Coherent spaces, Boolean rings and their applications

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- Dirac contour representation
- coherent spaces, coherent projectors (represented by a finite set of complex numbers)
- Boolean ring of finite sets of complex numbers (Stone's formalism)
- Application to classical gates
- Boolean ring of coherent spaces
- quantum CNOT gates with coherent states
- Discussion

Dirac contour representation

- h: harmonic oscillator Hilbert space
  $|A\rangle$:coherent state ($A \in \mathbb{C}$)
  $H(A) = \{|A\rangle\}$ one-dimensional subspace
  $\Pi(A) = |A\rangle\langle A|$ projector

- $|s\rangle = \sum s_N |N\rangle$:

$$|s\rangle \rightarrow s_k(z) = \sum_{N=0}^{\infty} \frac{s_N z^N}{\sqrt{N!}} \text{ (Bargmann)}$$

$$\langle s| \rightarrow s_b(z) = \sum_{N=0}^{\infty} \frac{s_N^* \sqrt{N!}}{z^{N+1}}.$$

  $s_b(z)$ converges $|z| > R$ (R depends on state)
  scalar product

$$\langle f|s\rangle = \oint_C \frac{dz}{2\pi i} f_b(z) s_k(z) = \sum_{N=0}^{\infty} f_N^* s_N.$$

  C contour enclosing singularities of $f_b(z)$.

- $s_k(z)$ and $s_b(z)$ related as

$$\oint_C \frac{dz}{2\pi i} s_b(z) \exp(\zeta^* z) = [s_k(\zeta)]^*$$

$$s_b(z) = \frac{1}{z} \int_0^{\infty} dt \exp(-t) \left[s_k\left(\frac{t}{z^*}\right)\right]^*.$$

- number state $|N\rangle$:

$$|N\rangle \;\to\; s_k(z) = \frac{z^N}{\sqrt{N!}}$$
$$\langle N| \;\to\; s_b(z) = \frac{\sqrt{N!}}{z^{N+1}}.$$

- coherent state $|A\rangle$:

$$|A\rangle \;\to\; s_k(z) = \exp\left(Az - \frac{1}{2}|A|^2\right)$$
$$\langle A| \;\to\; s_b(z) = \frac{\exp(-\frac{1}{2}|A|^2)}{z - A^*}; \quad |z| > |A|$$

  pole at A^*
  for convergence $|z| > |A| \leftrightarrow C$ should enclose pole

- in this paper:
  finite sums, of bra functions with **finite** set of poles each

  S_1, S_2 sets of poles of $s_b(z), f_b(z)$
  set of poles of $\lambda_1 s_b(z) + \lambda_2 f_b(z)$: $S_1 \cup S_2$ (or subset)

  not true in infinite sums

- operator $\Theta = \sum \Theta_{MN} |M\rangle\langle N|$

$$\Theta(z_1, z_2) = \sum \Theta_{MN} \sqrt{\frac{N!}{M!}} \frac{z_1^M}{z_2^{N+1}},$$

acts on ket states as

$$\Theta|s\rangle \; \rightarrow \; \oint_C \frac{d\zeta}{2\pi i} \Theta(z, \zeta) s_k(\zeta) = \sum \Theta_{MN} s_N |M\rangle,$$

acts on bra states as

$$\langle s|\Theta \; \rightarrow \; \oint_C \frac{d\zeta}{2\pi i} s_b(\zeta) \Theta(\zeta, z) = \sum \Theta_{MN} s_M^* \langle N|.$$

-

$$\mathbf{1} \; \rightarrow \; \Theta(z_1, z_2) = \frac{1}{z_2 - z_1}; \quad |z_2| > |z_1|$$

$$\Pi(A) = |A\rangle\langle A| \rightarrow \Theta(z_1, z_2) = \frac{\exp\left(A z_1 - |A|^2\right)}{z_2 - A^*}; |z_2| > |A|$$

$$|A_1\rangle\langle A_2| \; \rightarrow \; \Theta(z_1, z_2) = \frac{\exp\left(A_1 z_1 - \frac{1}{2}|A_1|^2 - \frac{1}{2}|A_2|^2\right)}{z_2 - A_2^*}$$

pole at A^*

Vourdas, Bishop, PRA53, (1996) R205
Vourdas, Bishop, JPA 31 (1998)8563

coherent spaces, coherent projectors

- **finite number of coherent states are linearly independent**

- $S = \{A_1, ..., A_n\}$ finite set of complex numbers
  $S^* = \{A_1^*, ..., A_n^*\}$
  coherent space

  $$H(S) = H(A_1, ..., A_n) = H(A_1) \vee ... \vee H(A_n)$$

  contains superpositions $\lambda_1 |A_1\rangle + ... + \lambda_n |A_n\rangle$
  in the Dirac contour repr:

  $$f_k(z) = \lambda_1 \exp(A_1 z) + ... + \lambda_n \exp(A_n z)$$
  $$f_b(z) = \frac{\lambda_1}{z - A_1^*} + ... + \frac{\lambda_n}{z - A_n^*}$$

  set of poles S^*

- Gram-Schmidt orthogonalization algorithm

  $$\Pi(A_1, A_2) = \Pi(A_1) + \frac{\Pi^\perp(A_1)\Pi(A_2)\Pi^\perp(A_1)}{\mathrm{Tr}[\Pi^\perp(A_1)\Pi(A_2)]}$$

  $\Pi^\perp(A_1) = \mathbf{1} - \Pi(A_1)$, and

  $$\Pi(A_1, ..., A_n) = \Pi(A_1, ..., A_{n-1}) + \frac{\Pi^\perp(A_1, ..., A_{n-1})\Pi(A_n)\Pi^\perp(A_1, ..., A_{n-1})}{\mathrm{Tr}[\Pi^\perp(A_1, ..., A_{n-1})\Pi(A_n)]}$$

- coherent projectors $\Pi(A_1,...,A_n)$ (rank n):
  resolution of the identity:

$$\frac{1}{n}\int_{\mathbb{C}}\frac{d^2A}{\pi}\Pi(A,A+d_2,...,A+d_n)=\mathbf{1}$$

  fixed $d_2,...,d_n$

- closure property:
  under displacement trs, and
  under time evolution with the Hamiltonian $a^\dagger a$
  they transform into projectors of same type:

$$D(z)\Pi(A_1,...,A_n)[D(z)]^\dagger=\Pi(A_1+z,...,A_n+z)$$

$$\exp(ita^\dagger a)\Pi(A_1,...,A_n)\exp(-ita^\dagger a)$$
$$=\Pi[A_1\exp(it),...,A_n\exp(it)]$$

- Coherent states eigenstates of a: $a^\ell\Pi(A_1)=A_1^\ell\Pi(A_1)$
  analogue

$$\Pi^\perp(A_1,...,A_i)a^\ell\Pi(A_1,...,A_i)=0$$
$$\mathrm{Tr}[a^\ell\Pi(A_1,...,A_n)]=\sum_{i=1}^{n}A_i^\ell.$$

- many subsets of coherent states: **total sets** analogue here:

  set of subspaces $\{h_i\}$ of h: total, if there is no state in h, which is orthogonal to all h_i

  use theory of growth and density of zeros of analytic functions:

  1. A set of coherent subspaces which is **uncountably** infinite, is a total set.
  2. $\{H(S_i)\}$ **countably** infinite set of coherent subspaces with $S_i = \{A_{i1}, ..., A_{ik_i}\}$. Relabel the A_{ij} as A_n (lexicographic order).
     - A_n converges to $A \rightarrow$ $\{H(S_i)\}$ total set of coherent subspaces.
     - $|A_n|$ diverges, and its density greater than $(2, 1) \rightarrow$ $\{H(S_i)\}$ is a total set of coherent subspaces.

- $$\mathrm{Tr}[x\Pi(A_1,...,A_n)] = \sqrt{2}\Re\left(\sum A_i\right)$$
  $$\mathrm{Tr}[p\Pi(A_1,...,A_n)] = \sqrt{2}\Im\left(\sum A_i\right)$$

- If $S = \{A_1, ..., A_n\}$, the $\Pi(S)$

  $$\Pi(A_1,...,A_n) = \sum_{j,k} G_{jk}(S)|A_j\rangle\langle A_k|$$

  $$\Theta(z_1,z_2) = \sum_{j,k} G_{jk}(S)\frac{\exp\left(A_j z_1 - \frac{1}{2}|A_j|^2 - \frac{1}{2}|A_k|^2\right)}{z_2 - A_k^*}$$

  $$|z_2| > \max(|A_1|,...,|A_n|).$$

  $G(S)$ inverse (exists) of the $n \times n$ matrix $g(S)$:

  $$g_{jk}(S) = \langle A_j|A_k\rangle = \exp\left(A_j^* A_k - \frac{1}{2}|A_j|^2 - \frac{1}{2}|A_k|^2\right).$$

  set of poles of $\Theta(z_1, z_2)$ is S^*.

  a finite set of complex numbers (poles), defines uniquely a coherent projector/coherent space

Boolean ring of finite sets of complex numbers

- Stone's formalism:
  set theory/Boolean algebra $\leftrightarrow$ rings $\leftrightarrow$ topology

- $\mathcal{L}$ set of all **finite** subsets of $\mathbb{C}$
  For $S_1, S_2 \in \mathcal{L}$ define partial order, disjunction, conjunction:

$$\begin{array}{rcl} S_1 \prec S_2 & \leftrightarrow & S_1 \subset S_2 \\ S_1 \vee S_2 & \leftrightarrow & S_1 \cup S_2 \text{ (logical } \mathbf{OR}) \\ S_1 \wedge S_2 & \leftrightarrow & S_1 \cap S_2 \text{ (logical } \mathbf{AND}) \end{array}$$

  $\mathcal{L}$ has 0 (least element): the empty set $\emptyset$
  $\mathcal{L}$ does not have 1 (greatest element): $\mathbb{C} \notin \mathcal{L}$

  cannot define complements ($\mathbb{C} \setminus S \notin \mathcal{L}$)
  complements important for logical **NOT**

  $\mathcal{L}$ is a distributive lattice
  $\mathcal{L}$ is **not** a Boolean algebra

- principal ideal $\mathcal{I}(R)$: all subsets of a finite set R
  $\mathcal{I}(R)$ has 1 (the set R)
  complements $\overline{S} = R \setminus S$ defined
  $\mathcal{I}(R)$ Boolean algebra

- **translate set theory into a ring (ordinary arithmetic)**
  in the set $\mathcal{L}$

  $$S_1 + S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1);\ \text{(logical } \mathbf{XOR}\text{)}$$
  $$S_1 \cdot S_2 = S_1 \cap S_2;\ \text{(logical } \mathbf{AND}\text{)}$$

  OR, **AND** replaced by **XOR**, **AND**

  $$S_1 \cup S_2 = S_1 + S_2 + (S_1 \cdot S_2).$$

  only finite sums and finite products

  $\mathcal{L}$ is closed under multiplication and addition
  addition, multipl: commutative, associative
  distributivity holds:

  $$S_1 \cdot (S_2 + S_3) = (S_1 \cdot S_2) + (S_1 \cdot S_3)$$

  $\emptyset$ is additive zero
  additive inverse of a set, is itself ($S_1 = -S_1$)

  $$S_1 + \emptyset = S_1; \quad S_1 + S_1 = \emptyset; \quad S_1 \cdot S_1 = S_1.$$

  multiplication is idempotent
  $\mathcal{L}$ commutative ring (without identity) and with idempotent multiplication

  ring with idempotent multiplication is commutative, and is called Boolean ring
  Boolean rings with identity: Boolean algebras
  $\mathcal{L}$ has no 1, it is not a Boolean algebra

- ideal $\mathcal{I}(R)$ within lattice theory, are also ideal within ring theory
  $\mathcal{I}(R)$: Boolean ring with R as 1: Boolean algebra
  complement of $S \in \mathcal{I}(R)$, is $\overline{S} = S + R = R \setminus S$

Application to classical gates

- some classical gates:
  OR, AND, XOR ($[\mathcal{I}(R)]^2 \to \mathcal{I}(R)$; not bijective)
  NOT ($\mathcal{I}(R) \leftrightarrow \mathcal{I}(R)$, bijective):

$$
\begin{aligned}
\mathcal{M}_{\text{OR}}(S_1, S_2) &= S_1 + S_2 + S_1 \cdot S_2 = S_1 \vee S_2 \\
\mathcal{M}_{\text{AND}}(S_1, S_2) &= S_1 \cdot S_2 = S_1 \wedge S_2 \\
\mathcal{M}_{\text{XOR}}(S_1, S_2) &= S_1 + S_2 \\
\mathcal{M}_{\text{NOT}}(S_1) &= R + S_1 = \overline{S}_1 = R \setminus S_1
\end{aligned}
$$

example: $R = \{A_1\}$ (binary)
notation:

$$\emptyset \;\to\; 0; \quad \{A_1\} \;\to\; 1$$

in	OR	AND	XOR
(0,0)	0	0	0
(1,0)	1	0	1
(0,1)	1	0	1
(1,1)	1	1	0

example: $R = \{A_1, A_2\}$ (2^2-ary)

$$\mathcal{M}_{\text{OR}}(\{1\},\{1,2\}) = \{1,2\}; \quad \mathcal{M}_{\text{OR}}(\{1\},\emptyset) = \{1\},$$
$$\mathcal{M}_{\text{AND}}(\{1\},\{1,2\}) = \{1\}; \quad \mathcal{M}_{\text{AND}}(\{1\},\emptyset) = \emptyset,$$
$$\mathcal{M}_{\text{XOR}}(\{1\},\{1,2\}) = \{2\}; \quad \mathcal{M}_{\text{XOR}}(\{1\},\emptyset) = \{1\},$$
$$\mathcal{M}_{\text{NOT}}(\{1\}) = \{2\}; \quad \mathcal{M}_{\text{NOT}}(\emptyset) = \{1,2\},$$

notation:

$\emptyset \rightarrow$ 0; $\{A_1\} \rightarrow$ 1; $\{A_2\} \rightarrow$ 2; $\{A_1, A_2\} \rightarrow$ 3

in	OR	AND	XOR
(0,0)	0	0	0
(1,0)	1	0	1
(2,0)	2	0	2
(3,0)	3	0	3
(0,1)	1	0	1
(1,1)	1	1	0
(2,1)	3	0	3
(3,1)	3	1	2
(0,2)	2	0	2
(1,2)	3	0	3
(2,2)	2	2	0
(3,2)	3	2	1
(0,3)	3	0	3
(1,3)	3	1	2
(2,3)	3	2	1
(3,3)	3	3	0

- reversible classical gates (bijective map):
  - CNOT gate (from $[\mathcal{I}(R)]^2$ to itself):

    $$\mathcal{M}(S_1, S_2) = (S_1, S_1 + S_2)$$

    S_1, S_2 control and target inputs

  - reversible

    $$\mathcal{M}(S_1, S_1 + S_2) = (S_1, S_2)$$

  - for fixed control input S_1:
    bijective map: target input $\rightarrow$ target output

  - example: $R = \{A_1\}$ (binary)
    notation:

    $$\emptyset \;\rightarrow\; 0; \quad \{A_1\} \;\rightarrow\; 1$$

in	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
out	$(0,0)$	$(0,1)$	$(1,1)$	$(1,0)$

  - also 2^n-ary case

Boolean ring of coherent spaces

- h_1, h_2 subspaces of h:

$$h_1 \vee h_2 = \text{span}(h_1 \cup h_2);\ \mathbf{OR}$$
$$h_1 \wedge h_2 = h_1 \cap h_2\ \ \mathbf{AND}$$

quantum **OR** $\neq$ classical **OR**

- $\mathcal{L}_{\text{coh}}$: set of coherent subspaces $H(S)$, S **finite**
$H(\emptyset) = \mathcal{O}$ (zero vector): element of $\mathcal{L}_{\text{coh}}$

$$H(S_1) \vee H(S_2) = H(S_1 \cup S_2)$$
$$H(S_1) \wedge H(S_2) = H(S_1 \cap S_2).$$

finite number of disjunctions and conjunctions
$\mathcal{L}_{\text{coh}}$ is closed under these operations

The $\mathcal{O}$ is the zero in this lattice.
no 1 in this lattice (h does not belong to $\mathcal{L}_{\text{coh}}$)

$\mathcal{L}_{\text{coh}}$, is a distributive lattice.
$\mathcal{L}_{\text{coh}} \simeq \mathcal{L}$; not Boolean algebra

- $\mathcal{L}_{\text{coh}}$ distributive sublattice of the Birkhoff-von Neumann (**non-distributive**) lattice

- principal ideal of all coherent subspaces of the coherent space $H(R)$:

$$\mathcal{I}_{\text{coh}}(R) = \{H(S) \in \mathcal{L}_{\text{coh}} \mid S \subset R\}.$$

Boolean algebra (1 is $H(R)$)

- In $\mathcal{L}_{\text{coh}}$ define

  $$H(S_1 + S_2) = H(S_1 \setminus S_2) \vee H(S_2 \setminus S_1).$$

  This is the logical XOR operation
  $H(S_1 + S_2)$ contains the vectors in $H(S_1 \setminus S_2)$, $H(S_2 \setminus S_1)$, and superpositions
  quantum **XOR** $\neq$ classical **XOR**

  $$H(S_1) + H(S_2) = H(S_1 + S_2)$$
  $$H(S_1) \cdot H(S_2) = H(S_1 \cdot S_2) = H(S_1) \wedge H(S_2).$$

  Only finite sums and finite products

- $\mathcal{L}_{\text{coh}}$ commutative ring (without identity) and with idempotent multiplication:

  $$H(S_1) \cdot H(S_1) = H(S_1)$$

  $\mathcal{L}_{\text{coh}}$ Boolean ring, isomorphic to $\mathcal{L}$.

  $$H(S_1) \vee H(S_2) = H(S_1) + H(S_2) + [H(S_1) \cdot H(S_2)]$$
  $$H(S_1) + \mathcal{O} = H(S_1)$$
  $$H(S_1) + H(S_1) = \mathcal{O}; \quad H(S_1) = -H(S_1)$$

quantum CNOT gates with coherent states

general quantum CNOT gate

$$|e\rangle \otimes |t\rangle \;\rightarrow\; |e\rangle \otimes (\mathcal{U}_T|t\rangle); \quad |e\rangle \in h_1; \quad |t\rangle \in h_2$$

$|e\rangle$ control input; $|t\rangle$ target input
previous work: orthogonal states; coherent states far from each other (almost orthogonal)

- quantum CNOT gate with coherent states
  binary example
  $H_A(A_1, A_2) \otimes H_B(B_1, B_2)$

- input

$$[\alpha_1|A_1\rangle + \alpha_2|A_2\rangle] \otimes [\beta_1|B_1\rangle + \beta_2|B_2\rangle]$$

- transformation:

$$U = \gamma_{1A}E_1(A_1, A_2) \otimes \mathcal{U}_{1T} + \gamma_{2A}E_2(A_1, A_2) \otimes \mathcal{U}_{2T}$$
$$\mathcal{U}_{1T} = g(B_1, B_2); \quad \mathcal{U}_{2T} = g(B_1, B_2) - 2\gamma_{2B}E_2(B_1, B_2)$$
$$[\mathcal{U}_{1T}, \mathcal{U}_{2T}] = 0.$$

$\gamma_{jA}, E_j(A_1, A_2)$ and $\gamma_{jB}, E_j(B_1, B_2)$: eigenvalues and eigenprojectors of $g(A_1, A_2)$ and $g(B_1, B_2)$

Discussion

- coherent spaces, coherent projectors
  defined uniquely by finite set of complex numbers
  (poles)

  language: Dirac contour representation

- finite sets of complex numbers
  distributive lattice
  Boolean ring (Stone's formalism)
  classical gates

- coherent spaces distributive lattice
  Boolean ring
  quantum CNOT gates with coherent states

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