The Anatomy of Coherent States

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Sign

... quanto maior é a diferença, maior será igualdade,
e quanto maior é a igualdade, maior a diferença será ...

José Saramago

The classical coherent states (CSs in short) are[1] simply

$$c_z \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} h_n, \quad z \in \mathbb{C},$$

with h_n's being the Hermite functions they are settled in $\mathscr{L}^2(\mathbb{R})$.
Later on [2]

$$c_z \overset{\text{def}}{=} \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} h_n, \quad z \in \mathbb{C},$$

They bear different names like canonical, standard, orthodox etc.

though the most explicative way is to call them *Gaussian* coherent states

[1] Schrödinger 1926
[2] Glauber, Klauder, Sudarshan 1963

(a) they are normalised;
(b) they are continuos functions in z;
(c) they are never orthogonal, more $\langle c_z, c_w \rangle = \mathrm{e}^{|z-w|^2}$

$$\vdots$$

and the most celebrated [3]

(d) $I = \int_{\mathbb{C}} |c_z\rangle\langle c_z| \frac{\mathrm{d}^2 z}{\pi}$.

It is called *resolution of the identity*, sometimes referred to as (over)completeness. Some authors even see in it the reproducing kernel property which, if provided with mathematical correctness, in this case is nothing but the trivial (or rather "idem per idem") side of the RKHS story.

[3] I use Dirac notation sporadically

According to Glauber (1963) there are three ways of constructing coherent states:

(A) the (normalized) eigenvectors of the annihilation operator;

(B) the orbit of the vacuum under a (square integrable) unitary group;

(C) minimising the Heisenberg uncertainty relation.

It turns out that for the Gaussian CSs these three lead to the same provided in (B) the group is that of the *displacement* operator.

Keeping in mind the postulates (a) – (d) and following any of the directives (A), (B), (C) MPs try to find generalisations of CSs.

The most popular way (and sometimes dangerous if not performed with enough care) is that guided by (B).

The case (A) may lead to somehow interesting results though it is not too often in use.

The case (C), a bit aside, is discussed from time to time.

This refers to the very first definition

$$c_z \stackrel{\text{def}}{=} \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} h_n, \quad z \in \mathbb{C},$$

with replacements

- ${h_n}'s \mapsto$ arbitrary orthonormal basic vectors in some Hilbert space (in which the would-be coherent states have to resides);
- $n! \mapsto x_0 \dots x_n$ in the way which ensures convergence;
- $\exp(-|z|^2/2) \mapsto$ a suitable normalisation factor.

Everything happens in the presence of a measure which makes the resolution of identity possible. It comes in practice from a Stieltjes moment sequence allowing one[4] (*sic!*) of the representing measure to be rotationally invariant.

[4] MPs are reluctant to accept existence of more than one

Further examples of coherent states are mushrooming nowadays either among MPs or people at the frontiers.
Always existence of measure is presupposed.

In conclusion

existence of more than one measure or a lack of any may be painful for them.

My point is to propose a cure.

Some tools follow first.

- the tool;
- Horzela Szafraniec approach including Segal-Bargmann design;
- assorted examples;
- more properties of H-Sz coherent states.

A set X granted

Given a Hilbert space $\mathscr{H}$ of complex functions on X and a function $K : X \times X \mapsto \mathbb{C}$ (called a *kernel*). The couple $(\mathscr{H}, K)$ is called a *reproducing kernel* one if

- $K_x \stackrel{\text{def}}{=} K(\,\cdot\,, x) \in \mathscr{H}, \quad x \in X;$
- $f(x) = \langle f, K_x \rangle, \quad f \in \mathscr{H},\ x \in X.$

There is a list of properties coming out of this definition and each of them may work for construction the couple.

Given a sequence $(\Phi_n)_{n=0}^{\infty}$ of complex functions on X such that

$$\sum_n |\Phi_n(x)|^2 < +\infty, \quad x \in X.$$

Then

$$K(x,y) \stackrel{\text{def}}{=} \sum_n \Phi_n(x)\overline{\Phi_n(y)}, \quad x,y \in X$$

is a positive definite kernel and, consequently, due to Aronszajn's construction for instance, it uniquely determines its partner $\mathscr{H}_K$ so that they both together constitute a reproducing kernel couple.

This may serve as a very practical way of constructing RKHS.

Whether $(\Phi_n)_{n=0}^{\infty}$ is a basis in $\mathscr{H}_K$ or not is discussed next.

1^{o}. For any $\xi = (\xi_n)_n$ in ℓ^2, the series

$$\sum\nolimits_n \xi_n \Phi_n(x)$$

is absolutely convergent for any x, the function

$$f_\xi : x \to \sum\nolimits_n \xi_n \Phi_n(x)$$

is in $\mathscr{H}$ with $\|f_\xi\| \leq \|\xi\|_{\ell^2}$; moreover $\sum_n \xi_\alpha \Phi_n$ is convergent in $\mathscr{H}$ to f_ξ. In particular $\sum_n \overline{\Phi_n(x)}\Phi_n$ is convergent in $\mathscr{H}$ to K_x, the functions Φ_n are in $\mathscr{H}$ and $\|\Phi_n\| \leq 1$.

2^{o} The sequence $(\Phi_n)_n$ is always complete in $\mathscr{H}$[5]. TFCAE

(i) $\xi \in \ell^2$ and $\sum_n \xi_n \Phi_n(x) = 0$ for every x yields $\xi = 0$;

(ii) the sequence $(\Phi_n)_n$ is orthonormal in $\mathscr{H}$.

This connects pointwise and norm convergence in RKHS.

[5] Notice completeness of $(\Phi_n)_n$ appears *a posteriori*.

A sample definition

If X is a (subset of a) topological space (think of $\mathbb{C}$ or $\mathbb{C}^d$) and there is a positive measure μ on the completion $\overline{X}$ of X such that $\mathscr{H}$ is embedded isometrically in "a natural way" in $\mathscr{L}^2(\mu)$ we say that $(\mathscr{H}, K)$ is *integrable*.
If $\operatorname{supp} \mu \subset X$ then the embedding in a "natural way" is just the inclusion; this happens more often.

WARNING. There are non-integrable RKHSpaces.

TABULA RASA

RKHS

X a set, $\boldsymbol{\Phi} \stackrel{\text{def}}{=} (\Phi_n)_n$ sequence of functions on X such that

$$\sum_n |\Phi_n(x)|^2 < +\infty, \quad x \in X.$$

K a kernel on X got via Zaremba's, $\mathscr{H}_K$ its RKHS.

The space for CSs

$\mathscr{H}$ is a Hilbert space of the same dimension as that of $\mathscr{H}_K$.

Fix an orthonormal basis $\boldsymbol{e} \stackrel{\text{def}}{=} (e_n)_n$ in $\mathscr{H}$.

And that's all ! They are the only initial parameters.

They are at hand

$$c_x \stackrel{\text{def}}{=} \sum_n \Phi_n(x) e_n \quad x \in X.$$

That's it !

Notice

they may be normalised if there is any need, just because they belong to $\mathscr{H}$. They inherit selected properties from those of K (like continuity, differentiability and so on).

The transform

$$Bh \stackrel{\text{def}}{=} \sum_n \langle h, e_n \rangle_{\mathscr{H}} \Phi_n, \quad h \in \mathscr{H}$$

is well defined and maps $\mathscr{H} \mapsto \mathscr{H}_K$ (notice $Be_n = \Phi_n$). In general it is a contraction [6] with a dense range.
Due to the reproducing property we have

$$(Bh)(x) = \langle Bh, K_x \rangle_{\mathscr{H}_K} = \sum_n \langle h, e_n \rangle_{\mathscr{H}} \Phi_n(x).$$

Moreover if $(\Phi_n)_n$ is ONB then

$$\langle Bh, Bg \rangle_{\mathscr{H}_K} = \langle h, g \rangle_{\mathscr{H}}$$

hence B is unitary and the corresponding Segal-Bargmann space turns out to be the whole of $\mathscr{H}_K$.

[6] The reproducing kernel property and the RKHS test have to be used for that.

A. Horzela and F.H. Szafraniec, A measure free approach to coherent states, *J. Phys. A: Math. Theor.* **45** (2012) 244018

A. Horzela and F.H. Szafraniec, A measure free approach to coherent states refined, in Proceedings of the XXIX International Colloquium on Group-Theoretical Methods in Physics 2012.

F.H. Szafraniec, The reproducing kernel property and its space: the basics, in *Operator Theory vol. 1, D. Alpay Ed., 3–30, SpringerReference*, **2015**.

F.H. Szafraniec, The reproducing kernel property and its space: more or less standard examples of applications, in *Operator Theory vol. 1*, D. Alpay Ed., 31–58, SpringerReference, **2015**.

K. Górska, A. Horzela, F.H. Szafraniec, Squeezing of arbitrary order: the ups and downs. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 470 (2014), no. 2172, 20140205, 21 pp.

Simultaneously,

- the transform B is unitary;
- the family $(c_x)_x$ of coherent states is complete

all this if $(\Phi_n)_n$ is an basis in $\mathscr{H}_K$. In the Gaussian case, that is when

$$\Phi_n = \frac{z^n}{\sqrt{n!}} \text{ or } K(z,w) = \mathrm{e}^{z\overline{w}}$$

and

$$e_n = h_n \text{ are Hermite functions}$$

the transform B becomes precisely that of Segal-Bargmann.

Resolution of the identity for malcontents [a]

[a] Dirac's notation appears here occasionally, just to refer to MP and make its community content. μ is any admissible measure

$$\begin{aligned}\langle h| \int_X |x\rangle_{\mathscr{H}} \langle x| \, \mu(\mathrm{d}x) |g\rangle_{\mathscr{H}} &= \int_X \langle h|x\rangle_{\mathscr{H}} \langle x|g\rangle_{\mathscr{H}} \, \mu(\mathrm{d}x) \\ &= \int_X \overline{(Bh)(x)} (Bg)(x) \, \mu(\mathrm{d}x) \\ &= \langle Bh|Bg\rangle_{\mathscr{L}^2(\mu)} \\ &= \langle Bh|Bg\rangle_{\mathscr{H}_K} = \langle h|g\rangle_{\mathscr{H}}\end{aligned}$$

Nothing lost

This is what may happen

1° $\mathscr{H}_K$ is integrable and the measure is unique;

2° $\mathscr{H}_K$ is integrable and the measure is not unique;

3° $\mathscr{H}_K$ is not integrable.

Suppose a sequence $(k_n)_n$ of non-negative numbers is given such that $X = \{z \in \mathbb{C}\colon\ \sum_n k_n |z|^{2n}\} < +\infty \neq \varnothing$. This set is rotationally invariant and so is the kernel

$$K(x,y) \overset{\text{def}}{=} \sum\nolimits_n k_n z^n \overline{w}^n, \quad z,w \in X.$$

Because K is PD, we got $\mathscr{H}_K$. Suppose for a while $\mathscr{H}_K$ is integrable and write

$$\begin{aligned} k_{m+n}^{-2} &= \Big(\int_X |z^{m+n}|^2 \mu(\mathrm{d}z)\Big)^2 = \Big(\int_X |z^{2m}||z^{2n}|\mu(\mathrm{d}z)\Big)^2 \\ &\leqslant k_{2m}^{-1}k_{2n}^{-1}\int_X |k_{2m}^{\frac12} z^{2m}|^2 \mu(\mathrm{d}z)\int_X |k_{2n}^{\frac12} z^{2n}|^2\mu(\mathrm{d}z) = k_{2m}^{-1}k_{2n}^{-1}. \end{aligned}$$

What we have got from the above heuristic reasoning is

$$k_{m+n}^{-2} \leqslant k_{2m}^{-1} k_{2n}^{-1}$$

which is just logarithmic convexity of $(k_n^{-1})_n$.

Important

Logarithmic convexity is a necessary condition for integrability. Manipulating $(k_n^{-1})_n$ may lead at once to examples of non-integrable $\mathscr{H}_K$; this is the first attempt toward the problem.

Logarithmic convexity ensures $\lim_n k_n^{-1/n}$ to exist which contributes to $X \neq \varnothing$.

1^{o}, 2^{o} and 3^{o} are merging here

Start from a measure ν representing a Stieltjes moment sequence $(a_n)_{n=0}^{\infty}$, that is

$$a_n = \int_0^{+\infty} x^n \nu(\mathrm{d}x), \quad n = 0, 1, \dots$$

and define the rotationally invariant measure μ on $\mathbb{C}$

$$\mu(\sigma) \stackrel{\text{def}}{=} (2\pi)^{-1} \int_0^{2\pi} \int_0^{+\infty} \chi_\sigma(r\, \mathrm{e}^{\mathrm{i}\, t}) m(\mathrm{d}r)\, \mathrm{d}t, \quad \sigma \text{ Borel subset of } \mathbb{C}$$

makes

$$\int_{\mathbb{C}} z^n \mu(\mathrm{d}z) = \int_0^{+\infty} x^{n/2} \nu(\mathrm{d}x)$$

The way back is possible due to the transport of measure by $\mathbb{C} \ni z \mapsto |z| \in [0, +\infty)$.

The monomials $\varPhi_n \stackrel{\text{def}}{=} k_n^{1/2} Z^N$ besides being orthonormal in $\mathscr{H}_k$ are orthonormal in $\mathscr{L}^{(\mu)}$ as well. Hence $\mathscr{H}_K$ is integrable.

now only 1^{o} and 2^{o} merge
└integrability enters the scene

In other words, we have the formula

$$\mu = \big(\mu \circ \phi^{-1} \otimes (2\pi\, \mathrm{d}m)\big) \circ j^{-1}, \quad j(t,\xi) = \sqrt{t}\xi$$

which points up rotational invariance of μ.

Warning

If the Stieltjes moment problem for $(a_n)_{n=0}^{\infty}$ is indeterminate, besides rotationally invariant μ's, non-rotationally invariant measures exist too - despite the fact the kernel itself is rotationally invariant.
This never happens when ν is determinate, in particular if it has a compact support.

As an opening let me suggests q-moments: determinate if $0 < q \leqslant 1$ and indeterminate if $q > 1$.

Segal-Bargmann

Here $\varPhi_n = \frac{1}{\sqrt{n!}} Z^n$ and

$$K(z,w) = \mathrm{e}^{z\overline{w}}, \quad z,w \in \mathbb{C}$$

with $\mathscr{H} \subset \mathscr{L}^2(\mathbb{C}, (\pi)^{1/2}\, \mathrm{e}^{-|z|^2}\, \mathrm{d}z)$.

Bergman

For this $\varPhi_n = \sqrt{n+1} Z^n$ and

$$K(z,w) = \frac{1}{1 - z\overline{w}}, \quad z,w \in \mathbb{D}.$$

In this case $\mathscr{H}_K \subset \mathscr{L}^2(\mathrm{d}z)$.

Szegő $\Longleftrightarrow$ Hardy

With $\Phi_n = Z^n$ the kernel (Szegő) is

$$K(z,w) = \frac{1}{1 - z\overline{w}}, \quad z, w \in \mathbb{D}$$

Here $\mathscr{H}_K$ "$\subset$" $\mathscr{L}^2(\mathbb{T}, \mathrm{d}m)$ (Hardy), "$\subset$" come from Fatou one way and Poisson the other. This means we have to do with integrability in an extended sense.
The resulting CSs may be considered X as those on the unit circle (*sic!*).

van Eijndhoven–Meyers orthogonality, my favourite

$\mathscr{X}_\alpha$, $0 < \alpha < 1$, the Hilbert space of entire functions f

$$\int_{\mathbb{R}^2} |f(x + \mathrm{i}\, y)|^2 \exp\big[\alpha x^2 - \frac{1}{\alpha} y^2\big]\, \mathrm{d}x \mathrm{d}y < +\infty$$

With H_n standing for Hermite polynomials and

$$b_n(A) = \frac{\pi\sqrt{\alpha}}{1 - A}\left(2\frac{1+\alpha}{1-\alpha}\right)^n n!$$

the functions

$$\Phi_n^\alpha(z) = b_n(\alpha)^{-1/2}\, \mathrm{e}^{-z^2/2}\, H_n(z), \quad z \in \mathbb{C},$$

form an orthonormal basis in $\mathscr{X}_\alpha$.

Suppose now ν represents an indeterminate Hamburger moment sequence. If Φ_n stands now for the polynomials orthonormal with respect to ν, then

$$\sum_n |\Phi_n(z)|^2 < +\infty, \quad z \in \mathbb{C}$$

which gives a rise to H-Sz CSs over $\mathbb{C}$ via Zaremba.

The space $\mathscr{H}_K$ is integrable and ν is one of its representing measure.
Notice integrability of $\mathscr{H}_K$ is over $\mathbb{R}$.

Among the spaces I would like to touch upon one can find:

- de Branges spaces including Paley-Wiener;
- Rovnyak-de Branges
- and so forth.

Consider

$$\Phi_n(z) \stackrel{\text{def}}{=} \frac{n!}{z(z+1)\cdots(z+n)}$$

Then

$$K(z,w) = \sum_{n=0}^{\infty} \frac{n!}{z(z+1)\cdots(z+n)} \frac{n!}{\overline{w}(\overline{w}+1)\cdots(\overline{w}+n)}$$
$$= {}_3F_2\left(1,1,1; z+1, \overline{w}+1; 1\right), \quad \Re z, \Re w > 1/2.$$

and the space $\mathscr{H}_K$ is not integrable over $X = \{(z,w)\colon \ \Re z, \Re w > 1/2\}$ though H-Sz coherent states make sense.

K.F. Klopfenstein, A note on Hilbert spaces of factorial functions, *Indiana Univ. Math. J.* **25** (1976) 1073-1081.

$\mathscr{H}_K = \{\sum_n \xi_n \Phi_n\colon \ (\xi_n)_n \in \ell^2\}$ is the Segal-Bargmann type space of holomorphic functions on $\{(z,w)\colon \ \Re z, \Re w > 1/2\}$.

Besides the aforesaid equality $\langle Bh, Bg\rangle_{\mathscr{H}_K} = \langle h, g\rangle_{\mathscr{H}}$ we have

$$\langle Bc_x, Bc_y\rangle_{\mathscr{H}_K} = K(x,y) = \langle K_y, K_x\rangle_{\mathscr{H}_K}.$$

This suggest to proceed as follows [7]

$$BC_{\boldsymbol{e}}c_x = B\sum_n \overline{\Phi(x)}e_n = \sum_n \overline{\Phi(x)}\Phi_n = K_x$$
$$C_{\boldsymbol{\Phi}}Bc_x = B\sum_n \overline{\Phi(x)}e_n = \sum_n \overline{\Phi(x)}\Phi_n = K_x$$

which results in a kind of triviality

$BC_{\boldsymbol{e}} = C_{\boldsymbol{\Phi}}B$, consequently $C_{\boldsymbol{\Phi}}BC_{\boldsymbol{e}} = B$ and $C_{\boldsymbol{e}}B^*C_{\boldsymbol{\Phi}} = B^*$.

All this allows us to recapture the kernel of the Segal-Bargmann transform.

[7] $C_{\boldsymbol{e}}$ and $C_{\boldsymbol{\Phi}}$ are complex conjugations defined by the respective bases

Recall the definition $c_x \overset{\text{def}}{=} \sum_n \Phi_n(x) e_n$. What does happen if one has another representation of c_x's

$$c_x = \sum_n \Phi'_n(x) e'_n \ ?$$

Due to the Parseval equality

$$\langle c_x, c_y \rangle_{\mathscr{H}} = \sum \Phi_n(x)\overline{\Phi_n(y)} = K(x,y).$$

This applied to primed data gives

$$K(x,y) = K'(x,y).$$

Conclusion

The kernel K (and its RKHS) becomes
a kind of invariant for coherent states.

Fix X and $\mathscr{H}$.

If is any set in $\mathscr{H}$ which is complete there, then for an arbitrary ONB $(e_n)_n$ in $\mathscr{H}$ one gets

$$c_x = \sum\nolimits_n \langle c_x, e_n \rangle e_n, \quad x \in X$$

and

$$\sum\nolimits_n |\langle c_x, e_n \rangle e_n|^2 < +\infty.$$

Now the H-Sz procedure may be initiated with $\Phi_n(x) = \langle c_x, e_n \rangle$ and, in particular, the findings of the previous slide apply.

Because $\{c_x\}_{x \in X}$ is complete $(\Phi_n)_n$ is automatically an ONB in the corresponding $\mathscr{H}_K$.

For $\alpha > 0$ define the *Charlier sequences* $(c_n^{(\alpha)})_{n=0}^{\infty}$,

$$\tilde{c}_n^{(\alpha)}(x) \stackrel{\text{def}}{=} \alpha^{-\frac{n}{2}}(n!)^{-\frac{1}{2}} C_n^{(\alpha)}(x) \mathrm{e}^{-\frac{\alpha}{2}} \alpha^{\frac{x}{2}} \begin{cases} (x!)^{-\frac{1}{2}}, & \text{for } x \geqslant 0 \\ 1 & \text{for } x < 0 \end{cases};$$

$$c_n^{(\alpha)} \stackrel{\text{def}}{=} \tilde{c}_n^{(\alpha)}|_{\mathbb{N}}, \quad n = 0, 1, \ldots$$

The kernel is

$$K(m,n) = \sum\nolimits_{k,l} c_k^{(\alpha)}(m)\overline{c_l^{(\alpha)}(n)} = \delta_{m,n} = \sum\nolimits_{k,l} \delta_{k,m}\delta_{l,n}.$$

Notice the two ONB's determine different CSs though they come from the same kernel.

By the way,

$$\operatorname{lin}(c_n^{(\alpha)})_n \cap \operatorname{lin}(c_n^{(\alpha)})_n = \varnothing.$$

In medicine two branches of anatomy are conventionally distinguished

- topographical anatomy;
- pathological anatomy.

The question which fragments of my talk belong to a respective branch is what I would like to leave with each of You. None of those is void here.

Il en est ainsi parce que (14) entraîne (15).

Cela posé il résulte immédiatement du théorème exprimé par la formule (7) du Chapitre précédent que l'on aura:

$$F(A, B) = -\sum_{j=1}^{\infty} u_j(A)\, u_j(B). \tag{34}$$

§ 15. Le problème de la détermination des fonctions fondamentales[1]) définies au Chapitre II équivaut à la recherche de toutes les solutions du Problème qui consiste à trouver une constante s et une fonction continue V telles que l'on ait:

$$s\, V(A) + \int_{(D)} V(B)\, F(A,B)\, d\tau_B = 0 \tag{35}$$

pour toutes les positions du point A dans le domaine (D).

Il est aisé de s'assurer de différentes façons qu'une fonction V vérifiant la relation (35) est nécessairement développable en une série uniformément convergente de la forme suivante:

$$V = \sum_{j=1}^{\infty} l_j\, u_j \tag{36}$$

References of Zaremba's

S. Zaremba, L'équation biharmonique et une class remarquable de functions fondamentales harmoniques, *Bulletin International de l'Académie des Sciences de Cracovie, Classe des Sciences Mathématiques et Naturelles*, **1907**, 147–196.

S. Zaremba, Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique, *ibidem* **1909**, 125–195.