathcal{D}} dz'^{\,2}\, (1-|z'|^2)^{2\eta-2}\, \mathcal{A}_f^w(z,z')\, \phi(z')\,,$$

where the kernel $\mathcal{A}_f^w$ is defined as

$$\mathcal{A}_f^w(z,z') = \frac{1}{C^{M^\omega}} \int_{\mathcal{D}} ds^2\, (1-|s|^2)^{2\eta-2}\, f(s)\, \frac{(1-\bar{z}s)^{-2\eta}}{(1-z'\bar{s})^{2\eta}}\, M^{w;\eta}\left(\frac{z-s}{1-z\bar{s}}, \frac{z'-s}{1-z'\bar{s}}\right)$$

Semiclassical portraits

Given a weight funct. w and a symmetric unit trace operator $M^{w,\eta}$ we can define the semiclassical or lower symbol o an operator A in $\mathcal{H}$

$$A \quad \to \quad \check{A}(z) := \mathrm{Tr}\,[A\, U^{\eta}(p(z))\, M^{w,\eta}\, U^{\eta\,\dagger}(p(z))] = \mathrm{Tr}\,[A\, M^{w,\eta}(p(z))]$$

If we have the operator A_f^w then

$$f(z) \quad \to \quad \check{f}(z) \equiv \check{A}_f^w(z) = \frac{1}{C^{M^{\omega}}} \int_{\mathcal{D}} \frac{ds^2}{(1-|s|^2)^2}\, f\left(\frac{s-z}{1-\bar{z}s}\right) \mathrm{Tr}\,[M^{w,\eta}(p(s))\, M^{w,\eta}]$$

Conclusions

- We study the possibilities that the $SU(1,1)$ IQ offers beyond the Perelomov CS $|z;\eta\rangle$
- The different choices for w can avoid singularities that can appear with the standard and usual choice
- This is the case for some applications of IQ on cosmology,
- e This part of the study has been focused to the construction of the weight function $w(|z|)$ on the disk $\mathcal{D}$
- The disk can be seen as the phase space of a particle on the anti de Sitter (1,1) space
- The next step will be to study the quantization of the fundamental observables that generate the $SU(1,1)$ symmetry at the classical level

$$k_0(z) = \frac{1+|z|^2}{1-|z|^2}\,, \quad k_1(z) = \mathrm{i}\frac{z-\bar{z}}{1-|z|^2}\,, \quad k_2(z) = \frac{z+\bar{z}}{1-|z|^2}\,.$$