Reproducing Pairs and Gabor Systems at Critical Density

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Reproducing Pairs

Frames: $\Psi = \{\psi_k\}_{k\in\mathcal{I}}$ is a frame for $\mathcal{H}$

$$A\|f\|^2 \leq \sum_{k\in\mathcal{I}} |\langle f,\psi_k\rangle|^2 \leq B\|f\|^2,\ \forall f \in \mathcal{H}$$

$$\Leftrightarrow$$

$$S_\Psi f = \textstyle\sum_{k\in\mathcal{I}}\langle f,\psi_k\rangle\psi_k \ \in \ GL(\mathcal{H})\ .$$

Many complete systems do not satisfy both frame bounds:
⇒ New concepts needed, e.g. semi-frames, or:

Definition 1 (Balazs & S., 2015)

Two families (Ψ,Φ) in $\mathcal{H}$ are called a **reproducing pair** if $S_{\Psi,\Phi} \in GL(\mathcal{H})$, where

$$\langle S_{\Psi,\Phi} f, g\rangle := \sum_{k\in\mathcal{I}}\langle f,\psi_k\rangle\langle\phi_k,g\rangle.$$

Reproducing Pairs

Examples:

- $\Phi = \{e_1, 2e_2, 1/3e_3, 4e_4 ...\}$ and $\Psi = \{e_1, 1/2e_2, 3e_3, 1/4e_4 ...\}$
- cross admissiblility a) nonstationary Gabor systems
  b) wavelets on $\mathbb{R}$ or S^2

New spaces replacing $\ell^2(\mathcal{I})$ are needed:

- Let $\mathcal{V}_\Phi(\mathcal{I}) := \{\xi : \mathcal{I} \to \mathbb{C}$ such that (1) holds$\}$,

$$\left| \sum_{k \in \mathcal{I}} \xi[k] \langle \phi_k, g \rangle \right| \leq c \left\| g \right\|, \forall \, g \in \mathcal{H}. \tag{1}$$

$\Rightarrow$ synthesis operator $D_\Phi \xi := \sum_{k \in \mathcal{I}} \xi[k] \phi_k$ is weakly well-defined.

- On $\mathcal{V}_\Phi(\mathcal{I}) / \text{Ker}\, D_\Phi$ one can introduce the inner product

$$\langle \xi, \eta \rangle_\Phi := \langle D_\Phi \xi, D_\Phi \eta \rangle.$$

Reproducing Pairs

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Gabor systems/ coherent states

Take the Weyl-Heisenberg group and consider the following representation on $L^2(\mathbb{R})$:

$$\pi(x,\omega)g(t) := e^{2\pi i\omega(t-x)}g(t-x), \ (x,\omega) \in \mathbb{R}^2$$

Gabor system: $G(g,a,b) := \{\pi(an,bm)g\}_{n,m\in\mathbb{Z}}$

- For $ab < 1$ oversampling: $\exists$ well-localized Gabor frames
- Interesting case: critical density $ab = 1$ (von Neumann lattice)

Theorem 2 (Amalgam Balian-Low Theorem (BLT))

If a Gabor system $G(g,a,b)$ at critical density (ab=1) is a frame then either $g \notin W_0(\mathbb{R})$ or $\hat{g} \notin W_0(\mathbb{R})$, where

$$W_0(\mathbb{R}) := \Big\{ f \in C(\mathbb{R}) : \sum_{n\in\mathbb{Z}} \operatorname{ess\,sup}_{x\in[0,1]} |f(x+n)| < \infty \Big\}.$$

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Outline

- We consider two cases: the reproducing partner is
  - a) another Gabor system ($\rightsquigarrow$ Zak transform methods, more BLTs)
  - b) an arbitrary system ($\rightsquigarrow$ reproducing pairs result)
- For b) we investigate the system $\mathcal{G} := G(\varphi, 1, 1)$: integer time-frequency shifts of the Gaussian

$$\varphi(t) = 2^{1/4} e^{-\pi t^2}$$

$\rightsquigarrow$ classical coherent state system on the von Neumann lattice

Zak transform methods

Zak transform:

$$Zf(x,\omega) := \sum_{k\in\mathbb{Z}} f(x-k)e^{2\pi i\omega k}$$

Properties:

- $Z : L^2(\mathbb{R}) \to L^2([0,1]^2)$ is an isometric isomorphism
- Diagonalization of Gabor frame operator: $Z(S_{g,\gamma}f) = \overline{Zg} \cdot Z_\gamma \cdot Z_f$

How to "trick" Balian-Low: Choose a nice window and calculate the dual (not well localized)

Choose $g \in L^2(\mathbb{R})$, s.t.: (i) $g, \hat{g} \in W_0(\mathbb{R})$, (ii) $1/Zg \in L^2([0,1]^2)$

$$\Rightarrow \ \gamma := Z^{-1}\left(1/\overline{Zg}\right) \in L^2(\mathbb{R}) \quad \& \quad S_{g,\gamma} = I$$

Example: for $(x,\omega) \in [0,1]^2$ set

$$Zg(x,\omega) = e^{2\pi i x\cdot\omega}\omega^{1/4}(1-\omega)^{1/4}$$

Limits of Zak transform approach

Proposition 3 (Some extensions of the BLT)

Let $\big(G(g,1,1), G(\gamma,1,1)\big)$ *be a reproducing pair, then*

- $g \notin M_2^2(\mathbb{R})$ **and** $\gamma \notin M_2^2(\mathbb{R})$ *[Daubechies & Janssen, 93]*
- $g \notin M_1^1(\mathbb{R})$ **and** $\gamma \notin M_1^1(\mathbb{R})$

$$\textit{where} \quad \|f\|_{M_s^p}^p := \int_{\mathbb{R}^2} |\langle f, \pi(x,\omega)\varphi\rangle|^p (1+|x|+|\omega|)^{ps} dx d\omega$$

Idea of proofs: One shows that if g satisfies the property then $1/Zg \notin L^2([0,1]^2) \Rightarrow$ not a reproducing pair. ☐

$\Rightarrow$ Rather mild decay on the TF-plane exclude the possibility of reproducing pairs consisting of **2 Gabor** systems

$\rightsquigarrow$ consider arbitrary dual systems

Canonical coherent states $\mathcal{G}$

Q1: Is the system $\mathcal{G}$ complete in $L^2(\mathbb{R})$? (von Neumann)
Yes: See [Perelomov 71, Bargmann et al., 71, Bacry et al., 75].

Q2: Is there a linear coefficient map $A : L^2(\mathbb{R}) \to \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$, s.t.

$$f = \sum_{k,l\in\mathbb{Z}} (Af)[k,l] T_k M_l \varphi, \ \forall f \in L^2(\mathbb{R}),$$

with convergence in some sense? (Gabor)
Yes: See [Janssen, 81].

Q4: Is there a dual window $\gamma \in L^2(\mathbb{R})$ for the Gaussian φ, s.t.

$$f = \sum_{k,l\in\mathbb{Z}} \langle f, T_k M_l \gamma \rangle T_k M_l \varphi, \ \forall f \in L^2(\mathbb{R}) \ ?$$

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New Question:

Q3: Can the coefficient map be calculated using inner products with an arbitrary family in $L^2(\mathbb{R})$, i.e., is there $\Psi := \{\psi_{k,l}\}_{k,l\in\mathbb{Z}} \subset L^2(\mathbb{R})$, s.t.

$$f = \sum_{k,l\in\mathbb{Z}} \langle f, \psi_{k,l}\rangle T_k M_l \varphi, \ \forall f \in L^2(\mathbb{R}) \ ?$$

Equivalent formulations:

- Is there a reproducing partner for $\mathcal{G}$?
- Is there a dual system for the upper semi-frame $\mathcal{G}$?
- Is there a dual system for the complete Bessel system $\mathcal{G}$?

Existence of reproducing partners

Theorem 4 (Antoine, Trapani & S., 2016)

Let $\Phi = \{\phi_k\}_{k\in\mathcal{I}} \subset \mathcal{H}$ *and* $\mathcal{E} = \{e_k\}_{k\in\mathcal{I}}$ *an ONB of* $\mathcal{H}$. *There exists* Ψ, *s.t.* (Ψ, Φ) *is a reproducing pair if and only if*

(A) Ran $D_\phi = \mathcal{H}$

(B) there exists a family $\{\xi_k\}_{k\in\mathcal{I}} \subset \mathcal{V}_\Phi(\mathcal{I})$, *s.t.*

$$D_\Phi \xi_k = e_k,\ \forall\ k \in \mathcal{I}, \quad \textit{and} \quad \sum_{k\in\mathcal{I}} |\xi_k[n]|^2 < \infty,\ \forall\ n \in \mathcal{I}.$$

In particular, $\psi_n := \sum_{k\in\mathcal{I}} \overline{\xi_k[n]} e_k$ *gives a reproducing partner.*

- (A) is not trivial: $\exists$ complete system, s.t. (A) does not hold
- Interpretation: (i) $(A) \Leftrightarrow Q2$, $(B) \Rightarrow Q3$
  (ii) $\{\xi_k\}_{k\in\mathcal{I}}$ is an ONB of a RKHS w.r.t. $\langle\cdot,\cdot\rangle_\Phi$

Illustration of condition (B)

$$\begin{array}{ccccccccl}
 & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\cdots & \xi_2[-2] & \xi_2[-1] & \xi_2[0] & \xi_2[1] & \xi_2[2] & \cdots & & \xi_2 \\
\cdots & \xi_1[-2] & \xi_1[-1] & \xi_1[0] & \xi_1[1] & \xi_1[2] & \cdots & & \xi_1\colon \quad D_{\mathcal{G}}\xi_1 = e_1 \\
\cdots & \xi_0[-2] & \xi_0[-1] & \xi_0[0] & \xi_0[1] & \xi_0[2] & \cdots & & \xi_0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}$$

Illustration of condition (B)

$$\begin{array}{ccc}
 & \begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} & \\
 & \begin{array}{ccccccc} \cdots & \xi_2[-2] & \xi_2[-1] & \xi_2[0] & \xi_2[1] & \xi_2[2] & \cdots \end{array} & \xi_2 \\
\sum & \boxed{\begin{array}{ccccccc} \cdots & \xi_1[-2] & \xi_1[-1] & \xi_1[0] & \xi_1[1] & \xi_1[2] & \cdots \end{array}} & \xi_1 \notin \ell^2(\mathbb{Z}) \\
 & \begin{array}{ccccccc} \cdots & \xi_0[-2] & \xi_0[-1] & \xi_0[0] & \xi_0[1] & \xi_0[2] & \cdots \end{array} & \xi_0 \\
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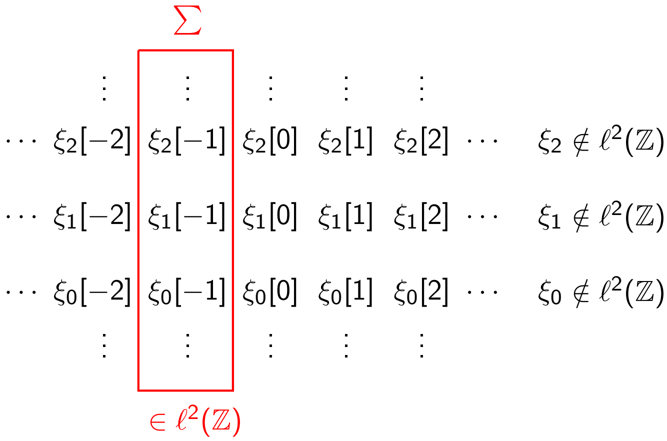

$$\begin{array}{cccccccc}
 & & \sum & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
\cdots & \xi_2[-2] & \xi_2[-1] & \xi_2[0] & \xi_2[1] & \xi_2[2] & \cdots & \xi_2 \notin \ell^2(\mathbb{Z}) \\
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 & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & & \in \ell^2(\mathbb{Z}) & & & & &
\end{array}$$

Illustration of condition (B)

$$
\begin{array}{ccc|c|cccc}
 & & & \sum & & & & \\
\cline{4-4}
 & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
\cdots & \xi_2[-2] & \xi_2[-1] & \xi_2[0] & \xi_2[1] & \xi_2[2] & \cdots & \xi_2 \notin \ell^2(\mathbb{Z}) \\
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 & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
 & & & & \in \ell^2(\mathbb{Z}) & & & &
\end{array}$$

Shift invariance

- Take a Gabor ONB, e.g. $\gamma = \chi_{[-1/2,1/2]}$. One has to solve for

$$D_{\mathcal{G}}\xi_{k,l} = \pi(k,l)\gamma = T_k M_l \gamma$$

- Let ξ_0 be s.t. $D_{\mathcal{G}}\xi_0 = \gamma$, then

$$\begin{aligned} T_k M_l \gamma = T_k M_l D_{\mathcal{G}}\xi_0 &= \sum_{n,m} \xi_0[n,m] T_{n+k} M_{m+k} \varphi \\ &= \sum_{n,m} \xi_0[n-k, m-l] T_n M_m \varphi = D_{\mathcal{G}}(\mathcal{T}_{k,l}\xi_0), \end{aligned}$$

where $\mathcal{T}_{k,l}$ denotes the index shift operator.

Hence: For some $p_{k,l} \in \operatorname{Ker} D_{\mathcal{G}}$, $\xi_{k,l}$ is given by

$$\xi_{k,l} = \mathcal{T}_{k,l}\xi_0 + p_{k,l}.$$

Bastiaans dual: Calculate ξ_0

Defining $\beta := Z^{-1}(1/Z\varphi)$ yields Bastiaans dual window

$$\beta(t) = K \cdot e^{\pi t^2} \cdot \sum_{k>|t|-1/2} (-1)^n e^{-\pi(k+1/2)^2} \notin L^2(\mathbb{R}),$$

[Janssen, 82]: Under some assumptions on $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the coefficient map is given by:

$$(Af)(k,l) = \langle f, T_k M_l \beta \rangle$$

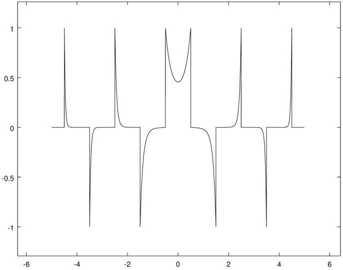

Figure : Bastiaans' dual window

Assumptions hold for γ

Thus, set $\xi_0[k,l] := \langle \gamma, T_k M_l \beta \rangle$

$$\Rightarrow \quad D_{\mathcal{G}}\xi_0 = \gamma$$

Characterization of Ker $D_{\mathcal{G}}$

- Take a closer look at ξ_0:

$$\xi_0[k,l] \asymp (1+k^2+l^2)^{-1}(-1)^{k+l}H_k$$

H_k converges quickly ⇝ choose appropriate elements in Ker $D_{\mathcal{G}}$!

- **[Janssen, 81]:** Every $p \in \text{Ker } D_{\mathcal{G}}$ is of the form

$$p[n,m] = (-1)^{n+m} \sum_{0 \le s+t \le N} c_{s,t} \cdot n^s \cdot m^t,$$

- Use constant polynomials, i.e. $p_{k,l}[n,m] = (-1)^{n+m} c_{k,l}$, where.

$$c_{k,l} := (2\pi)^{-1} e^{-\pi/4} \text{sgn}(k) \mathcal{F}(\mathcal{F}^{-1}(h_k)\varphi^{-1})[l],$$

where $h_k[l] = (-1)^{k+l}(k+il)^{-1}$.

Conclusion

Then, **for all** $(n, m) \in \mathbb{Z}^2$

$$\sum_{k,l\in\mathbb{Z}} |\xi_{k,l}[n,m]|^2 = \sum_{k,l\in\mathbb{Z}} |\xi_0[n-k, m-l] + (-1)^{n+m} c_{k,l}|^2 < \infty,$$

which by Theorem 4 yields the result:

Theorem 5 (One result, multiple languages)

- *There exists a reproducing partner for $\mathcal{G}$.*
- *There exists a dual system for the classical coherent states sampled on the von Neumann lattice.*
- *There exists a dual system for the complete Bessel sequence $\mathcal{G}$.*
- *The coefficient map A can be calculated using inner products.*

Some plots of $\psi_{k,l}$

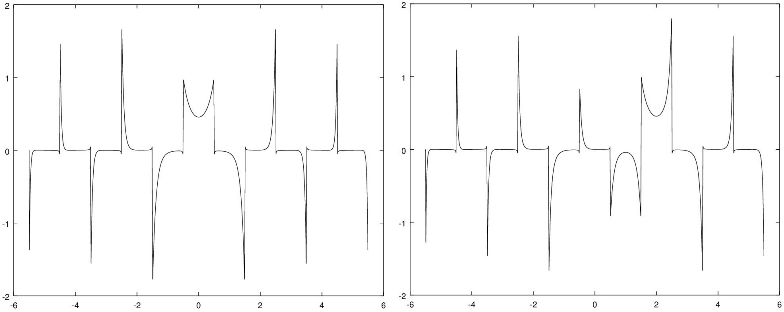

Figure : Left: $\psi_{0,0}$, Right: $\psi_{2,0}$

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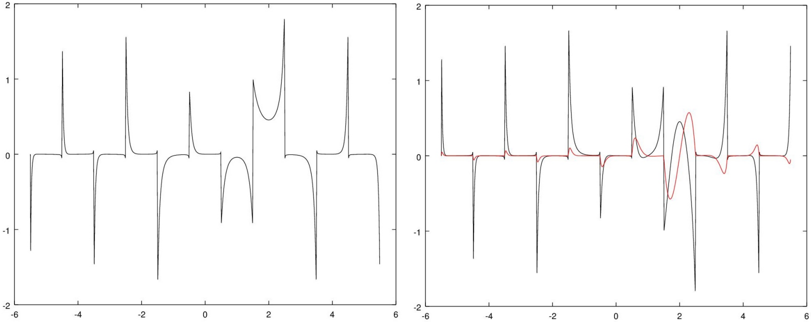

Figure : Left: $\psi_{2,0}$, Right: $\psi_{2,1}$

The End

Thank you for your attention!

Questions, Comments...

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